What is topological order, and is it present in the high temperature superconductors ?

Tata Institute of Fundamental Research, Mumbai

Subir Sachdev

January 10, 2017

Talk online: sachdev.physics.harvard.edu

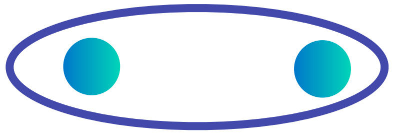
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

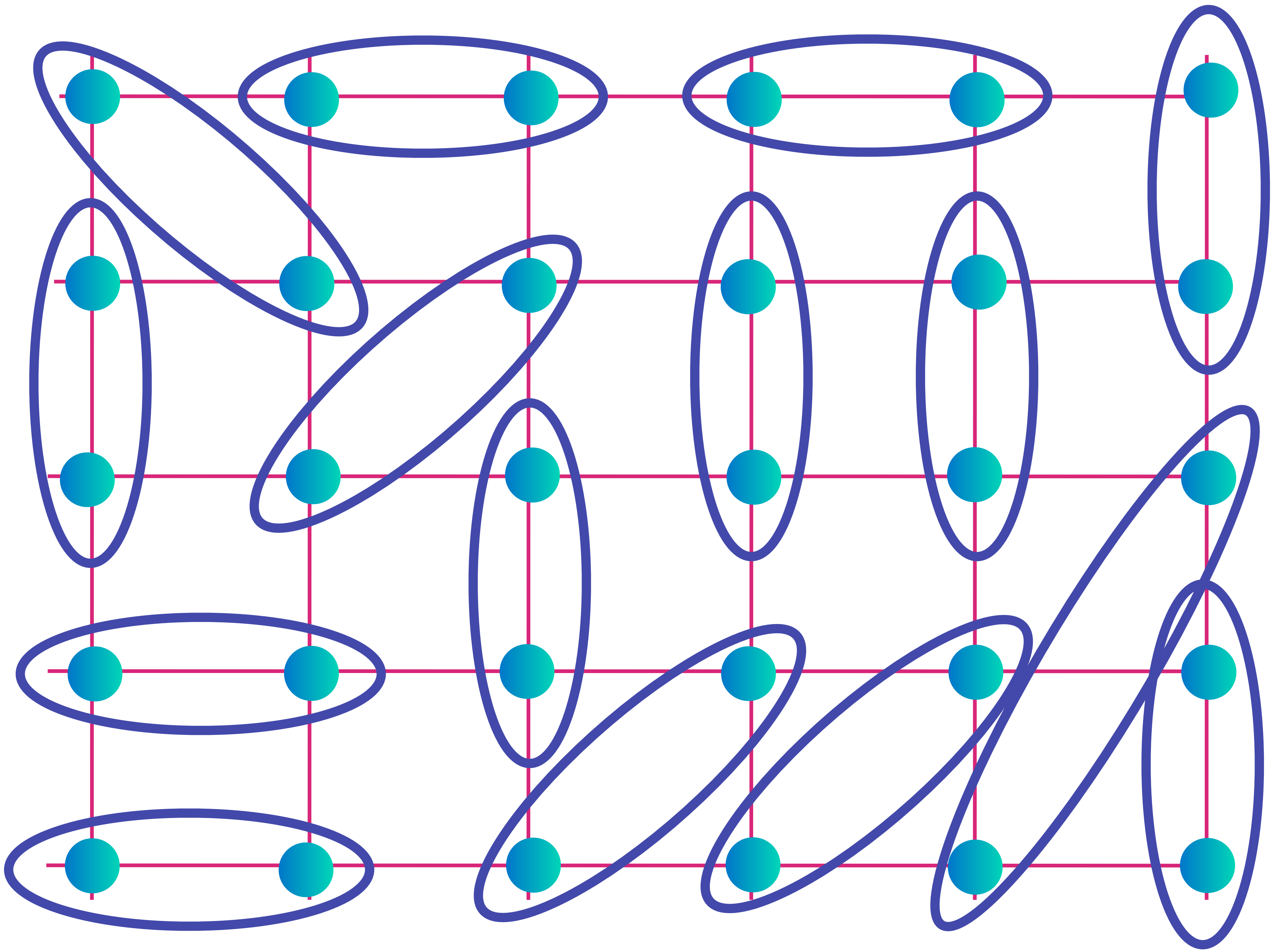
PHYSICS

HARVARD

1. Introduction to spin liquids and topological order
2. Topological order and phase transitions in a model of bosons on the square lattice
3. Survey of recent experiments in the cuprates
4. Model of a topological phase transition for the cuprates

Insulating spin liquid

 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) / \sqrt{2}$



Lattice realization of a topological quantum field theory (TQFT)

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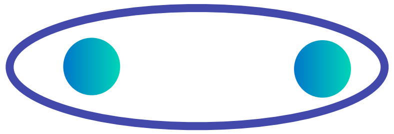
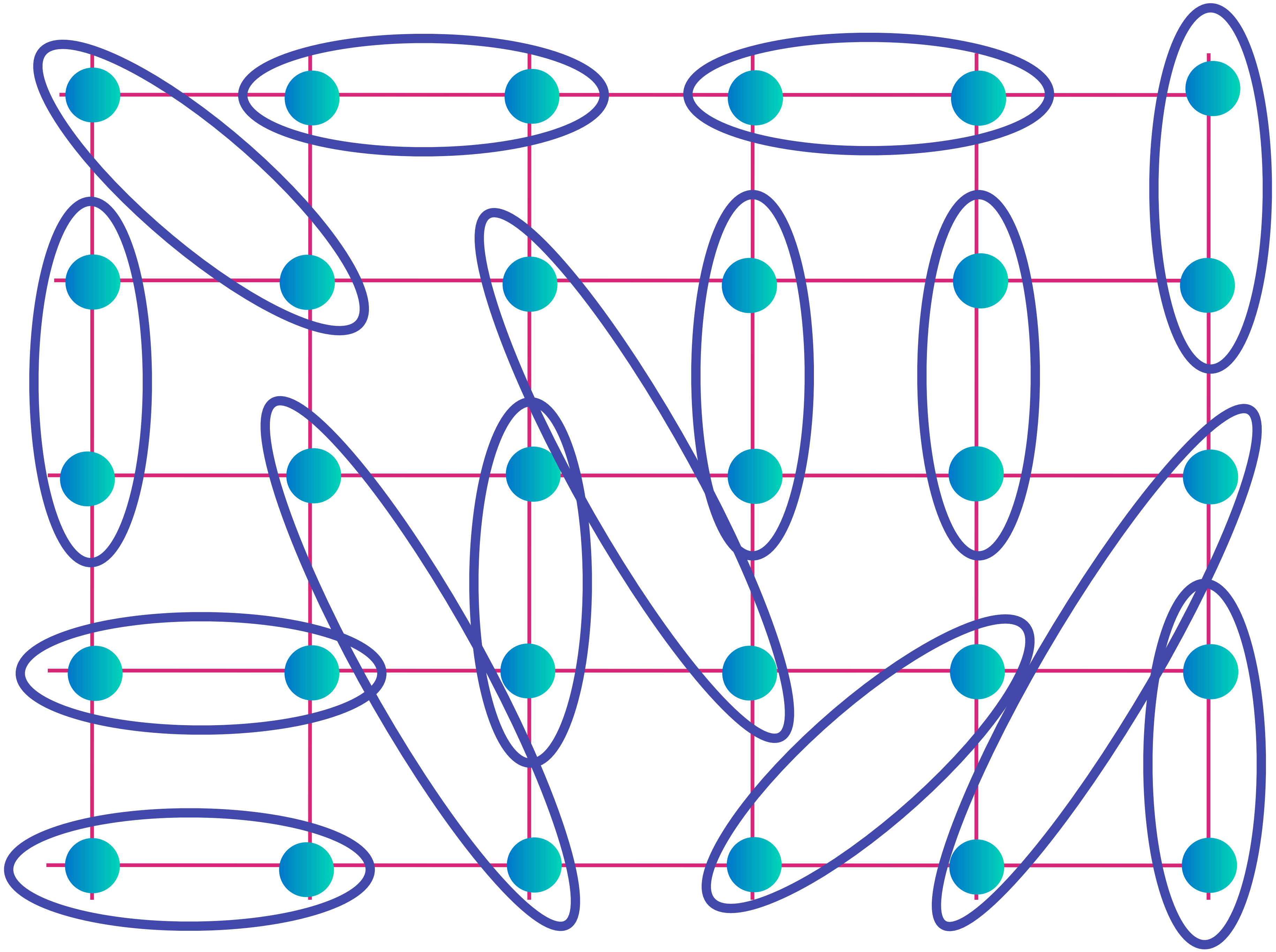
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) / \sqrt{2}$



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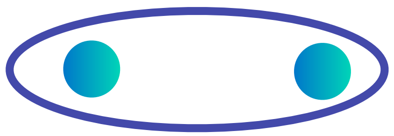
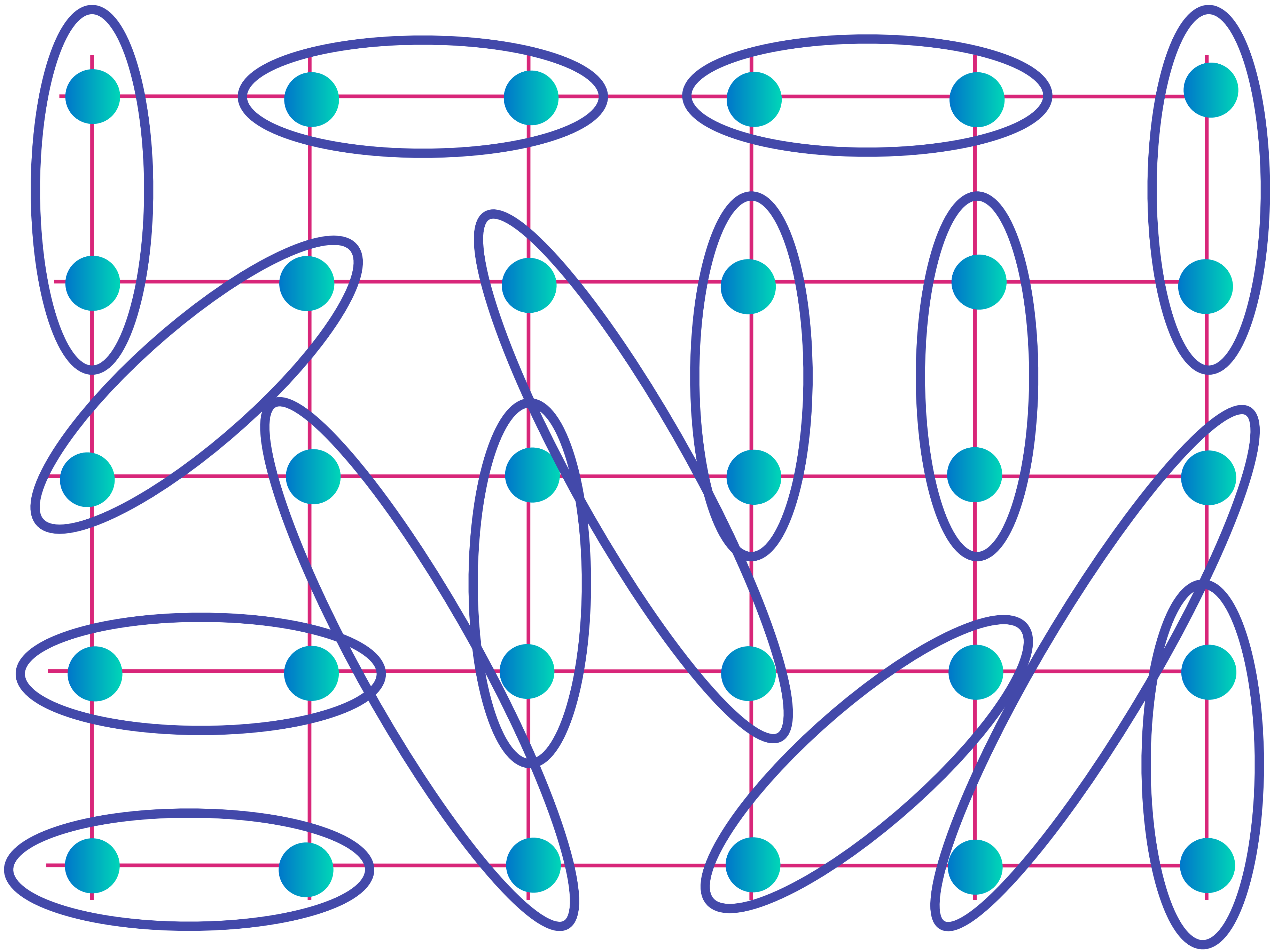
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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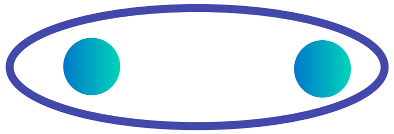
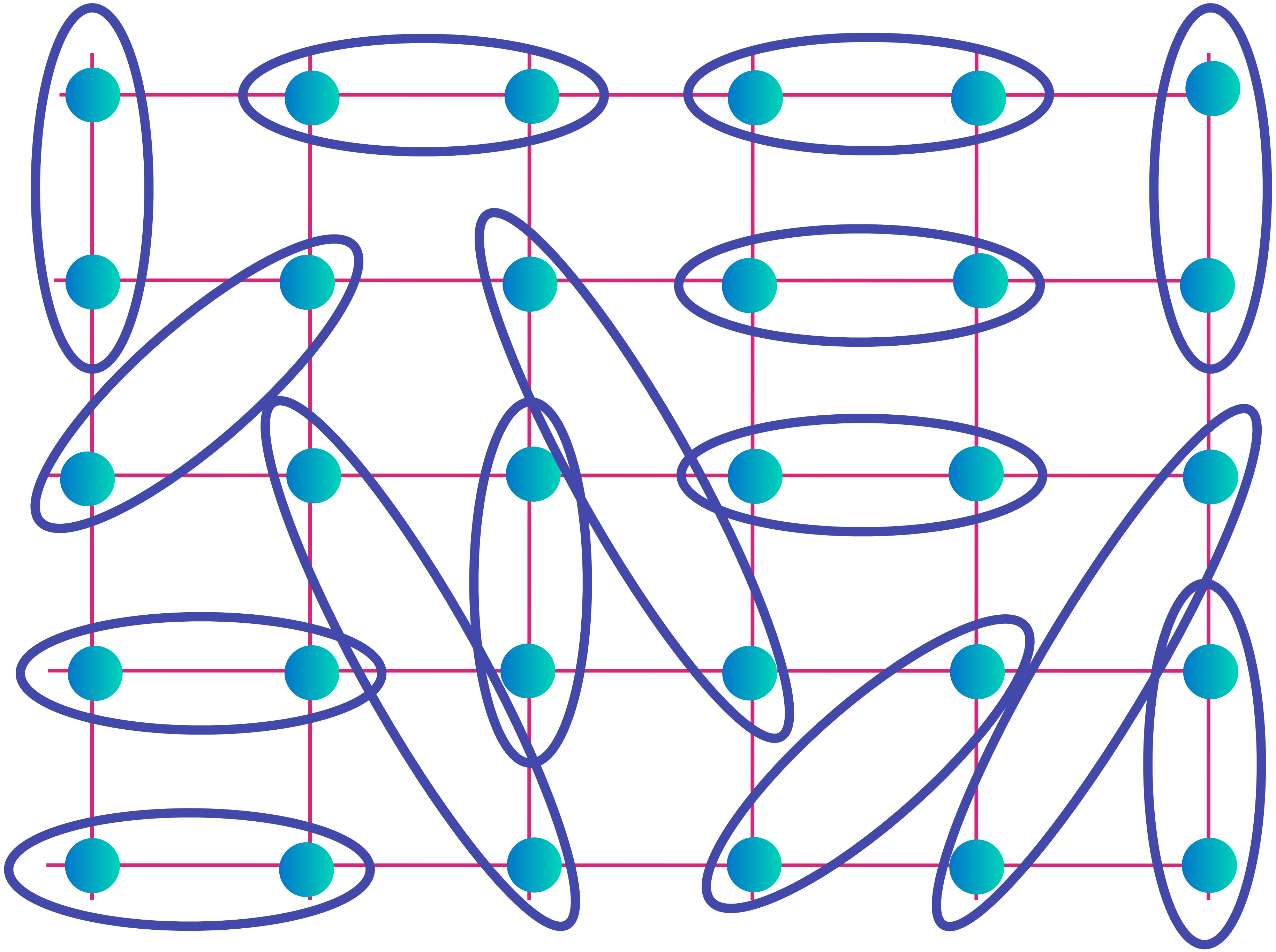
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) / \sqrt{2}$



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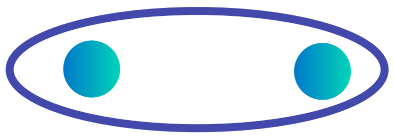
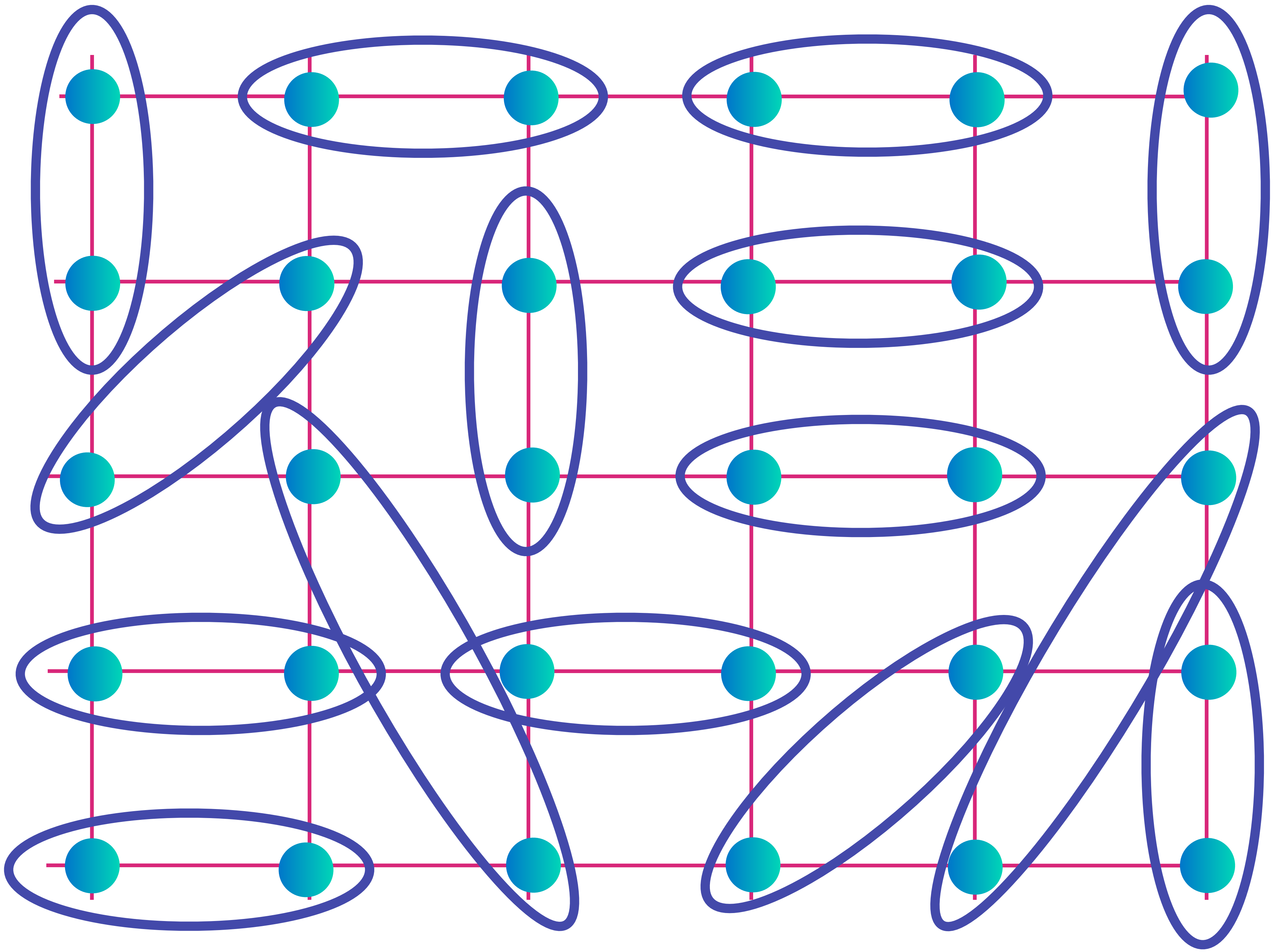
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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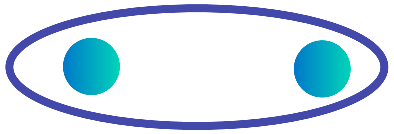

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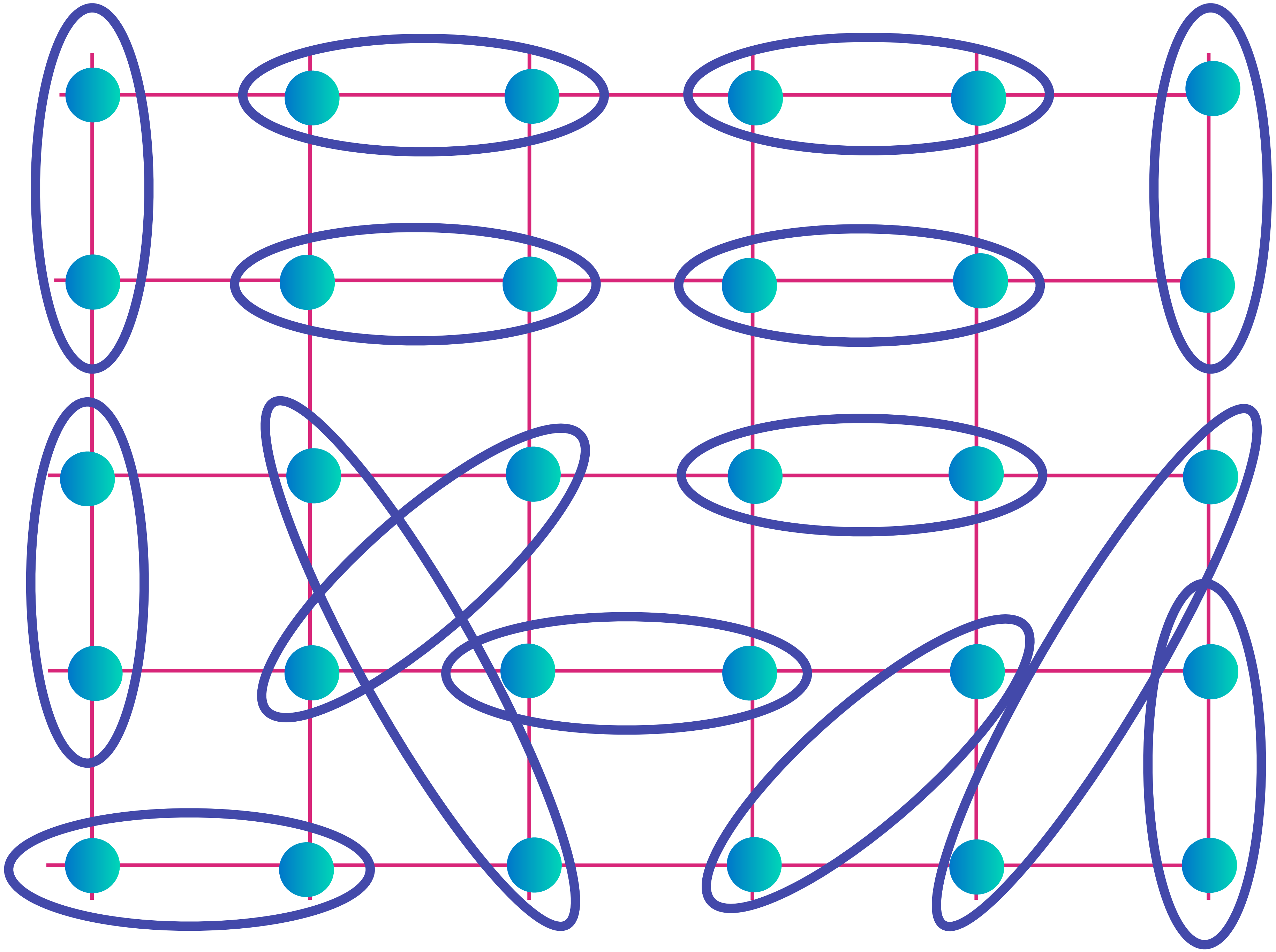

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Why is this a TQFT ?

Place insulator on a torus:

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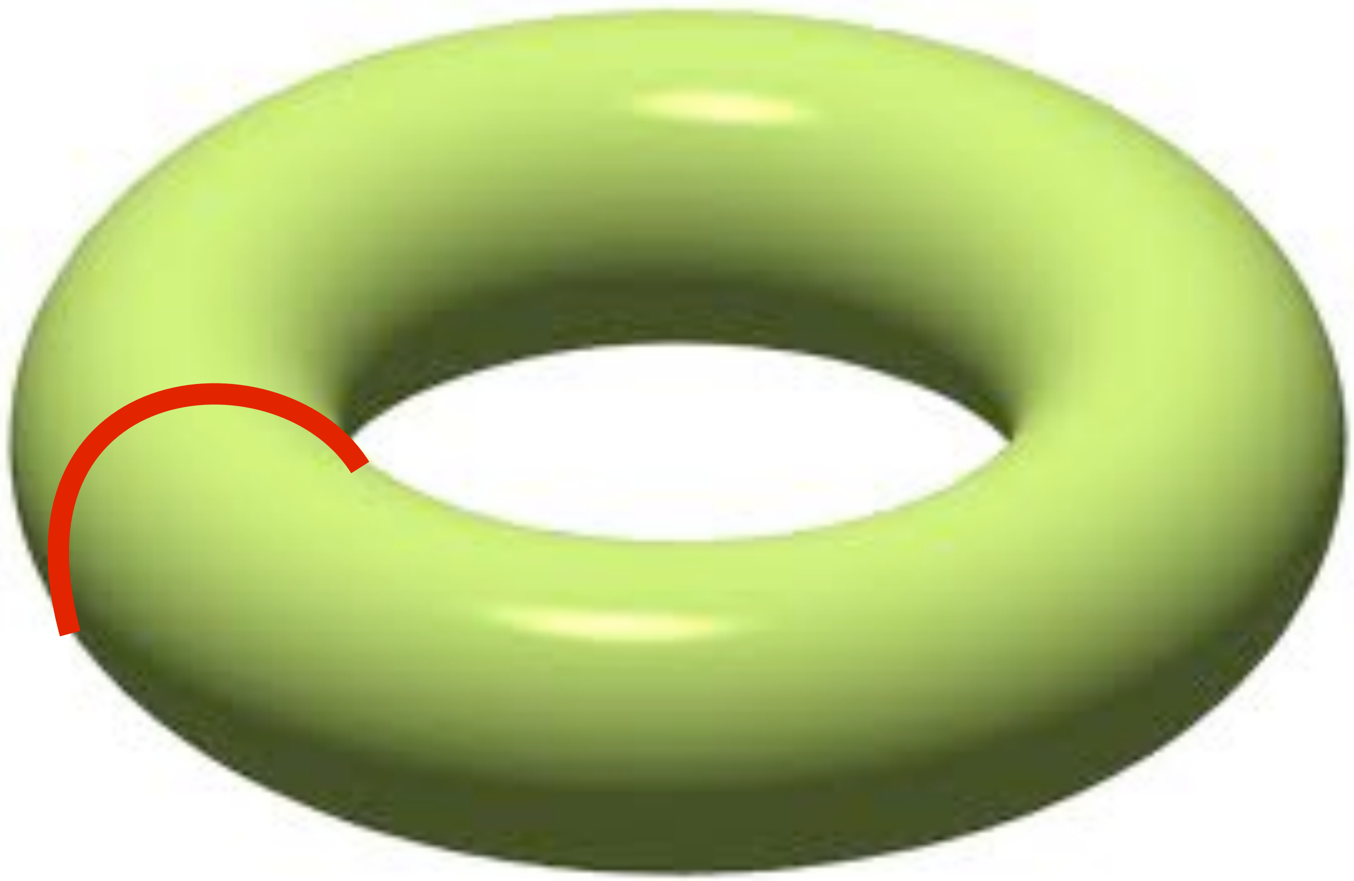

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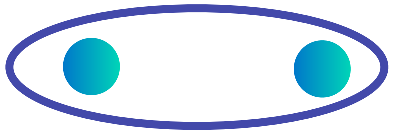
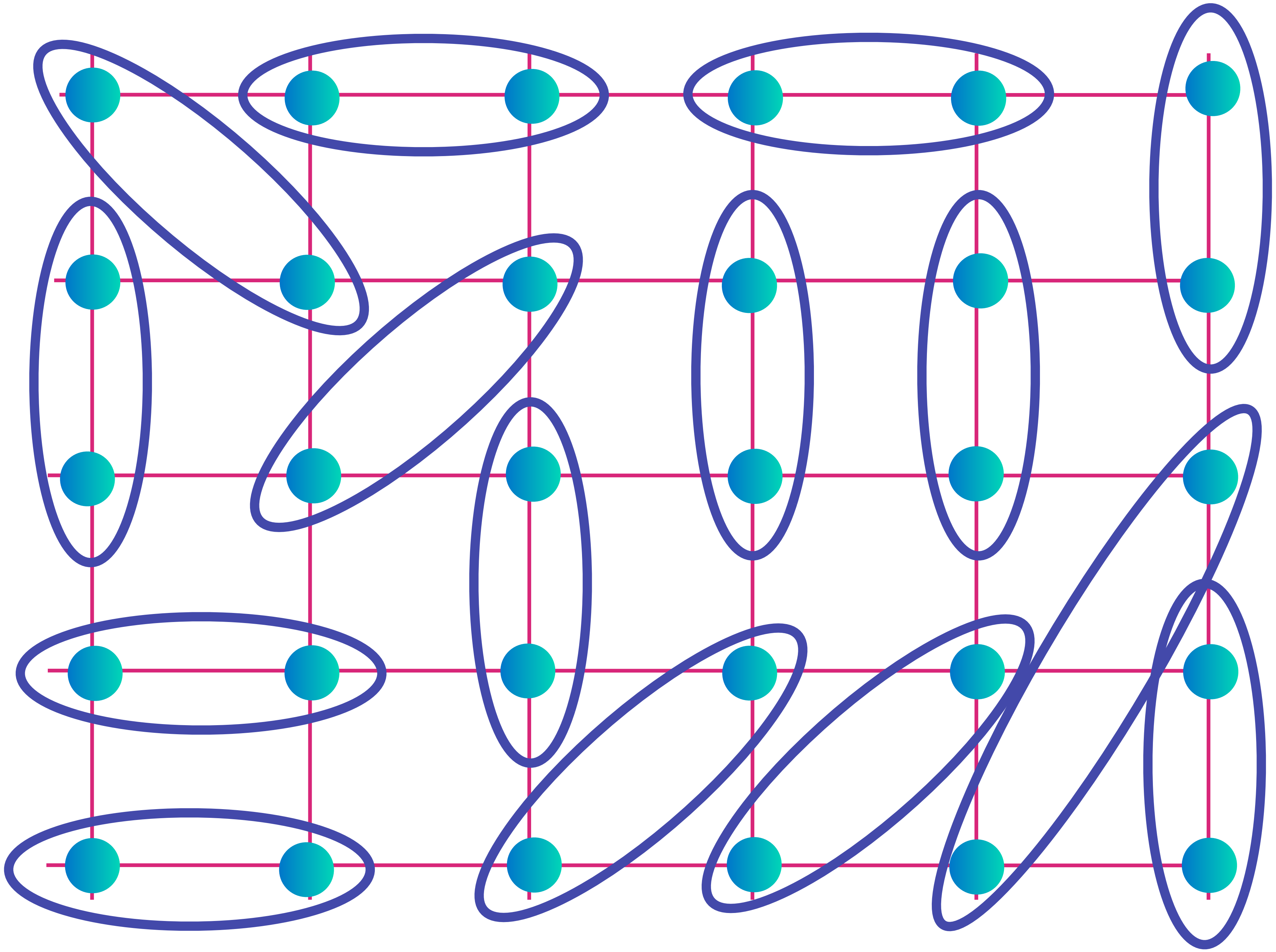
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$



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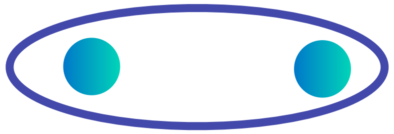

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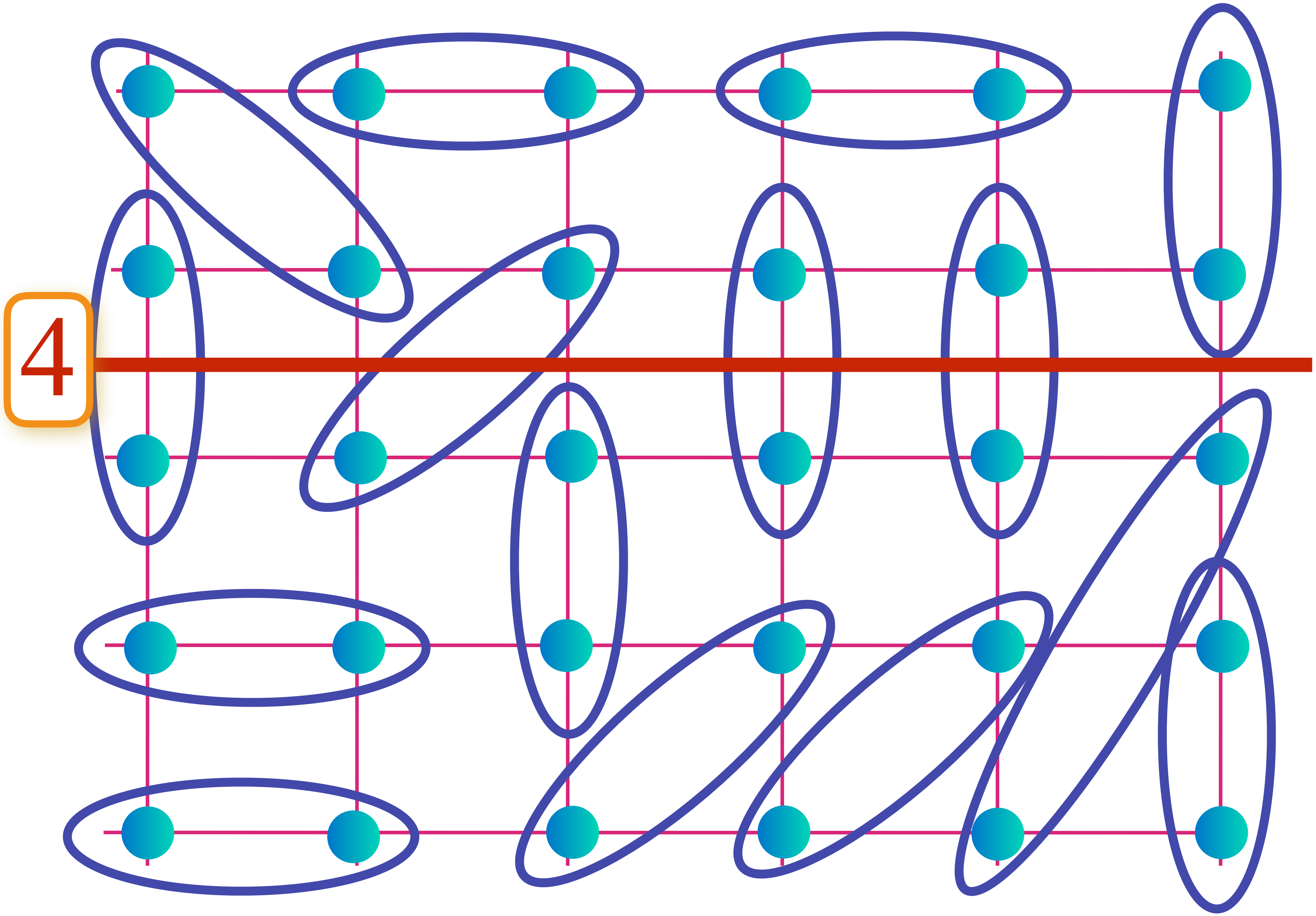

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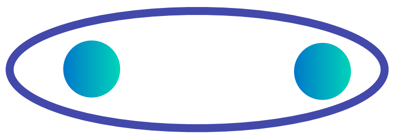

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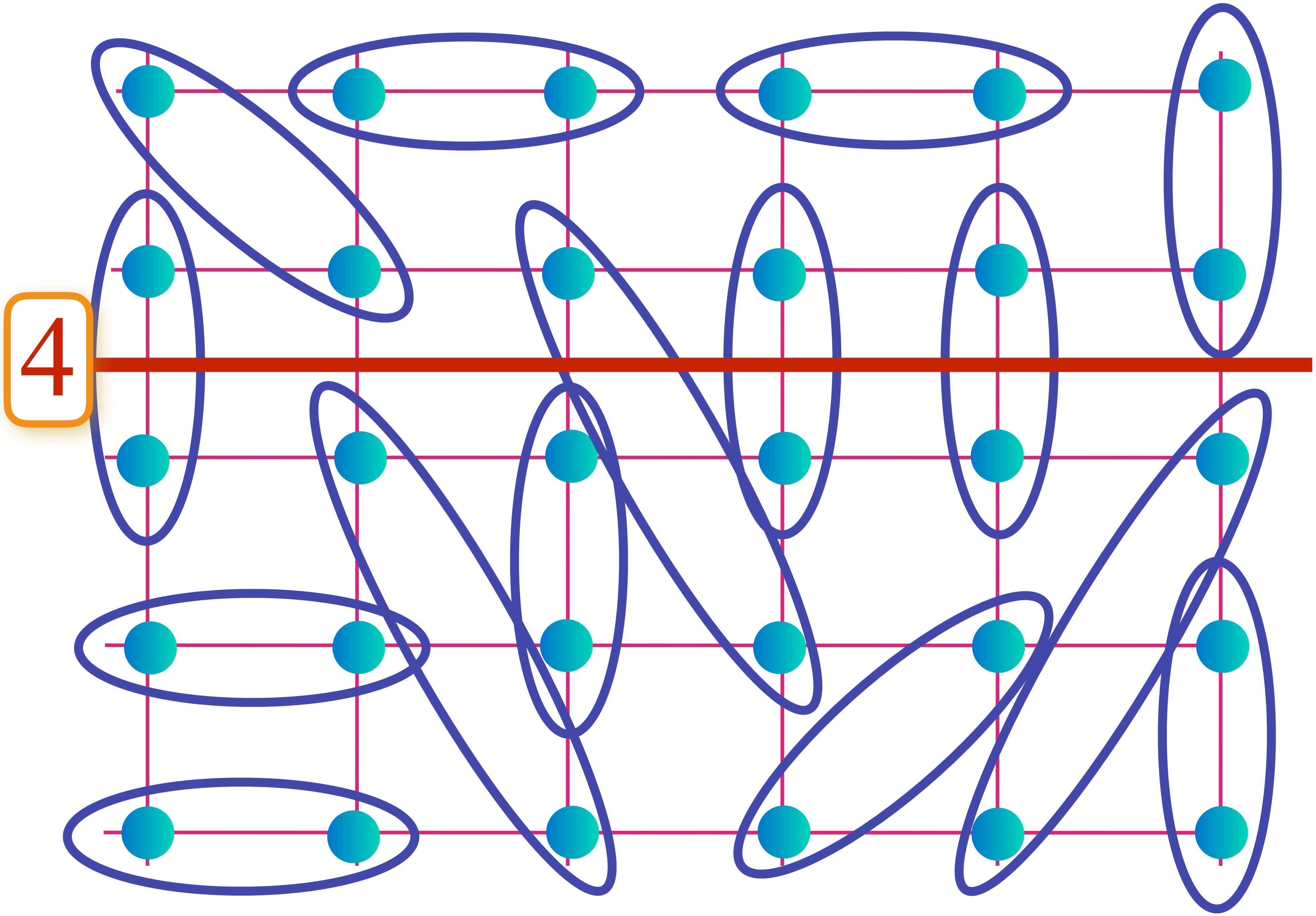

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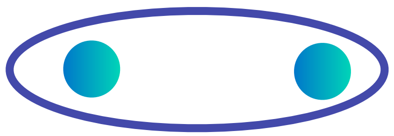
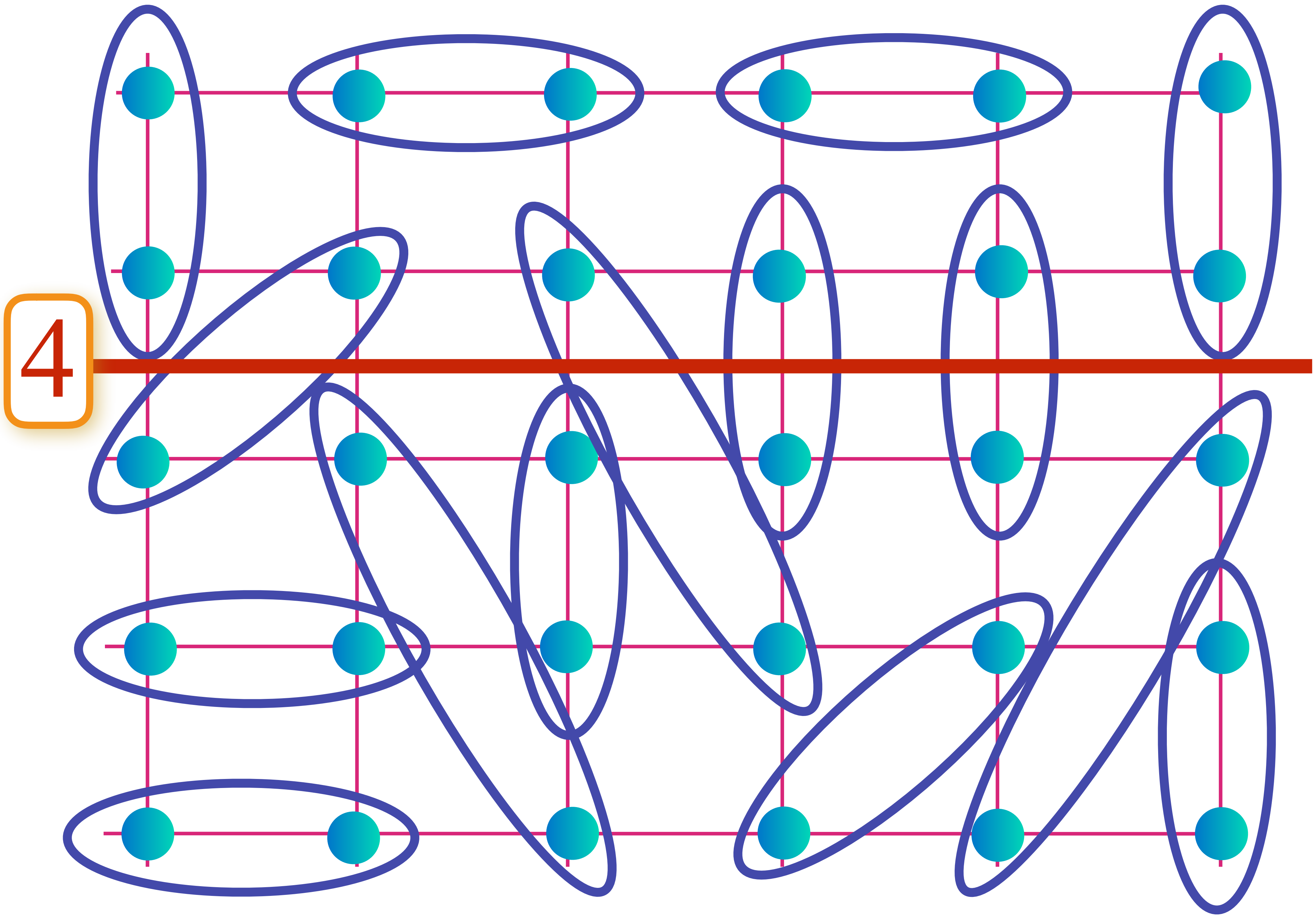
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) / \sqrt{2}$



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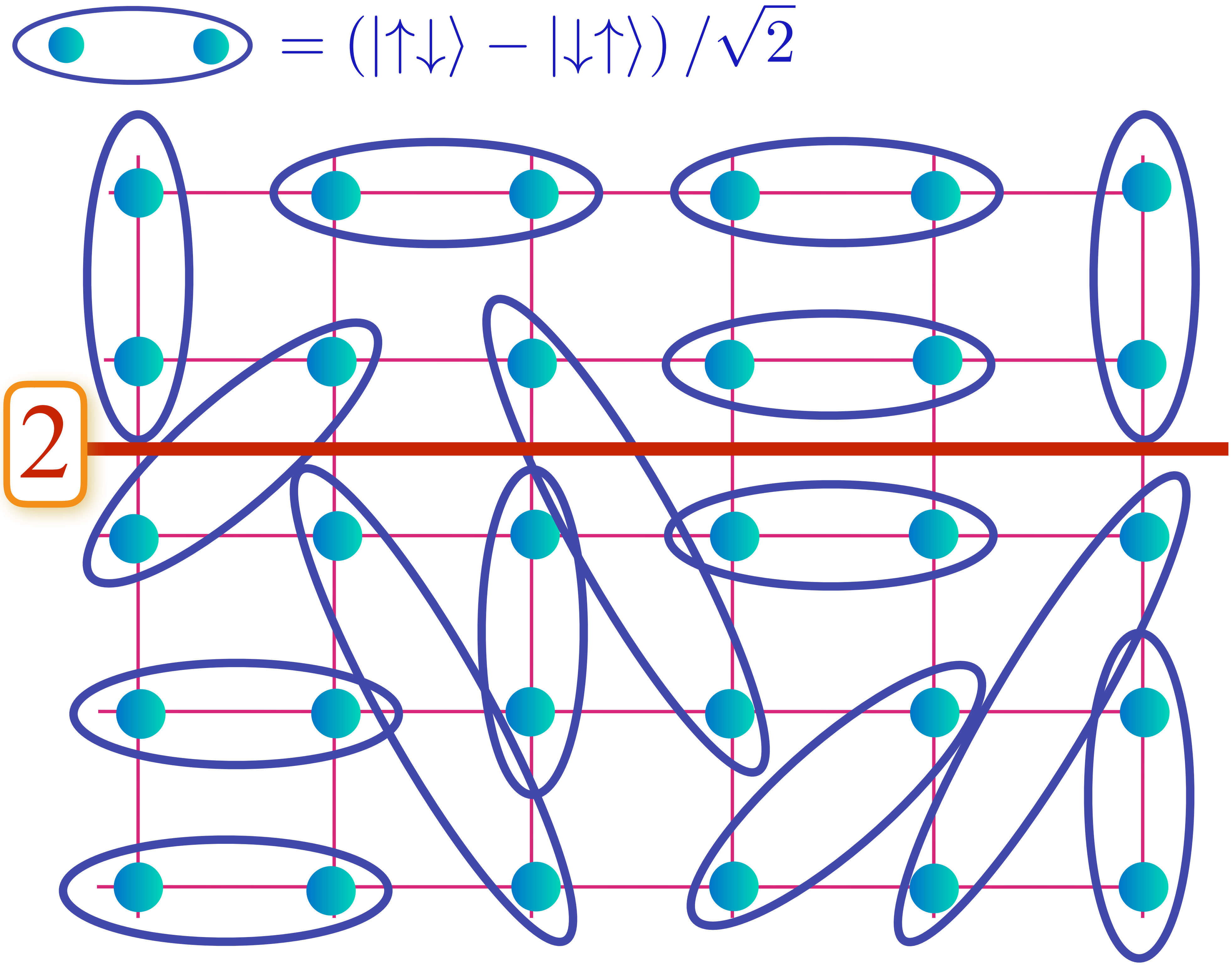

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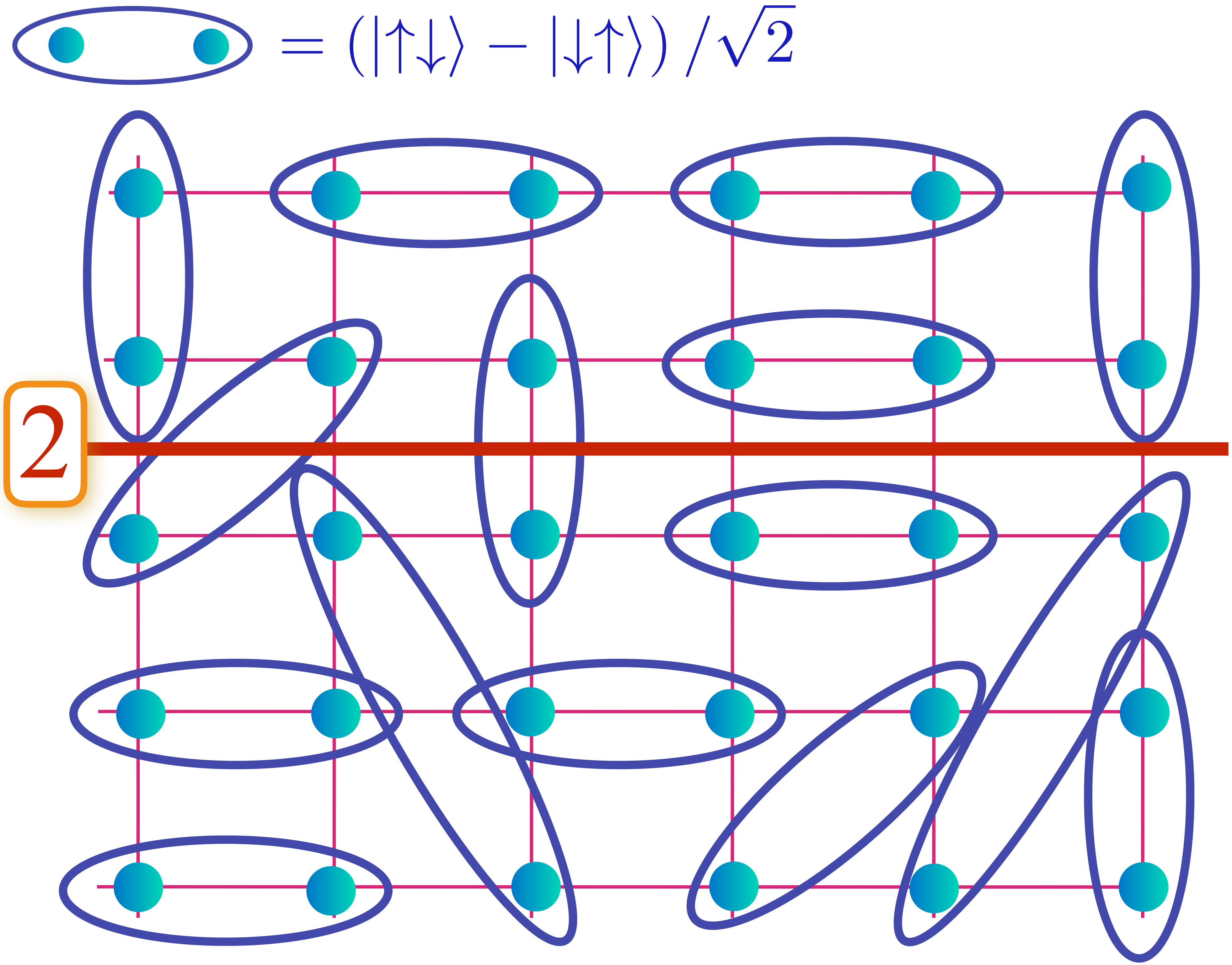

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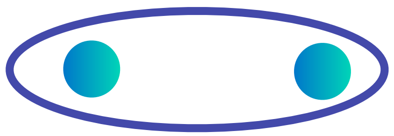

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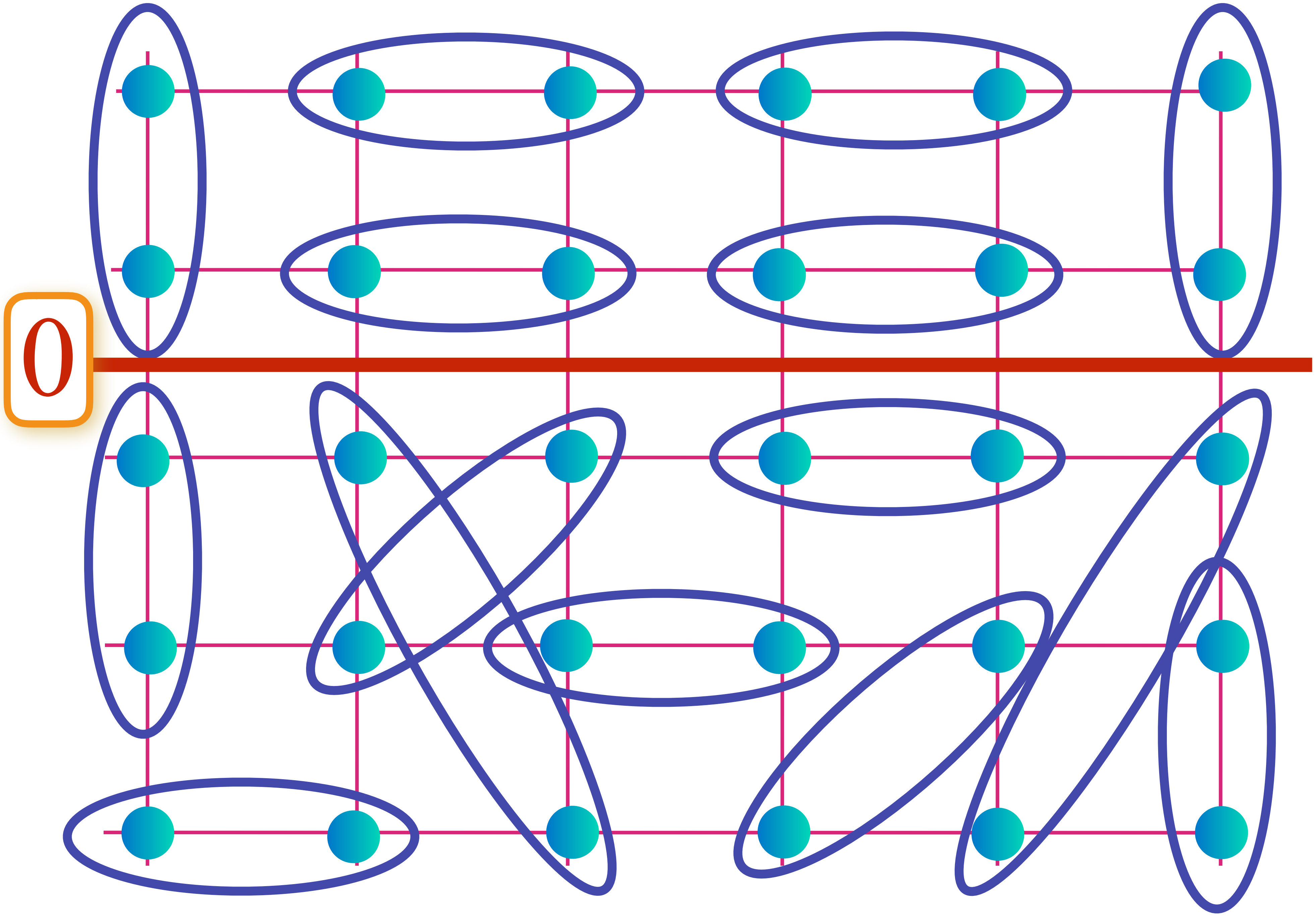

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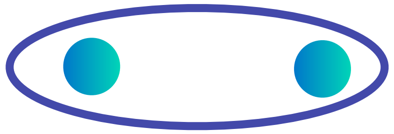
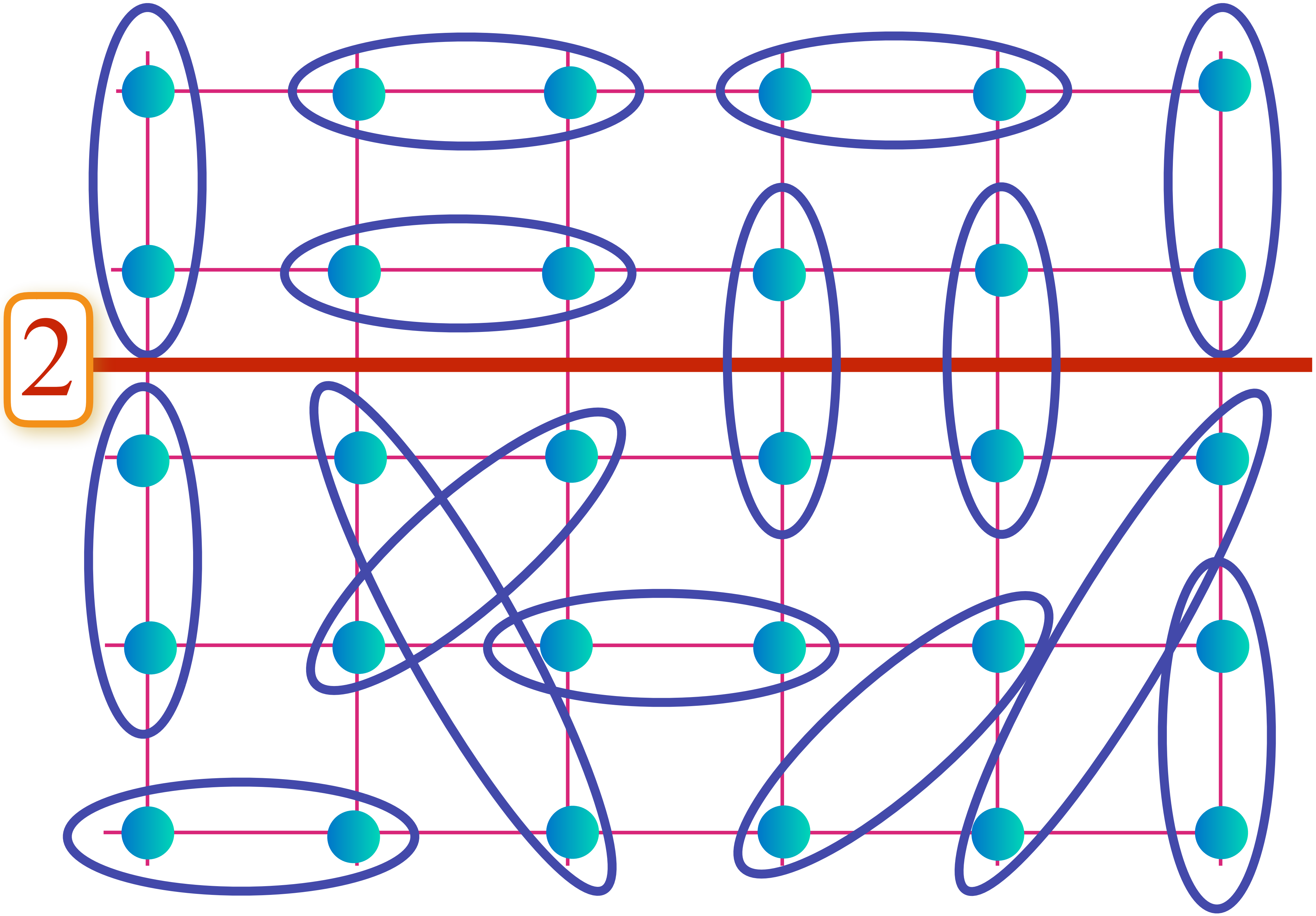
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) / \sqrt{2}$



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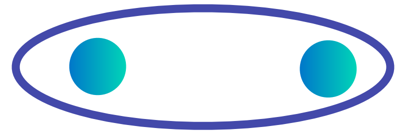
Introduce a pseudo spin, τ_ℓ^z on every link ℓ.



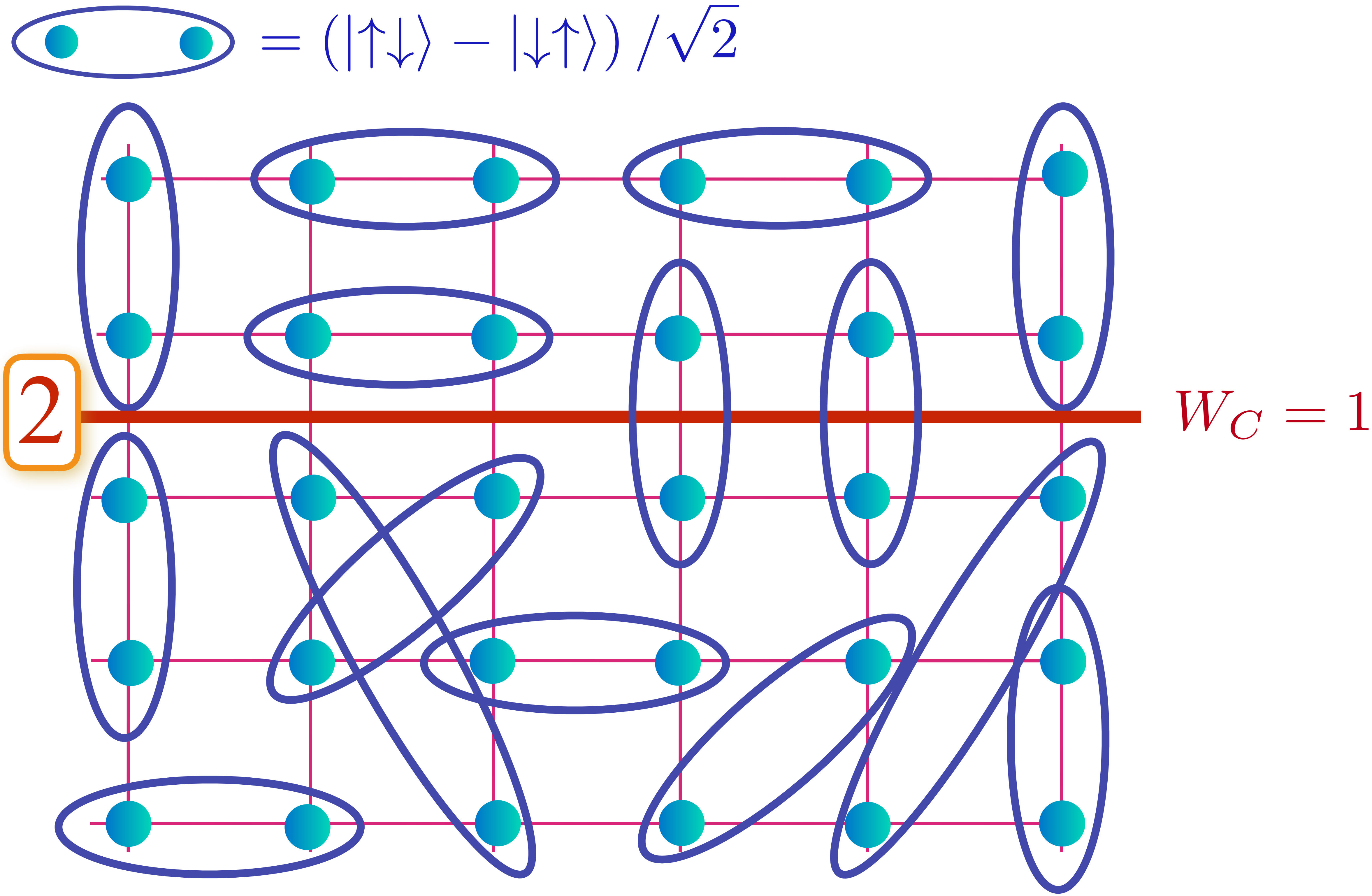
$= |-1\rangle$

$= |+1\rangle$

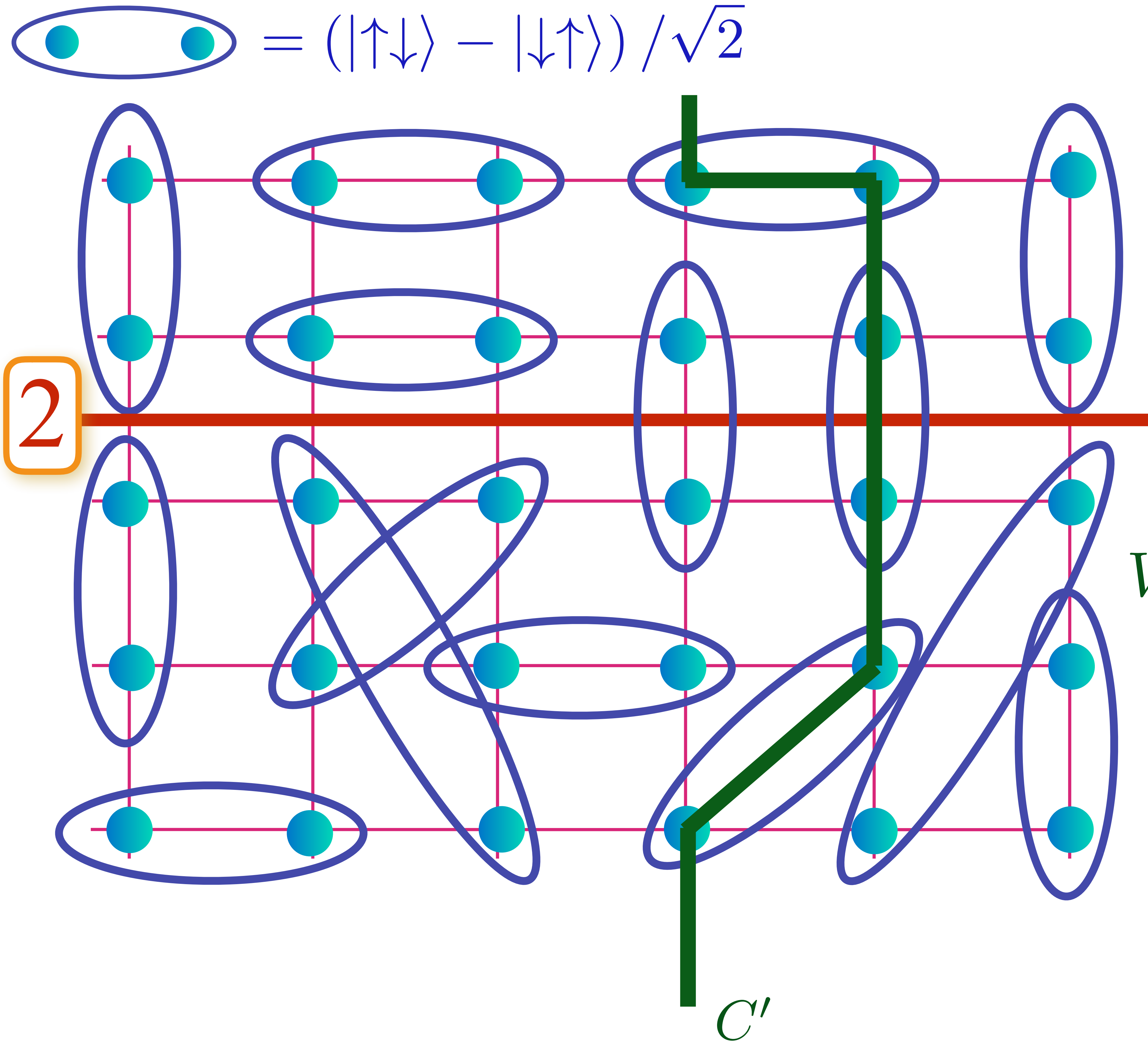
'Odd' dimer constraint: $$\prod_{\ell \text{ on site } i} \tau_\ell^z = -1.$$

Topologically conserved charge:

$$W_C = \prod_{\ell \text{ cuts contour } C} \tau_\ell^z = \pm 1.$$


$= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$
2
$W_C = 1$

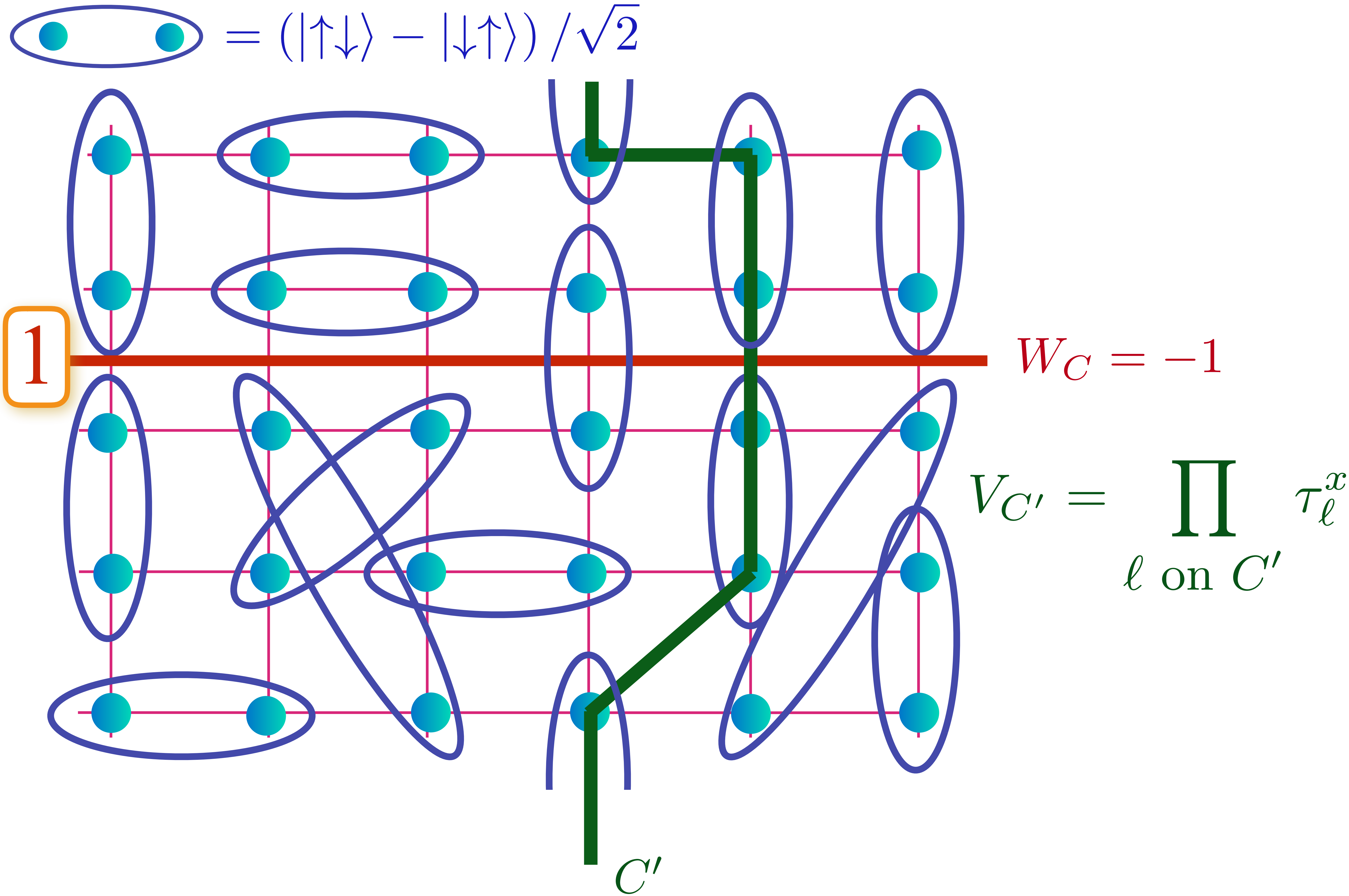
Operator to change W_C



$W_C = 1$

$$V_{C'} = \prod_{\ell \text{ on } C'} \tau^x_\ell$$

Operator to change W_C



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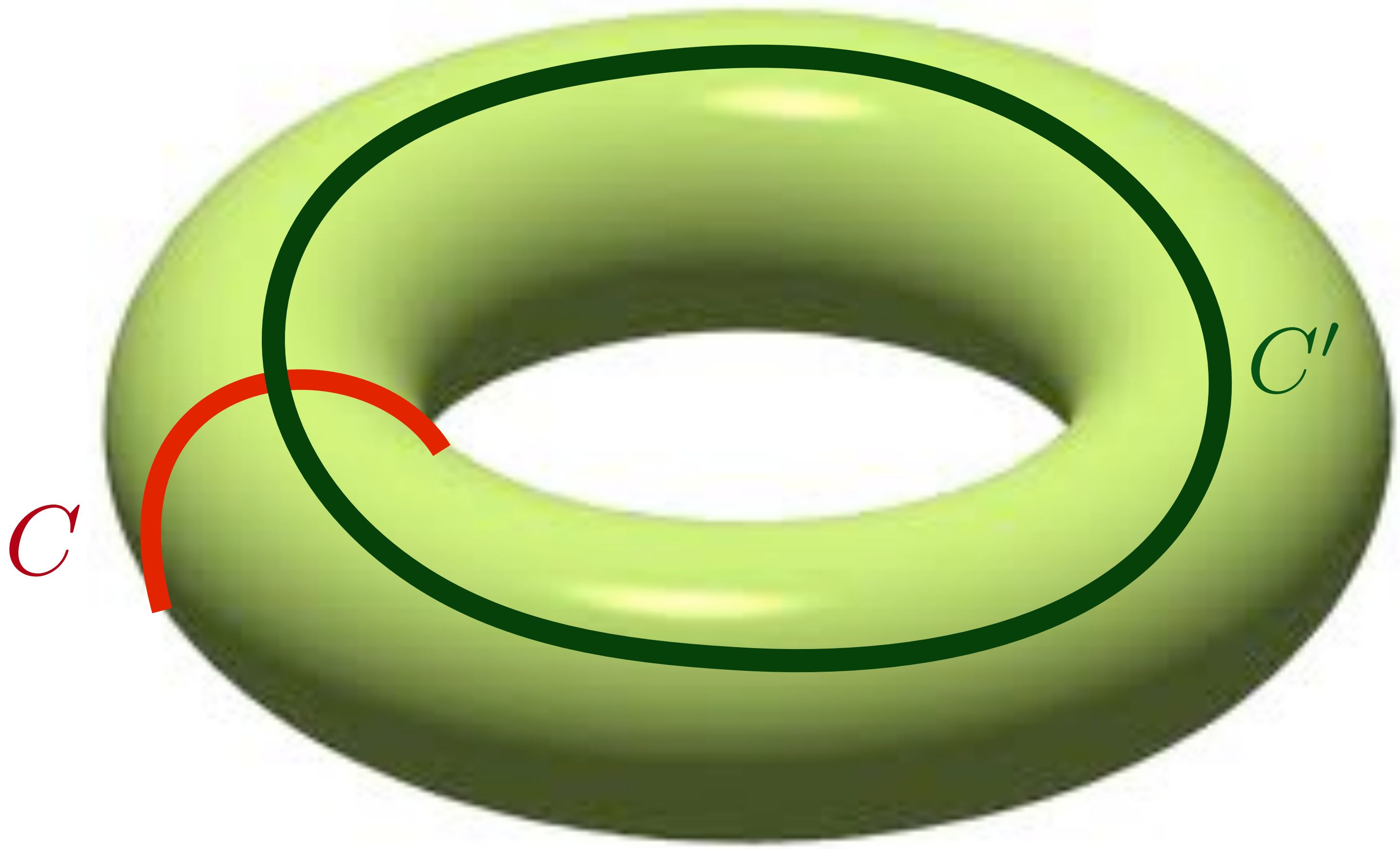

$$W_C V_{C'} = -V_{C'} W_C$$

$$V_C W_{C'} = -W_{C'} V_C$$

The TQFT

The $\mathbb{Z}_2$ spin liquid: Described by the simplest, non-trivial, topological field theory with time-reversal symmetry:

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} \int d^3x \, a^I \wedge da^J$$

where a^I, $I = 1, 2$ are U(1) gauge connections, and the K matrix is

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

The Wilson loops

$$V_C = \exp\left(i \int_C dx \cdot a^1 \right) \quad , \quad W_C = \exp\left(i \int_C dx \cdot a^2 \right)$$

obey $W_C V_{C'} = -V_{C'} W_C$ when C and C' wrap separate cycles of the torus.

N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991); X.-G. Wen, Phys. Rev. B 44, 2664 (1991);
S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000);
M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Annals of Physics **310**, 428 (2004)
T. H. Hansson, Vadim Oganesyan, S. L. Sondhi, Annals of Physics **313**, 497 (2004)

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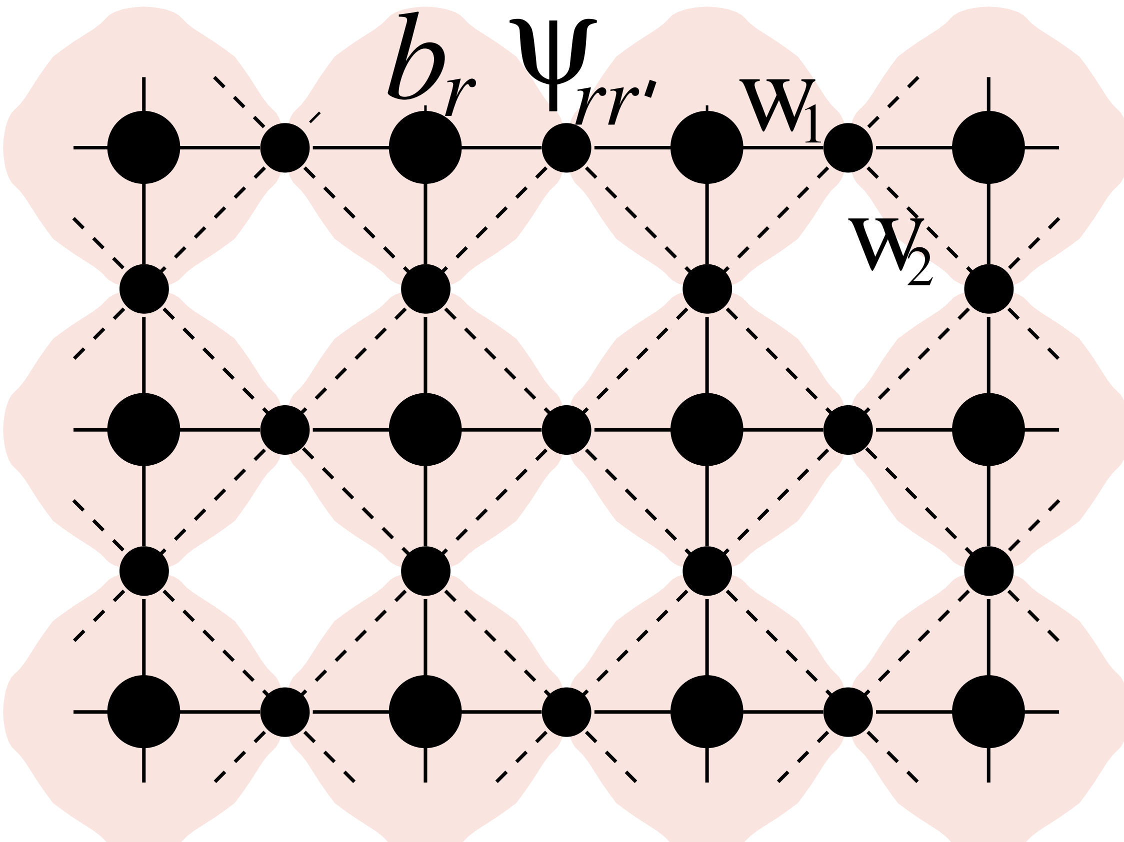
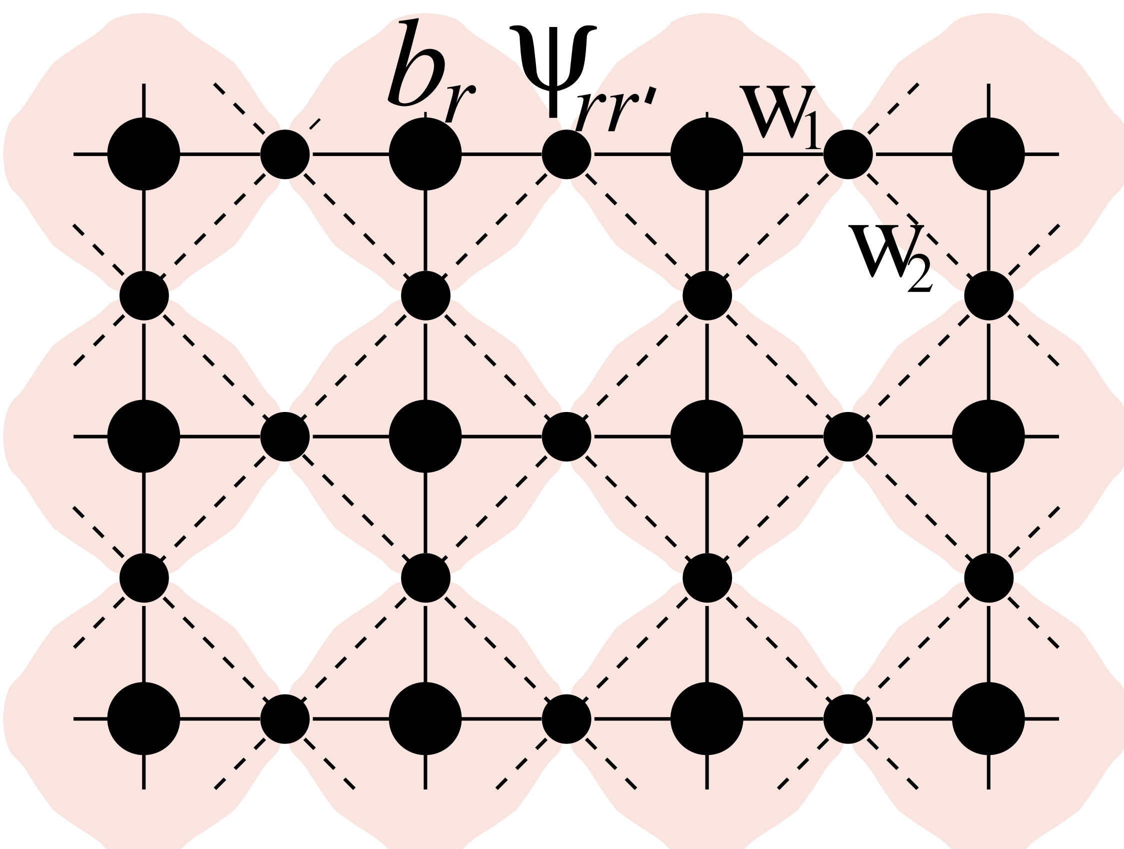
$$H = -w_1 \sum_{r, r' \in r} (b_r^\dagger \psi_{rr'} + \text{H.c.}) - w_2 \sum_{[rr'r'']} (\psi_{rr'}^\dagger \psi_{r'r''} + \text{H.c.}) + u_b \sum_r (n_r^b)^2 + u_\psi \sum_{\langle rr' \rangle} (n_{rr'}^\psi)^2 + U \sum_r N_r^2.$$

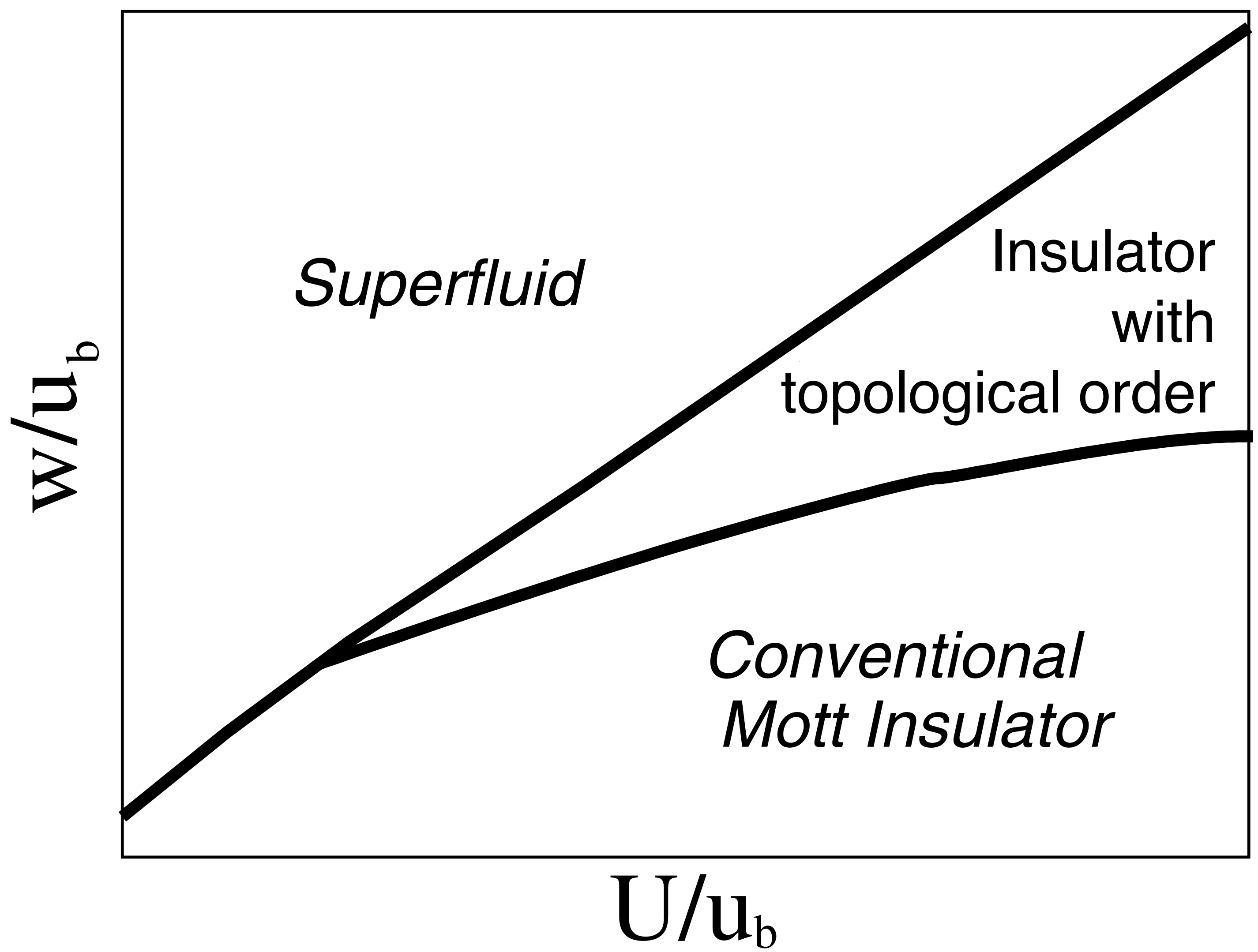


$$2N_r = 2n_r^b + \sum_{r' \in r} n_{rr'}^\psi.$$

Average of one boson per site: $\langle N_r \rangle = 1$

R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); O.I. Motrunich and T. Senthil, PRL **89**, 277004 (2002).





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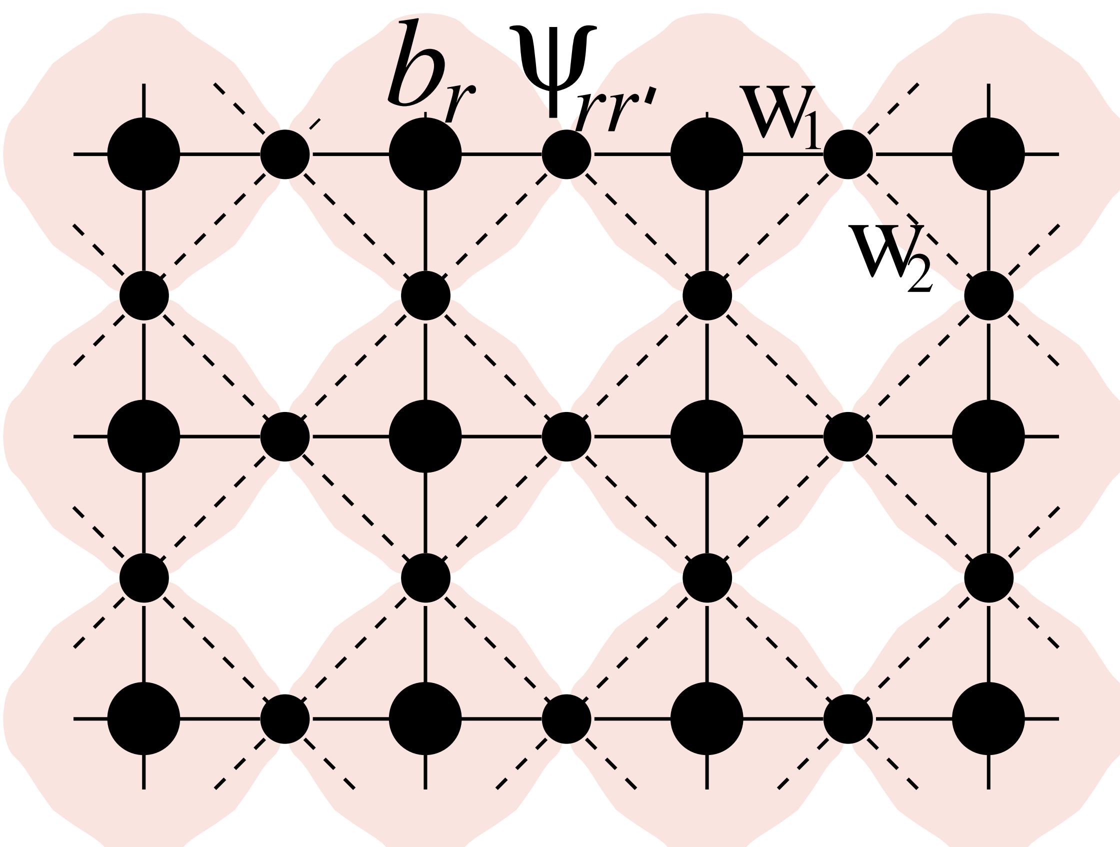

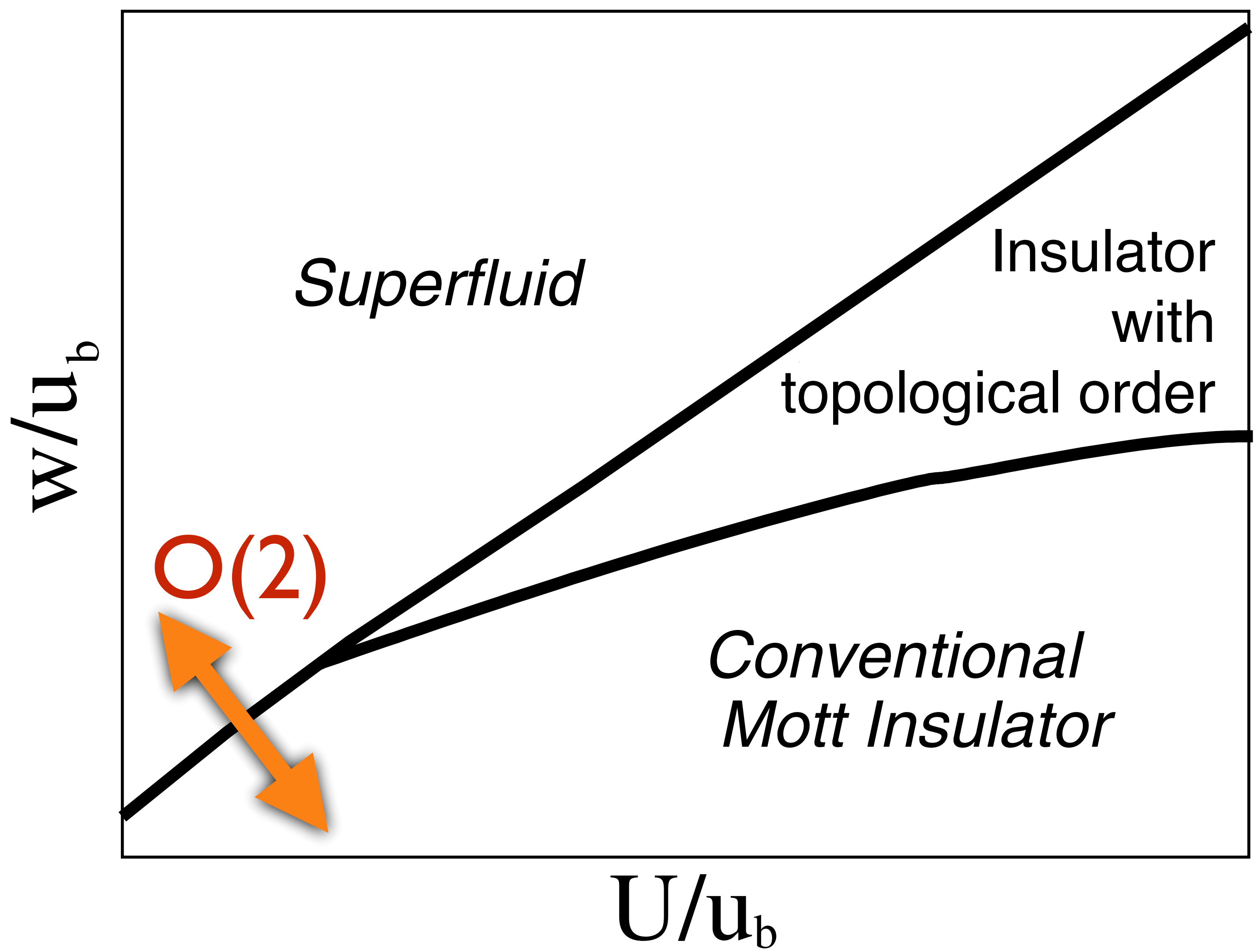

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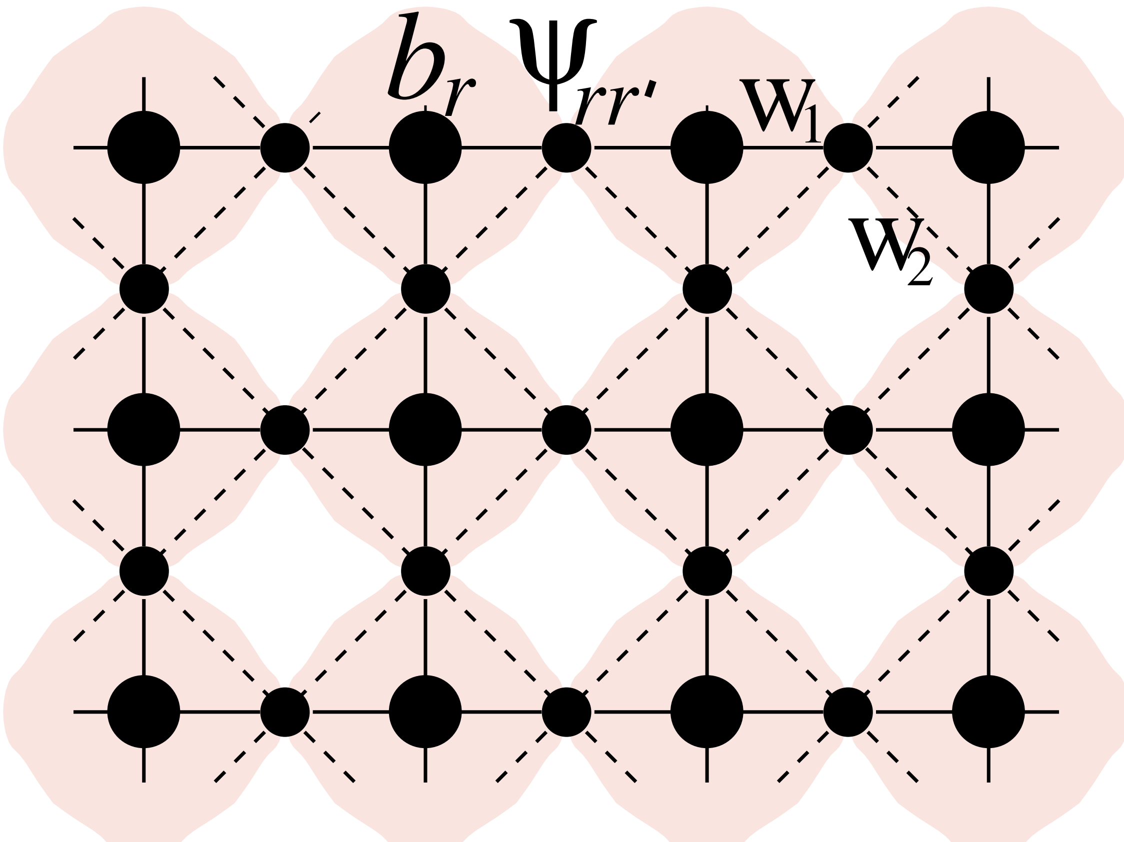
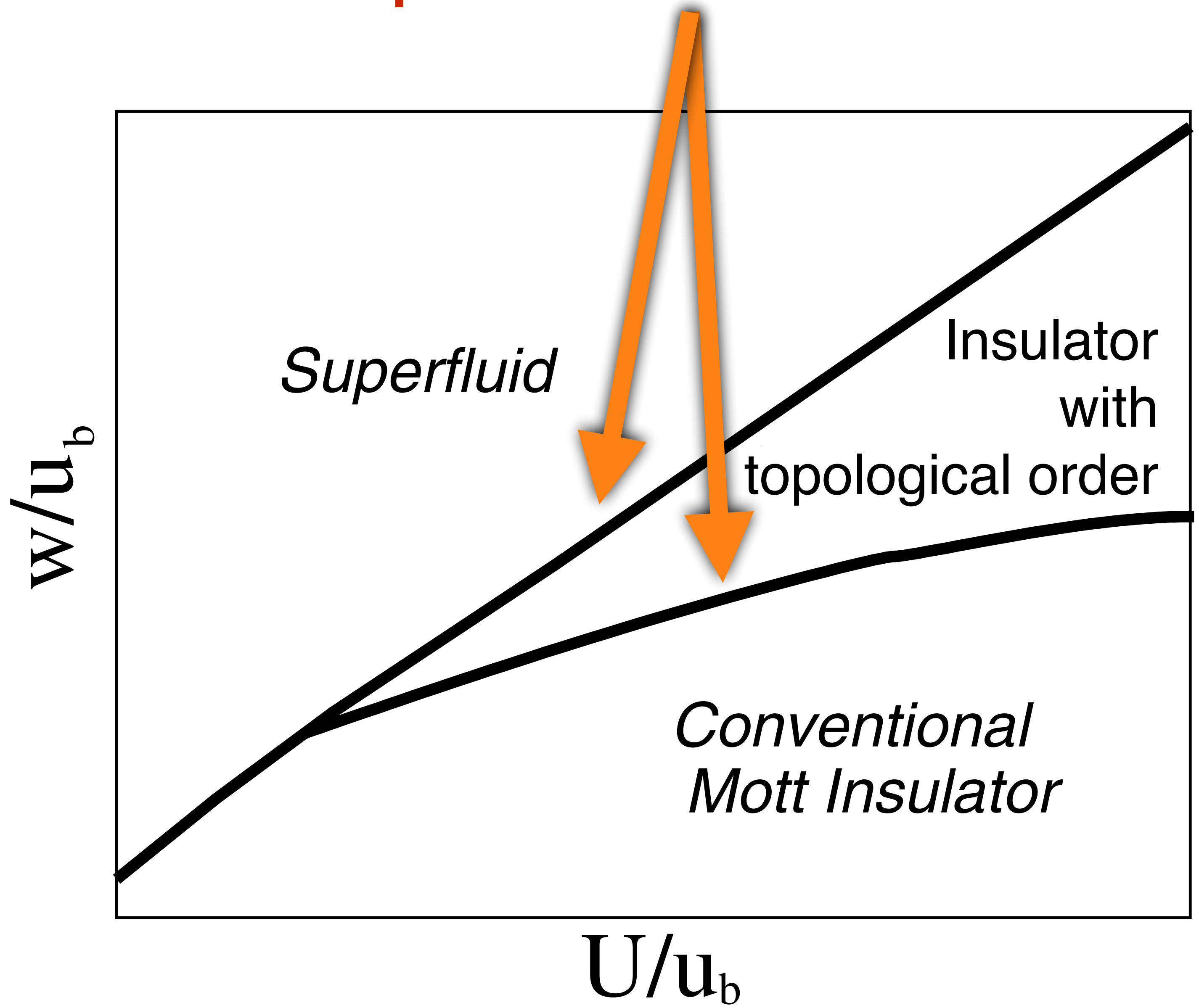
Topological phase transitions





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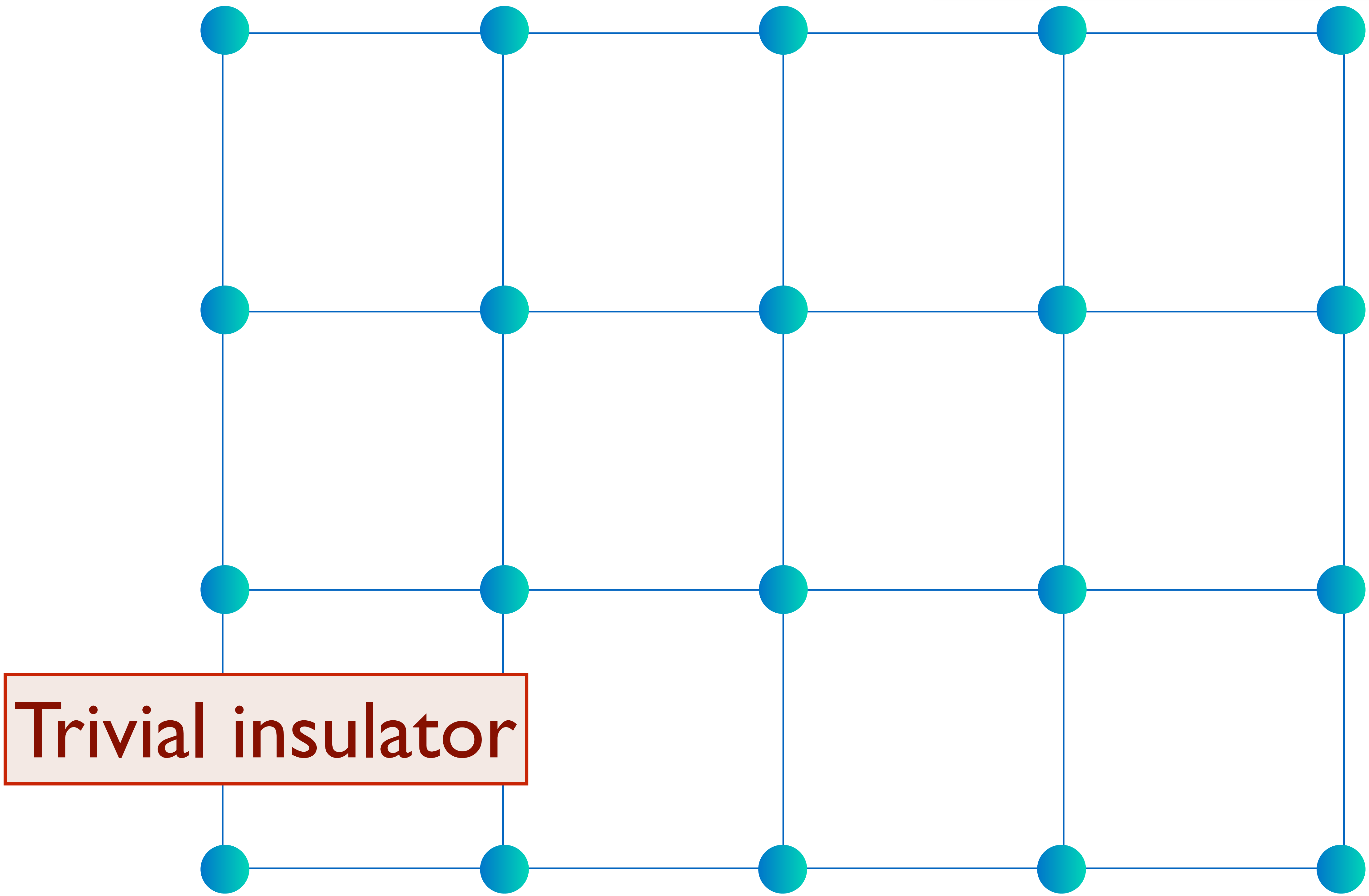
● $= b^\dagger$;

$N_r = 1$ for all r



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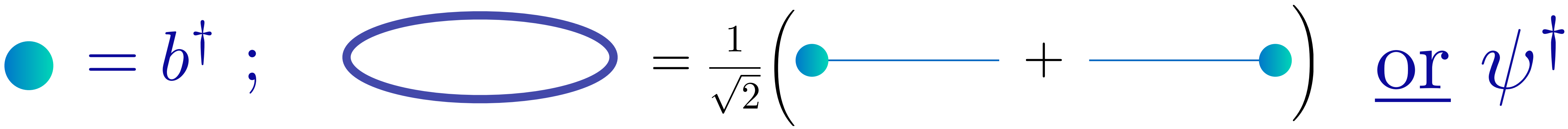

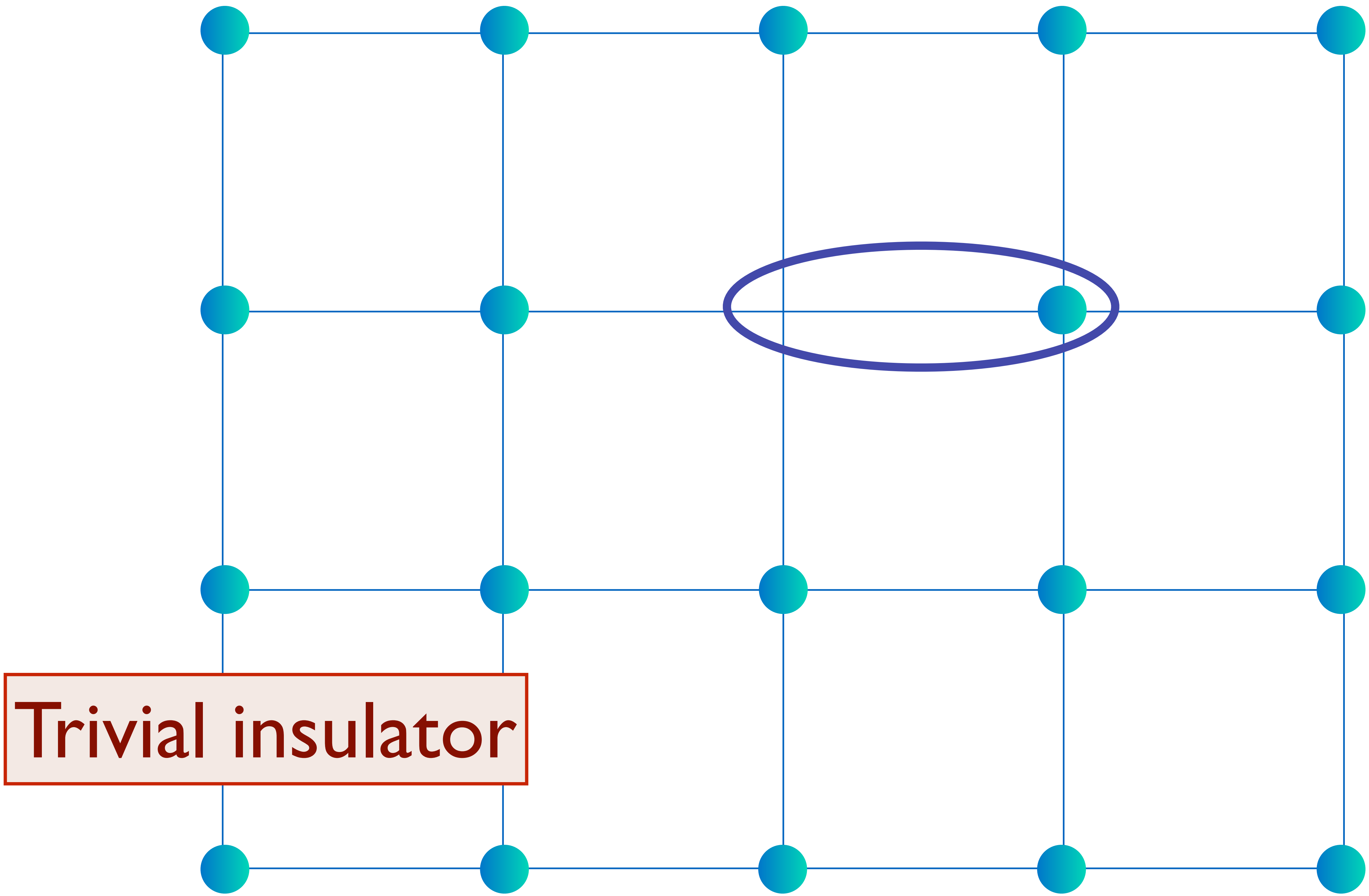

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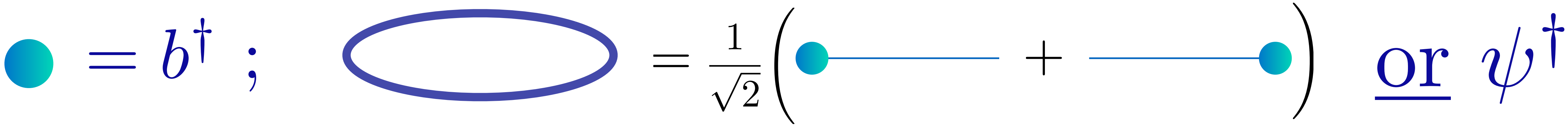

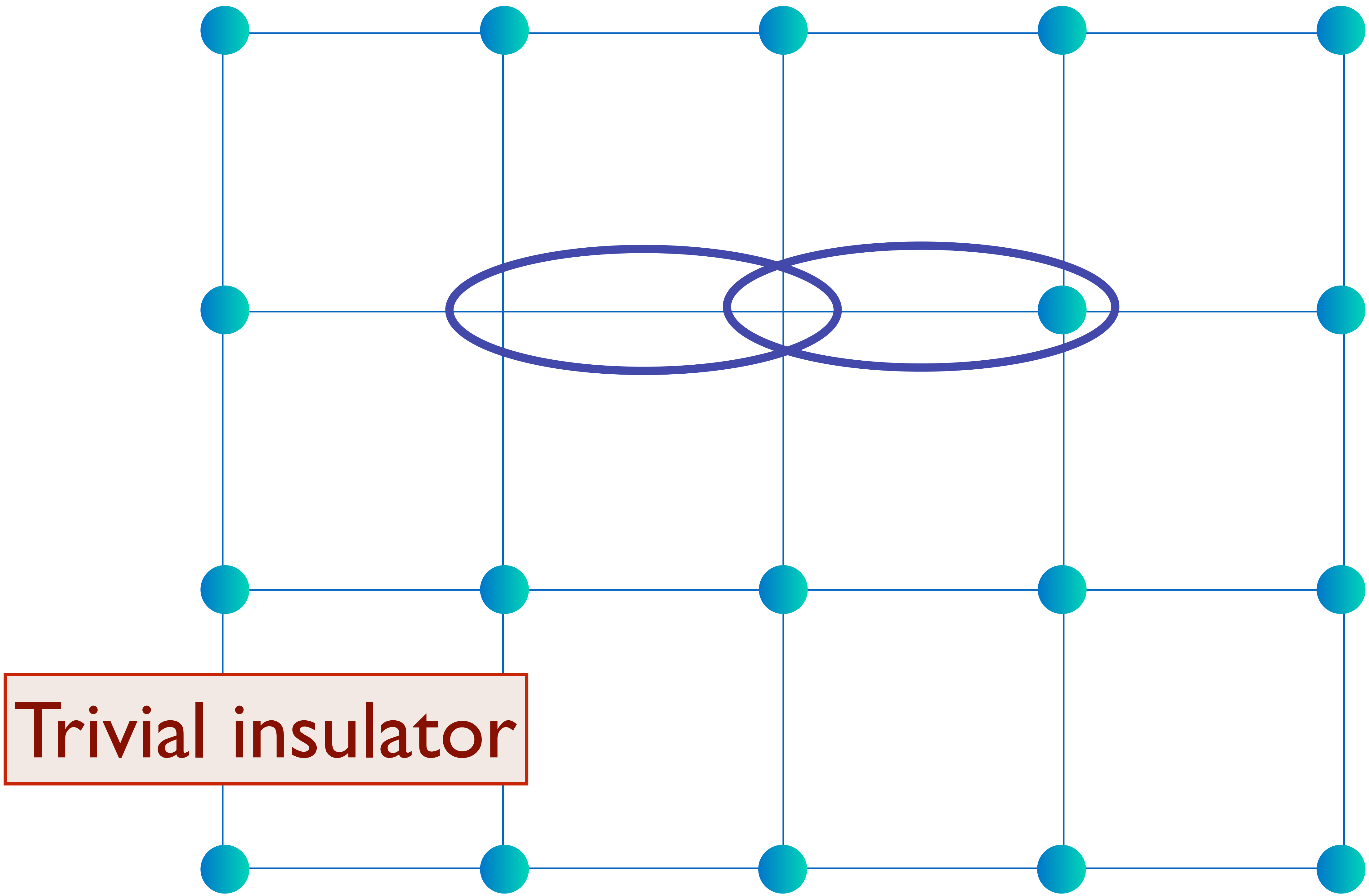

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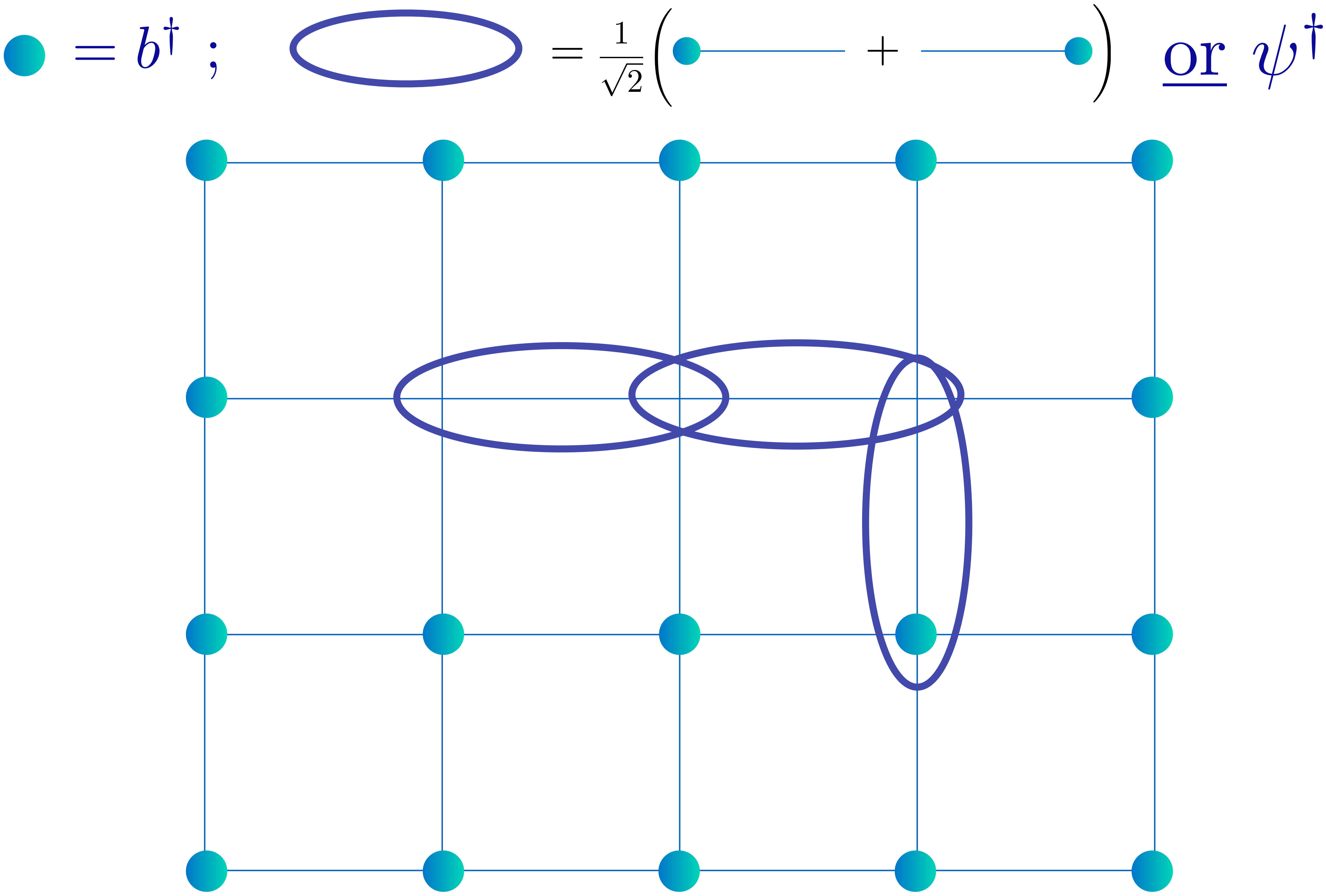

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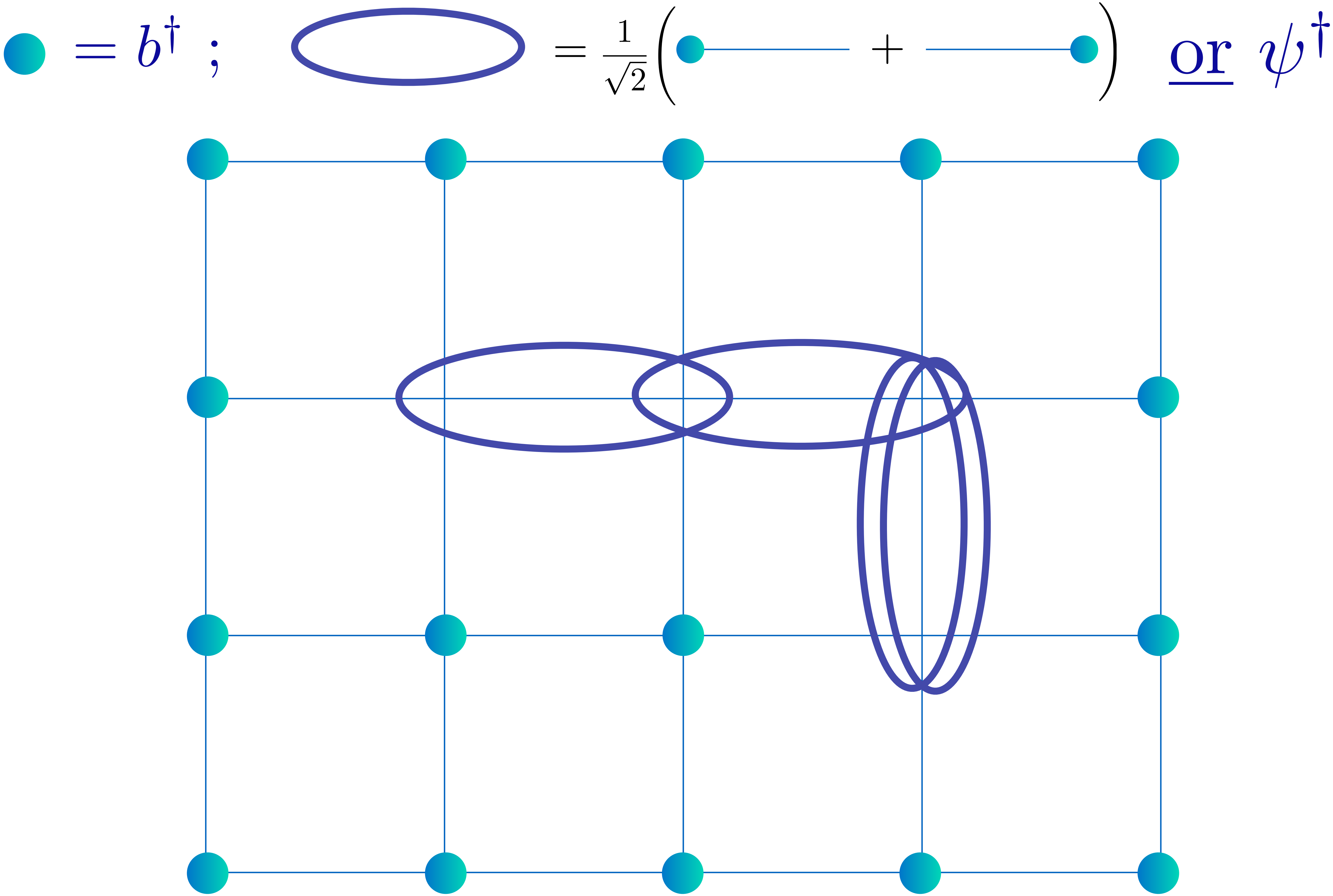

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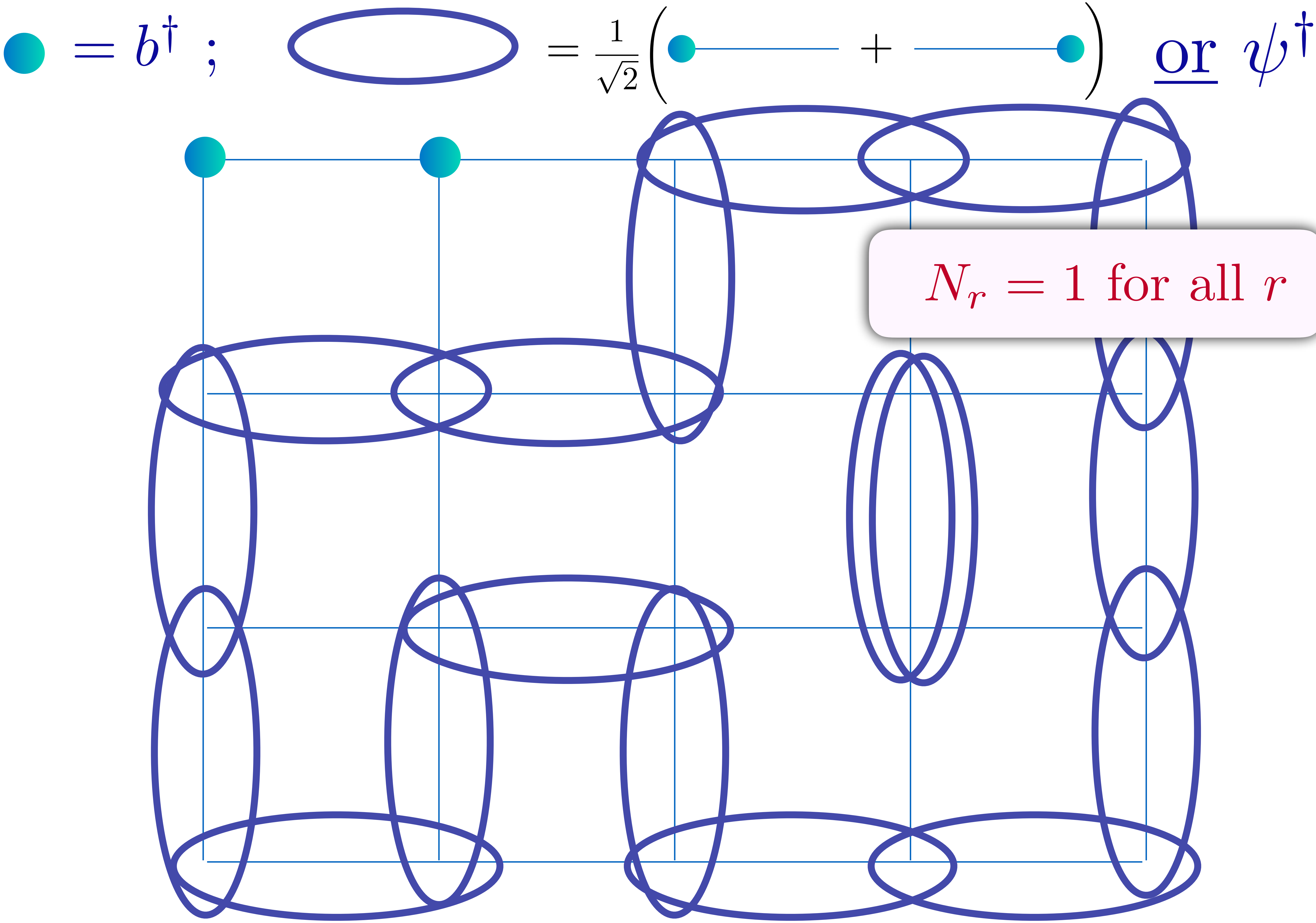
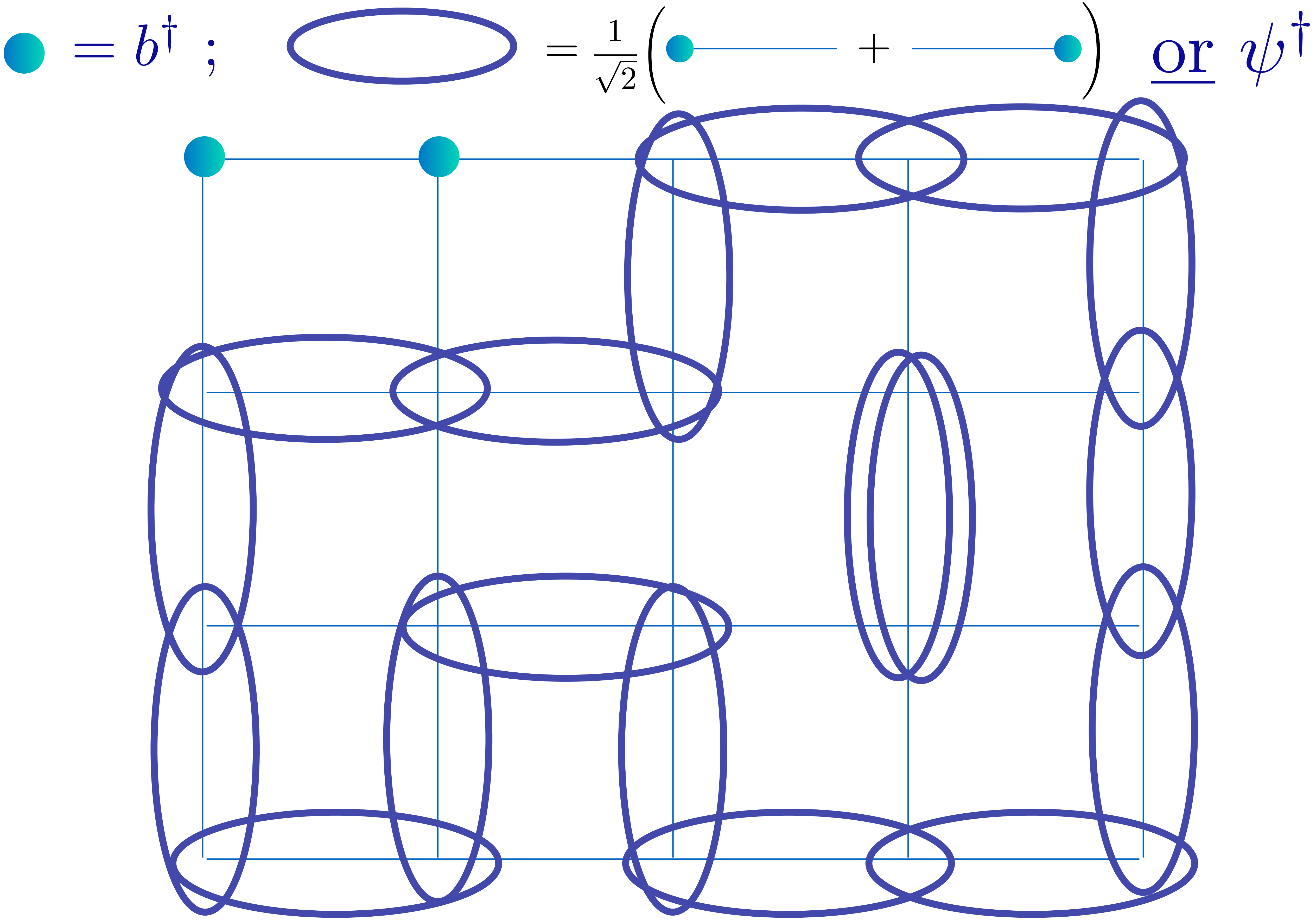


Average of one boson per site: $\langle N_r \rangle = 1$

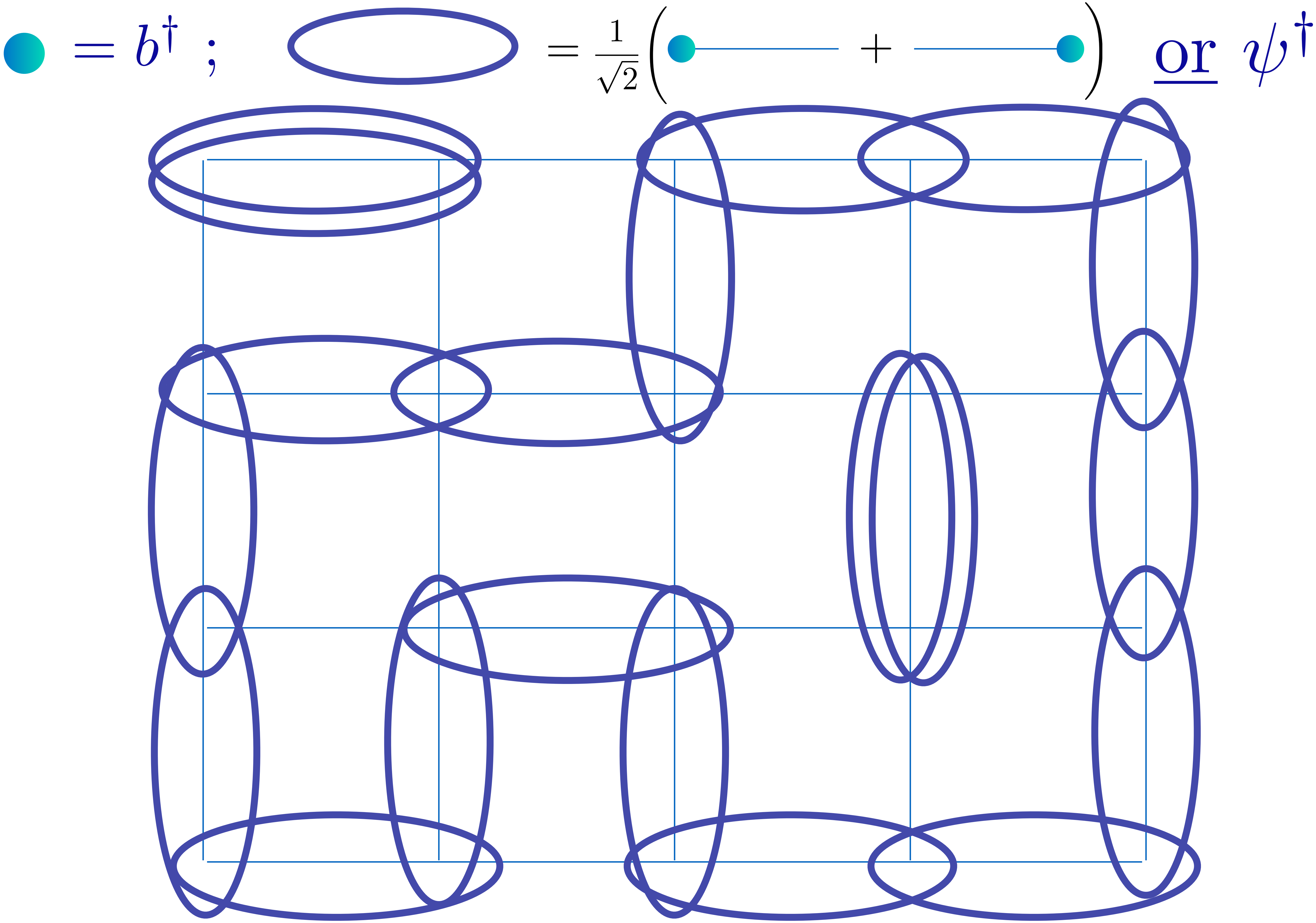
In the limit of large U, we expect to prefer states in which $N_r = 1$ at all r. By the usual procedure of perturbation theory in $1/U$, we obtain an effective Hamiltonian within the subspace of states with $N_r = 1$. To order $1/U$, this effective Hamiltonian is

$$
\begin{aligned}
H_{\text{eff}}^{(0)} = \; & - J_{\text{bond}} \sum_{\langle r r' \rangle} \left[(\psi_{rr'}^{\dagger})^2 b_r b_{r'} + \text{H.c.} \right] \\
& - K_{\text{ring}} \sum_{\square} (\psi_{12}^{\dagger} \psi_{23} \psi_{34}^{\dagger} \psi_{41} + \text{H.c.}), \\
& + u_{\psi} \sum_{\langle r r' \rangle} (n_{rr'}^{\psi})^2 \quad + u_b \sum_{r} (n_r^b)^2
\end{aligned}
$$

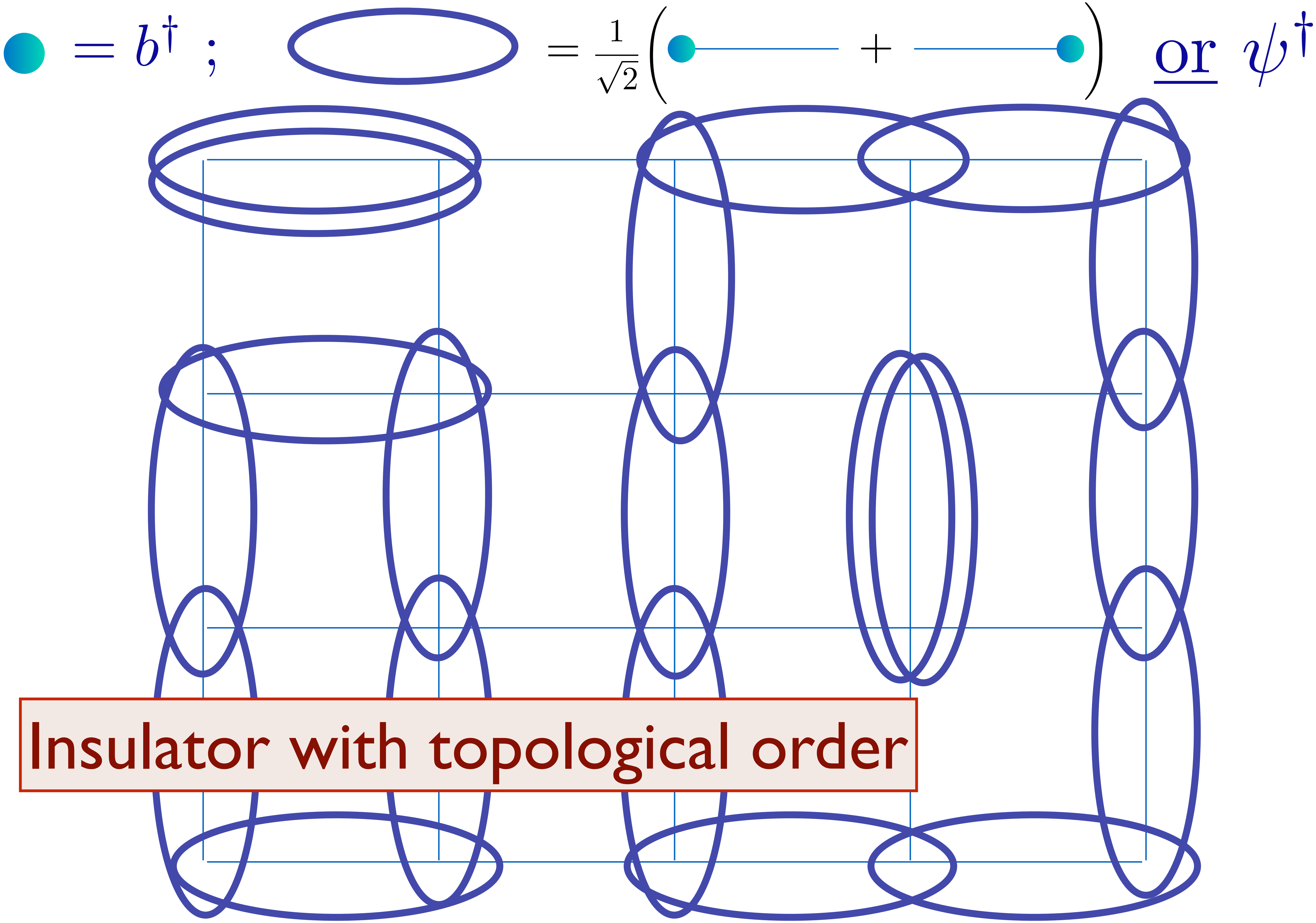
Bosons at unit density on the square lattice; $N_r = 1$ for all r



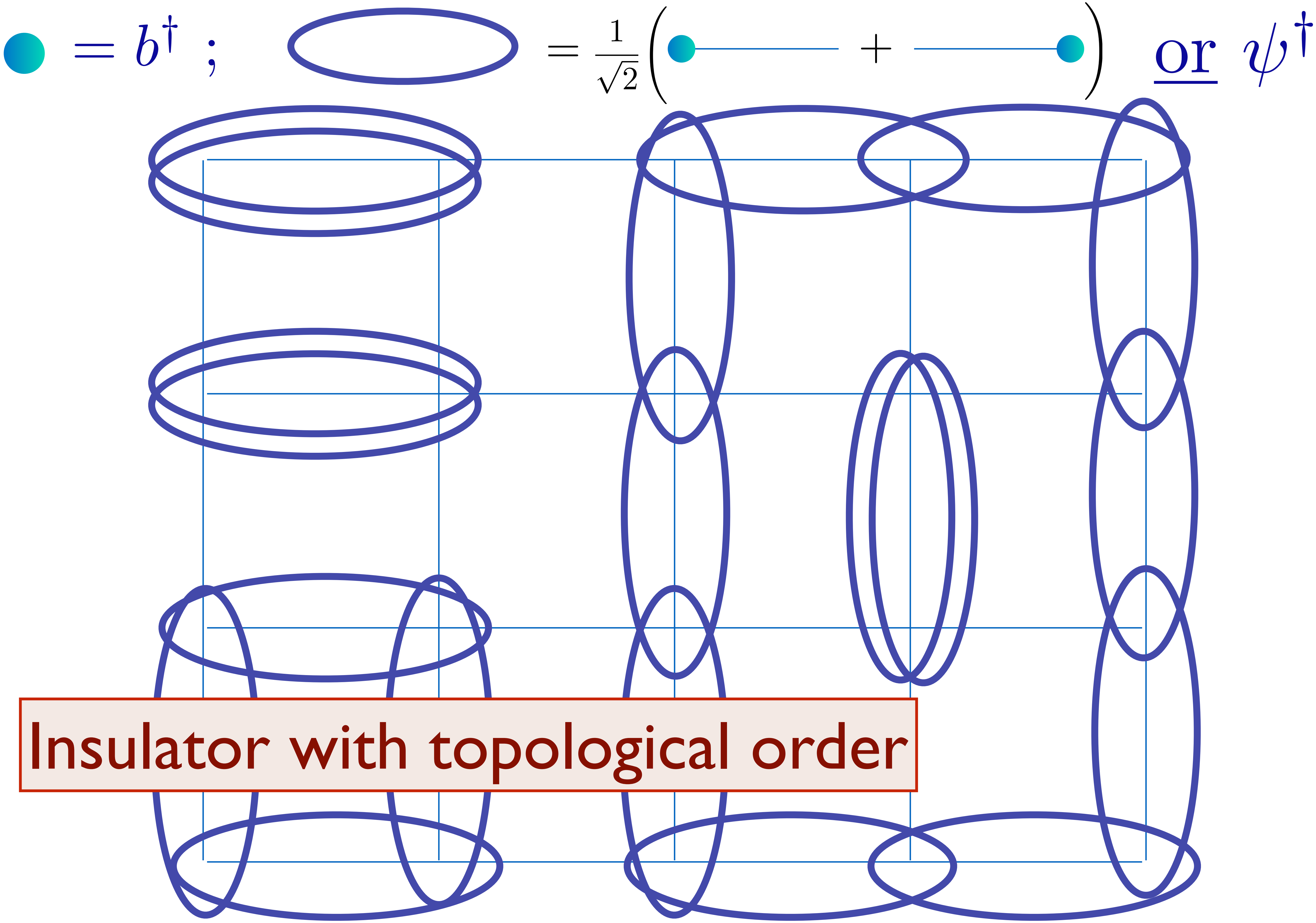
Bosons at unit density on the square lattice; $N_r = 1$ for all r



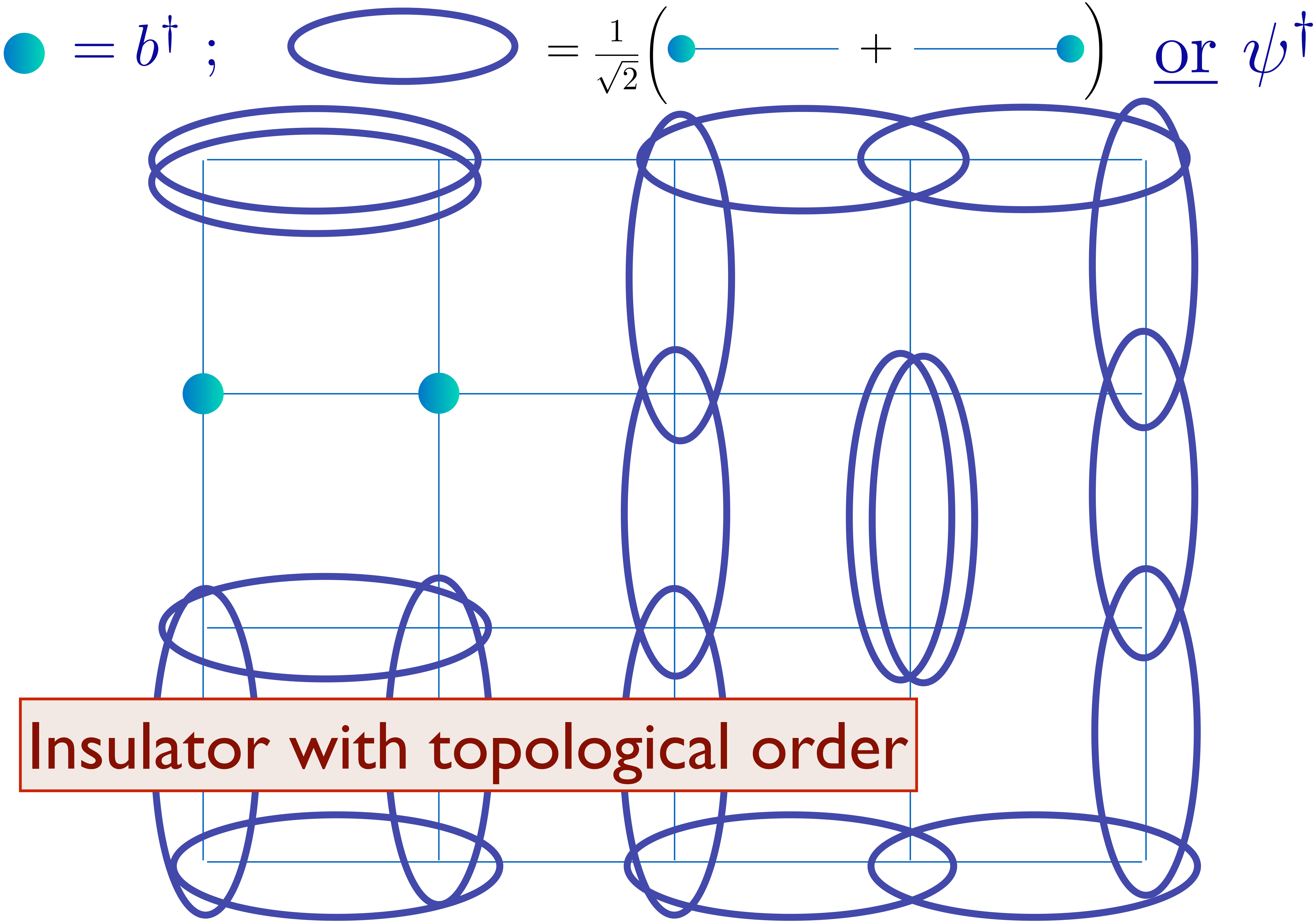
Bosons at unit density on the square lattice; $N_r = 1$ for all r



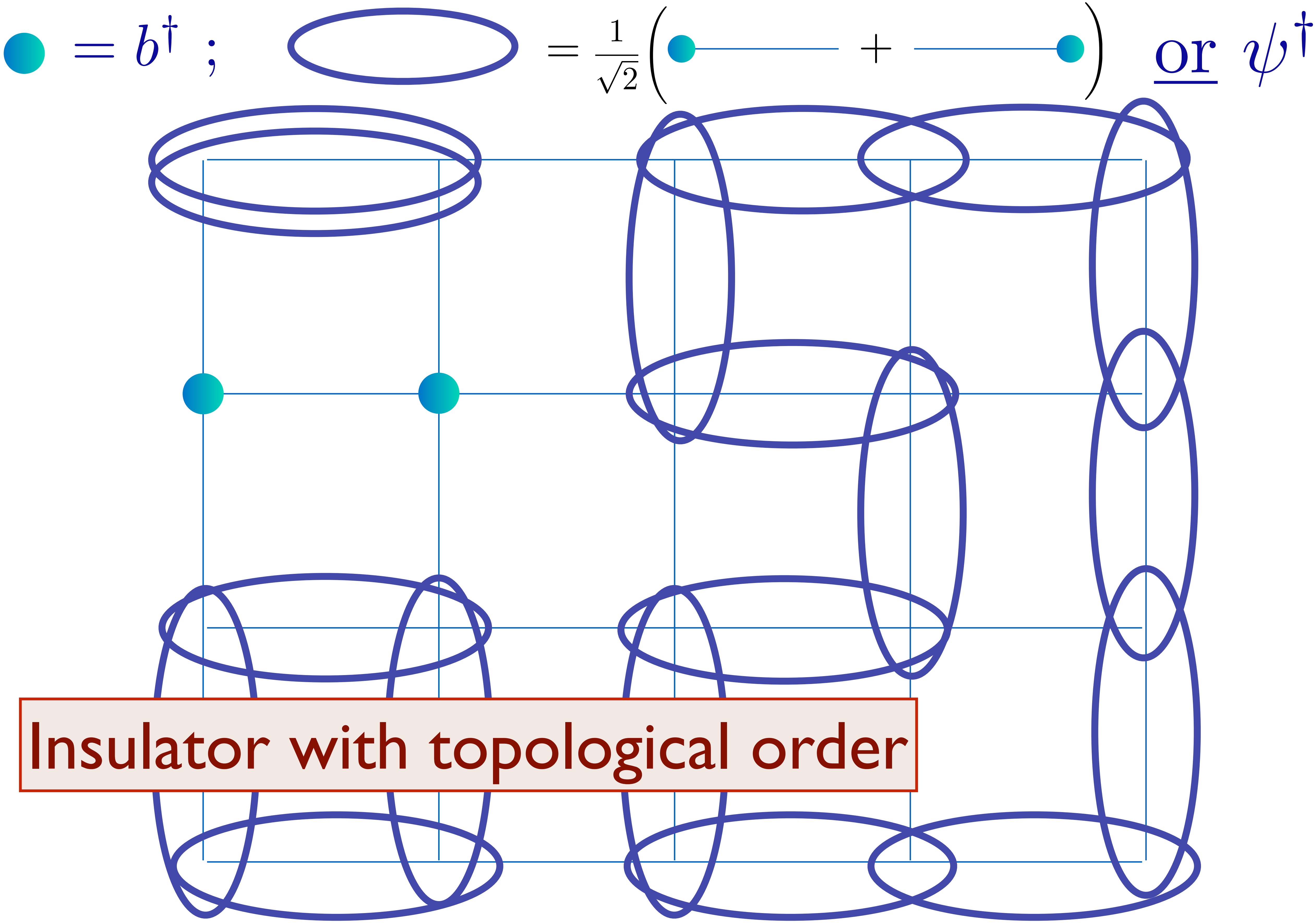
Bosons at unit density on the square lattice; $N_r = 1$ for all r



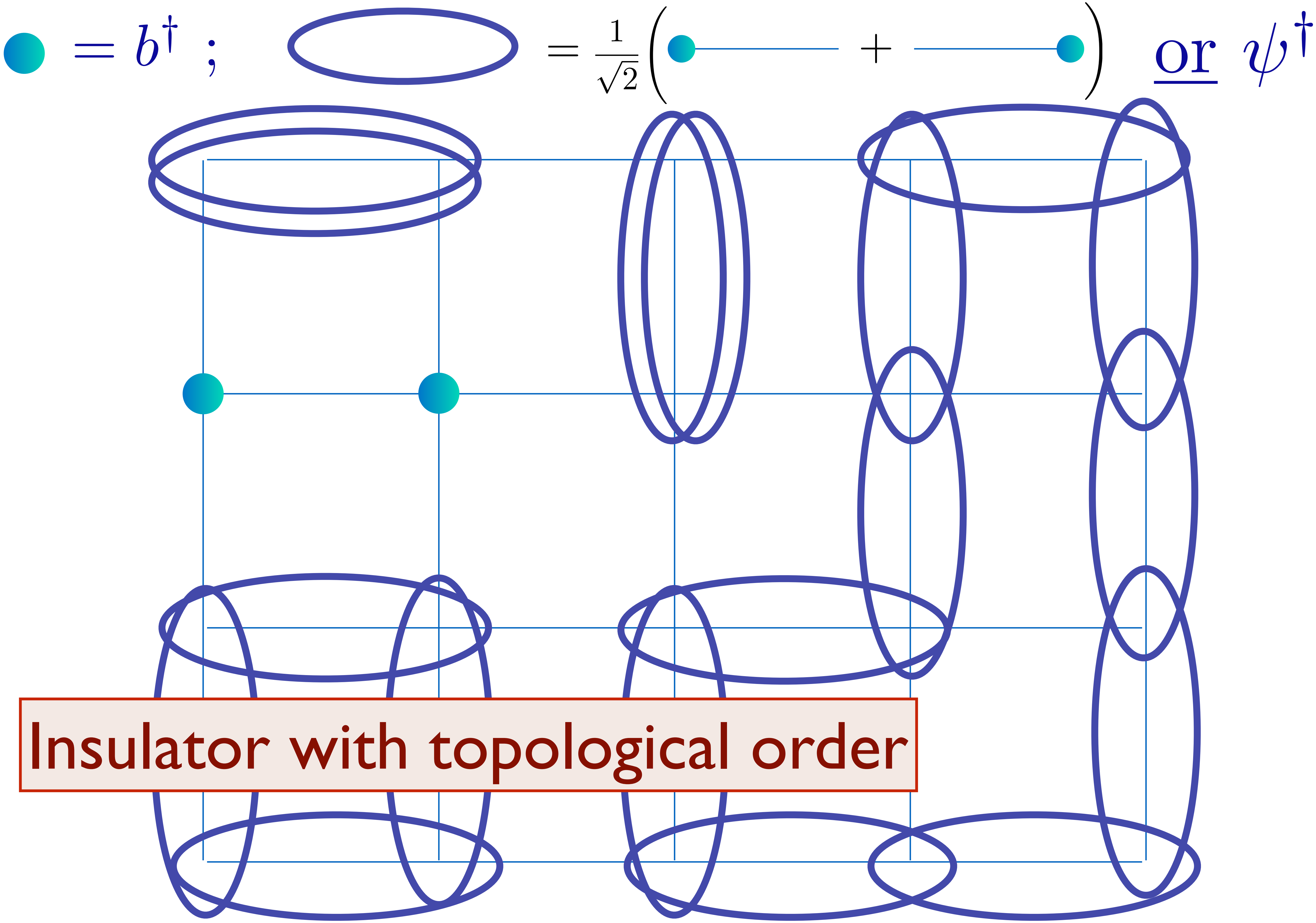
Bosons at unit density on the square lattice; $N_r = 1$ for all r



Bosons at unit density on the square lattice; $N_r = 1$ for all r



Bosons at unit density on the square lattice; $N_r = 1$ for all r



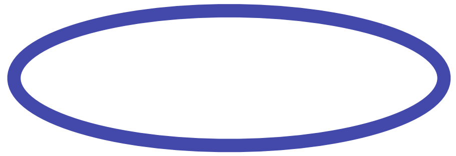
Bosons at unit density on the square lattice; $N_r = 1$ for all r

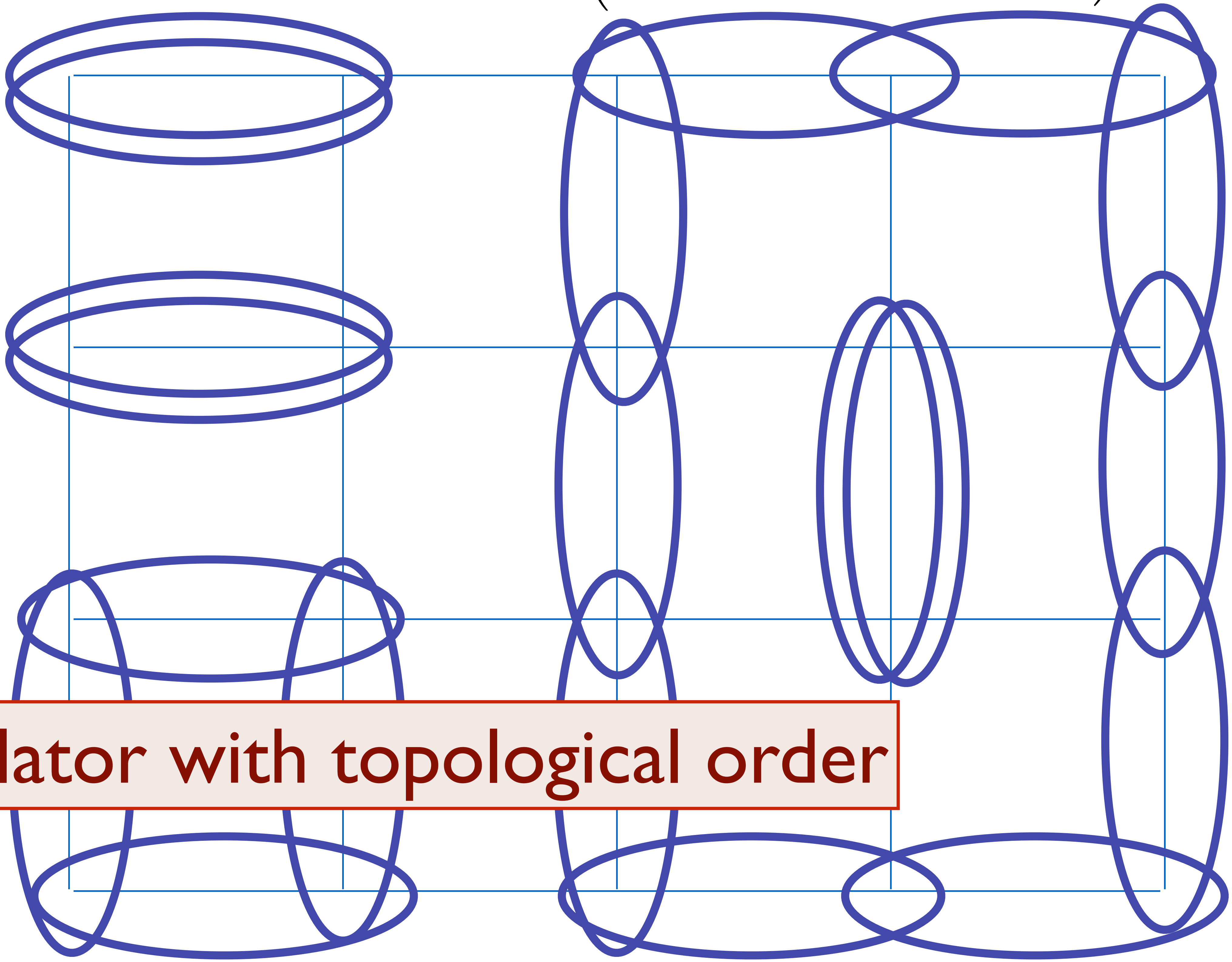
$$
\begin{aligned}
H_{\mathrm{eff}}^{(0)} = \quad & - J_{\mathrm{bond}} \sum_{\langle rr' \rangle} [(\psi_{rr'}^{\dagger})^2 b_r b_{r'} + \mathrm{H.c.}] \\
& - K_{\mathrm{ring}} \sum_{\square} (\psi_{12}^{\dagger} \psi_{23} \psi_{34}^{\dagger} \psi_{41} + \mathrm{H.c.}), \\
& + u_{\psi} \sum_{\langle rr' \rangle} (n_{rr'}^{\psi})^2 \quad + u_b \sum_{r} (n_r^b)^2
\end{aligned}
$$

This effective Hamiltonian has exactly the form of a U(1) lattice gauge theory. We define $b_r \sim e^{i\varepsilon_r \phi_r}$ where $\varepsilon_r = \pm 1$ on the two sublattices, and $\psi_{rr'} \sim e^{i\varepsilon_r a_{r\alpha}}$, where $r' = r + \hat{e}_\alpha$, $\alpha = x, y$. Then the above theory can be written on cubic spacetime lattice in a "relativistic" from with action

$$
\begin{aligned}
S \quad = \quad & -J \sum_{r} \cos(\Delta_\mu \phi_r - 2a_{r\mu}) \\
& - K \sum_{\square} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda)
\end{aligned}
$$

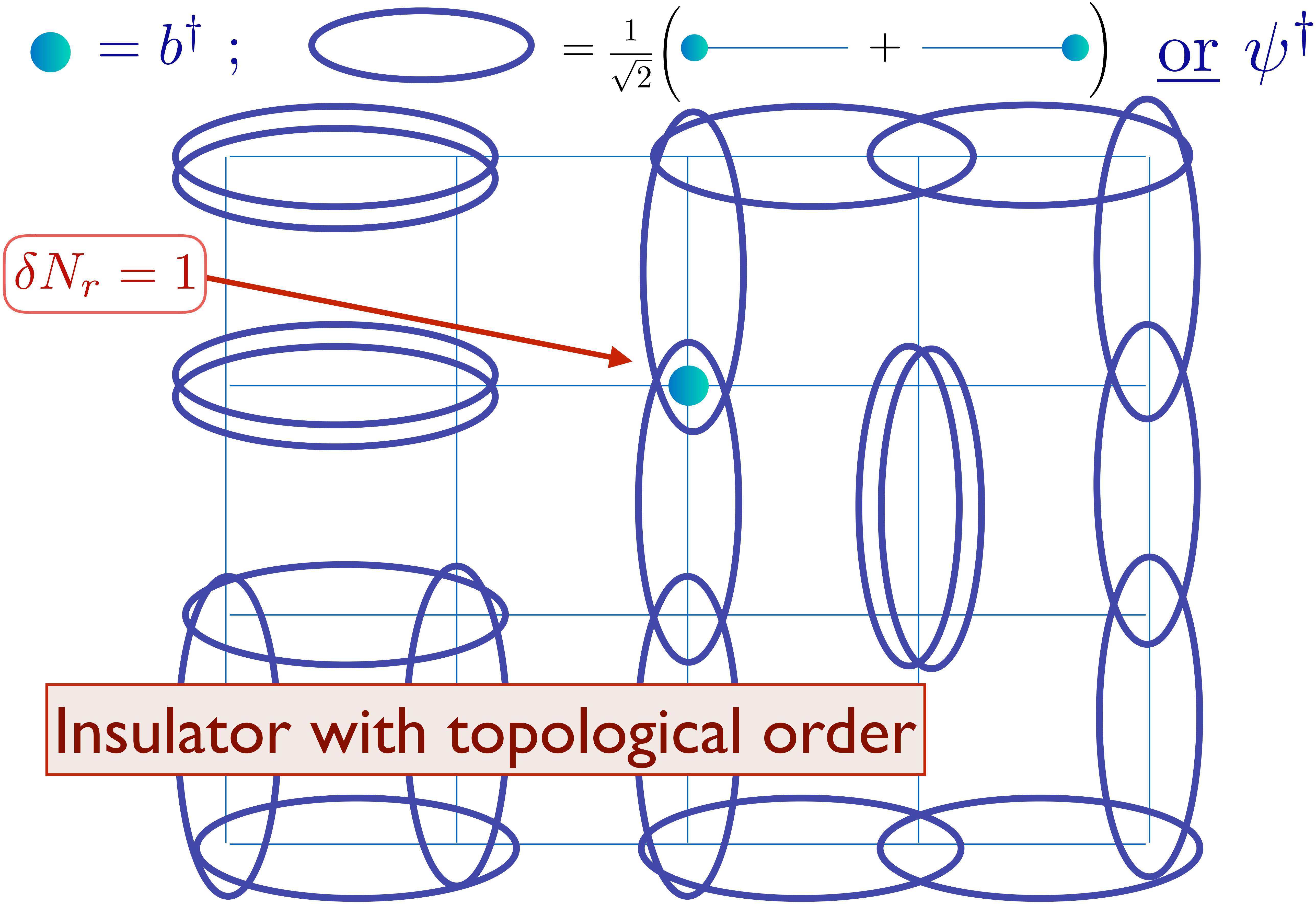
The boson $e^{i\phi_r}$ has U(1) gauge charge 2. The Gauss law for this lattice gauge theory is equivalent to the constraint $N_r = 1$ at all r.

$\bullet = b^\dagger$;  $= \frac{1}{\sqrt{2}}\left(\bullet\!\!-\!\!-\!\!-\; + \;-\!\!-\!\!-\!\!\bullet\right)$ or $\psi^\dagger$

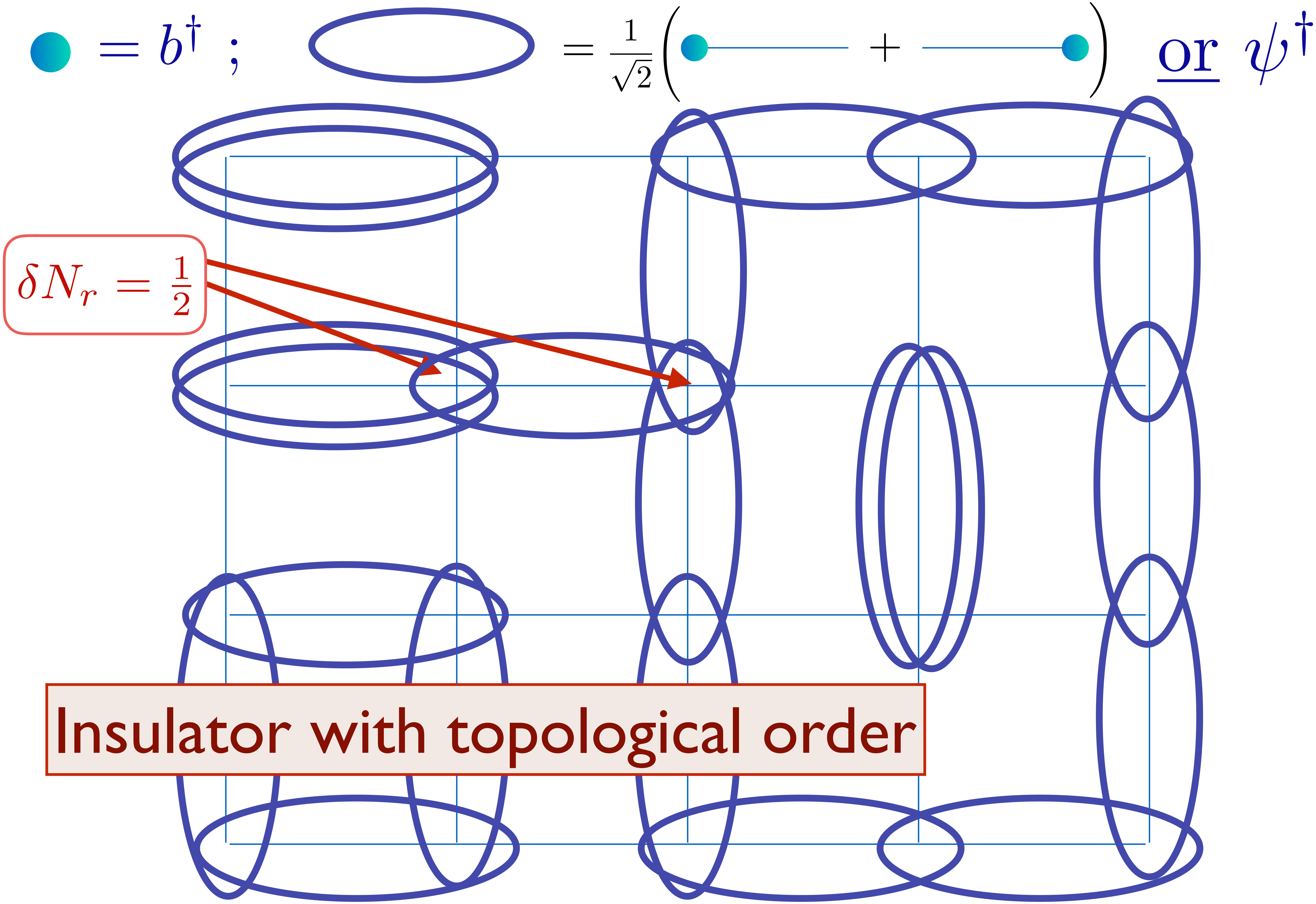


Insulator with topological order

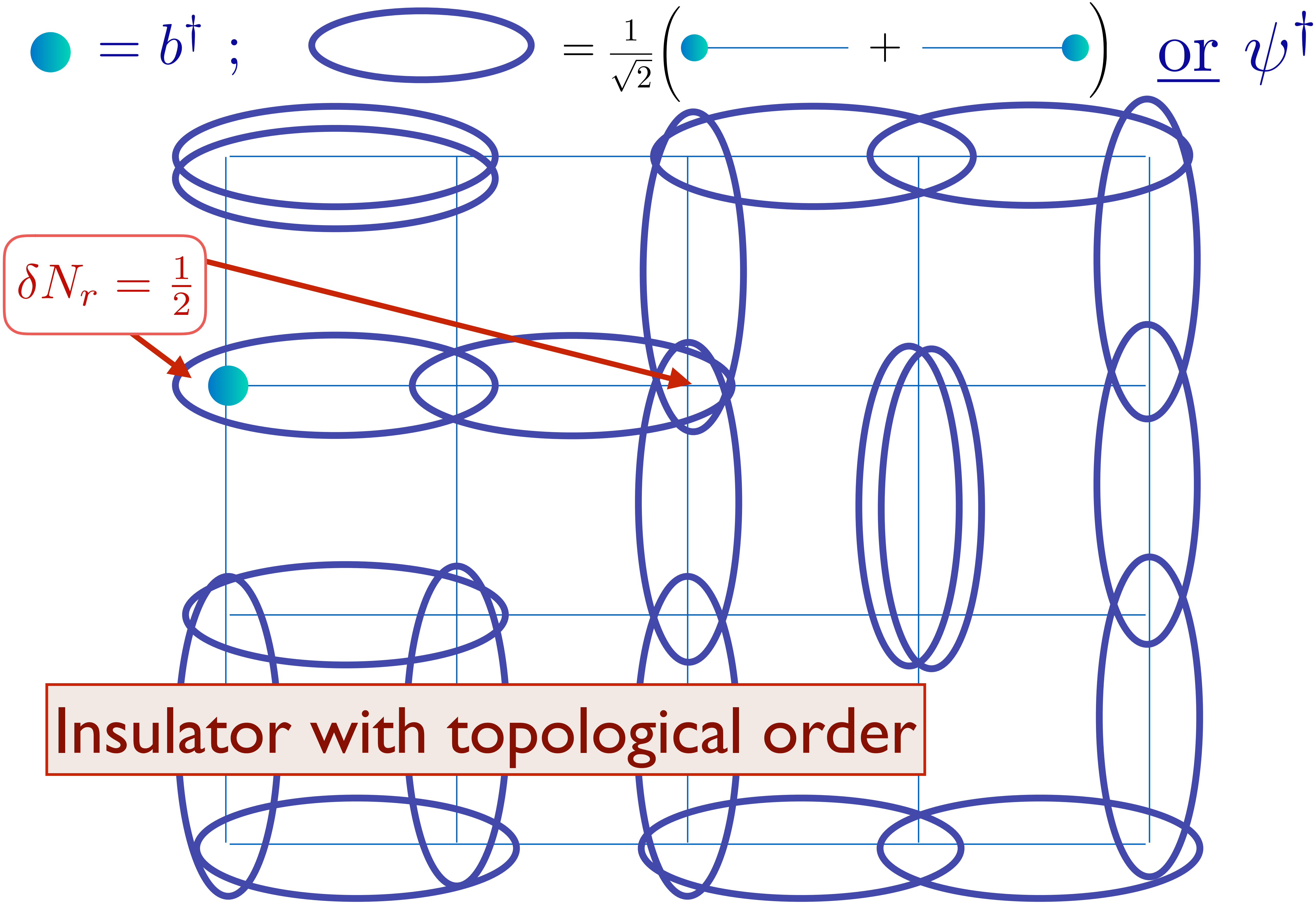
Bosons at unit density on the square lattice; $N_r = 1$ for all r



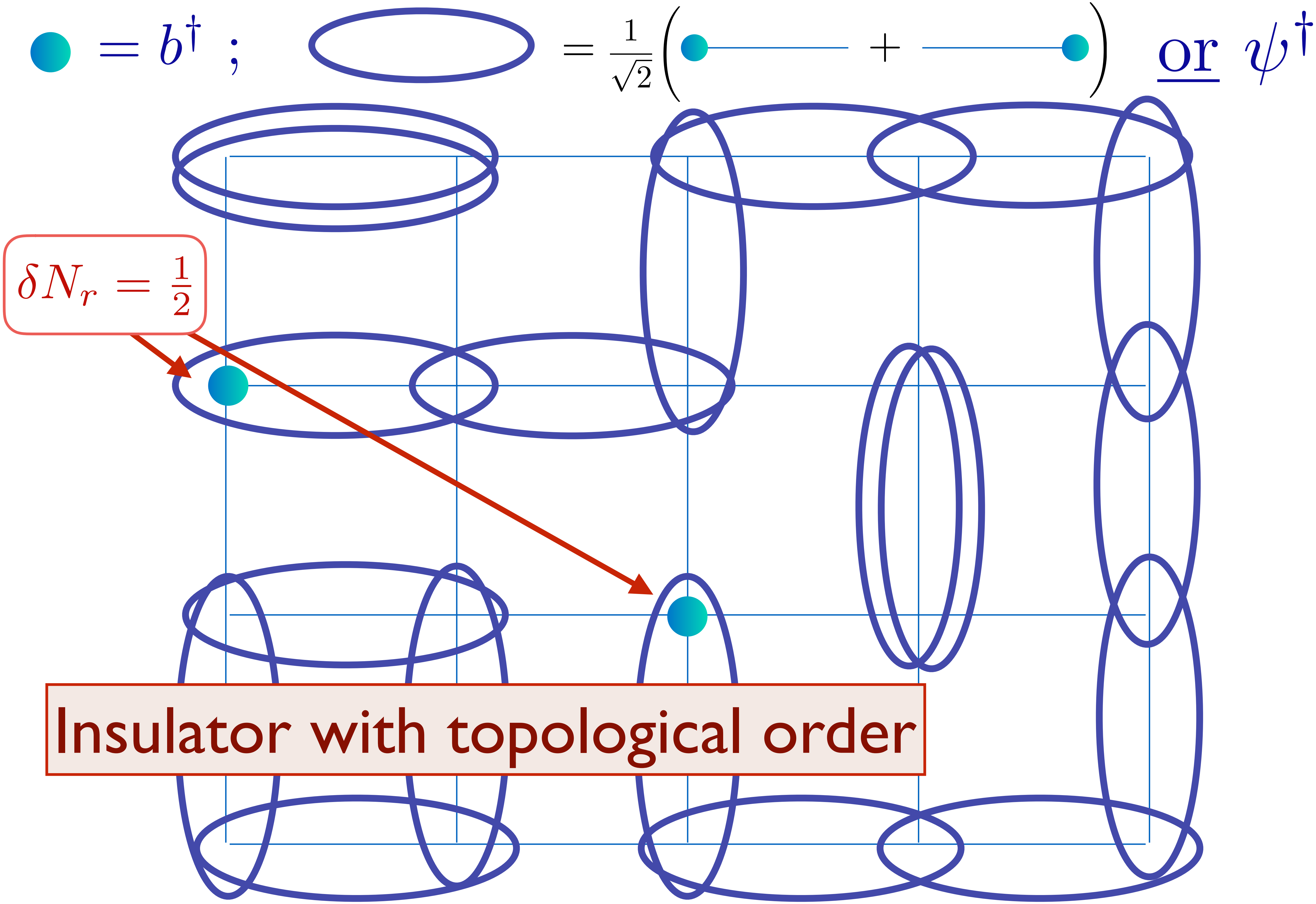
Add a boson on site.



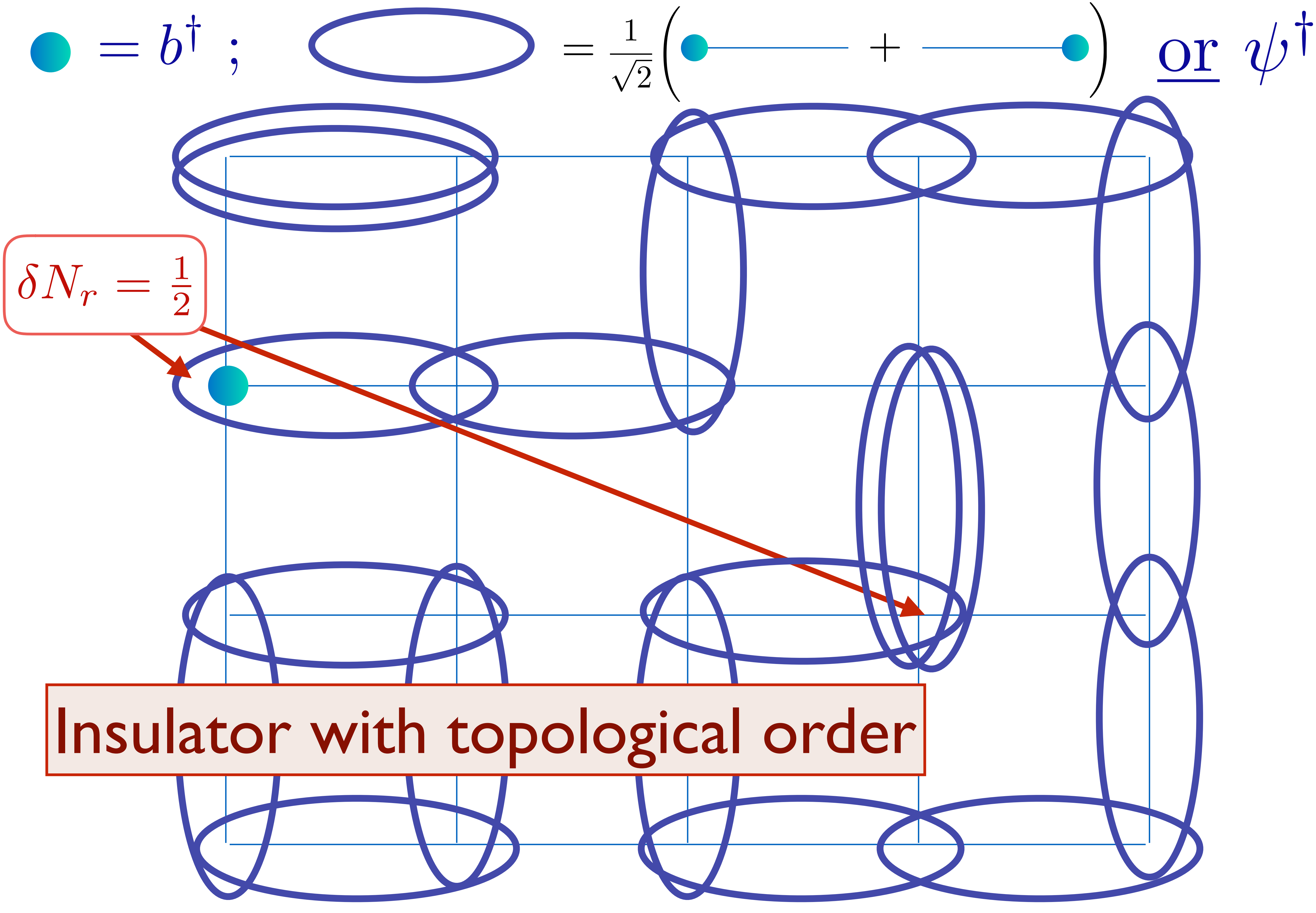
At large U, energy is lowered when
the boson to splits into 2 half bosons.



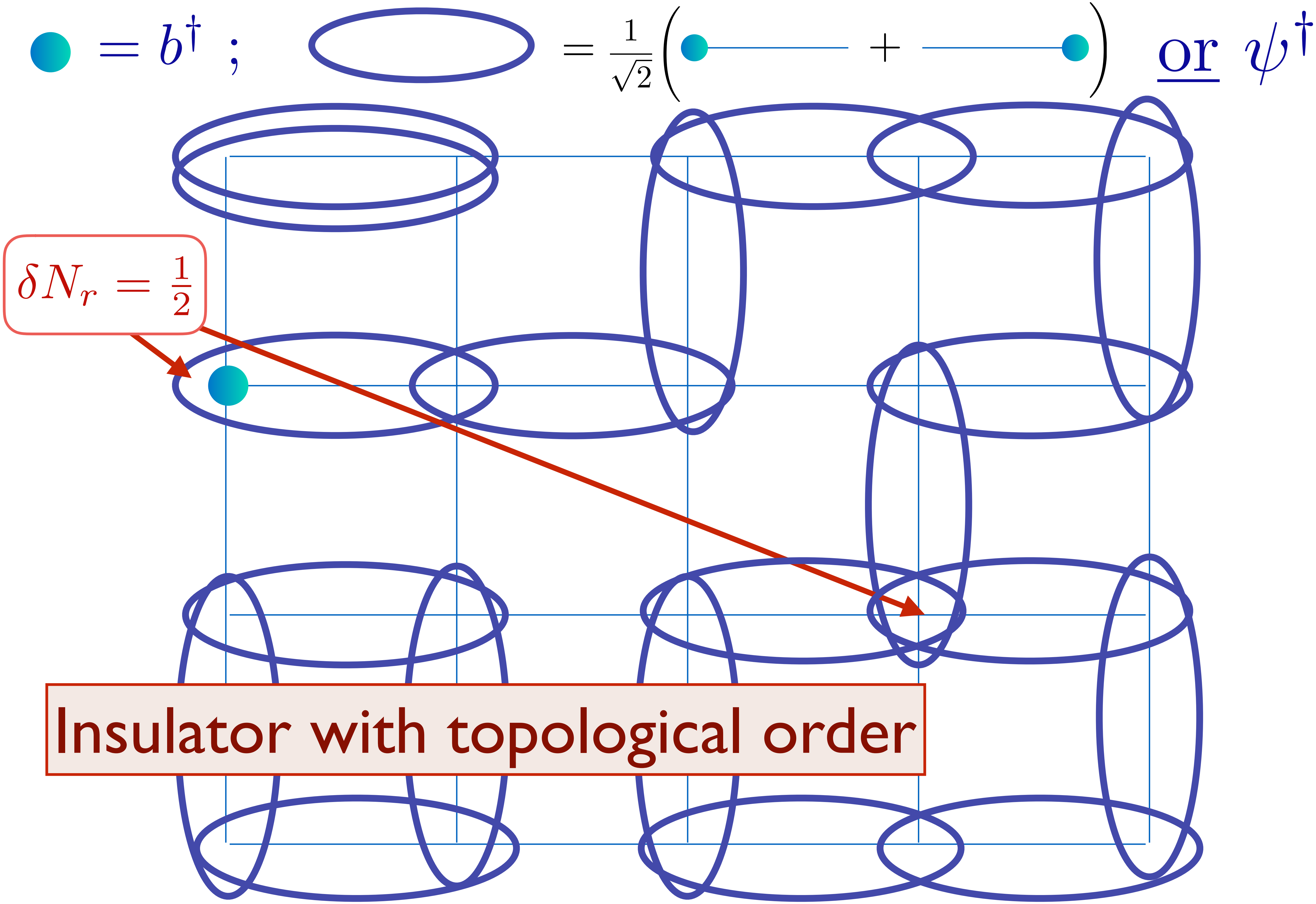
The half charge bosons can then move freely through the lattice



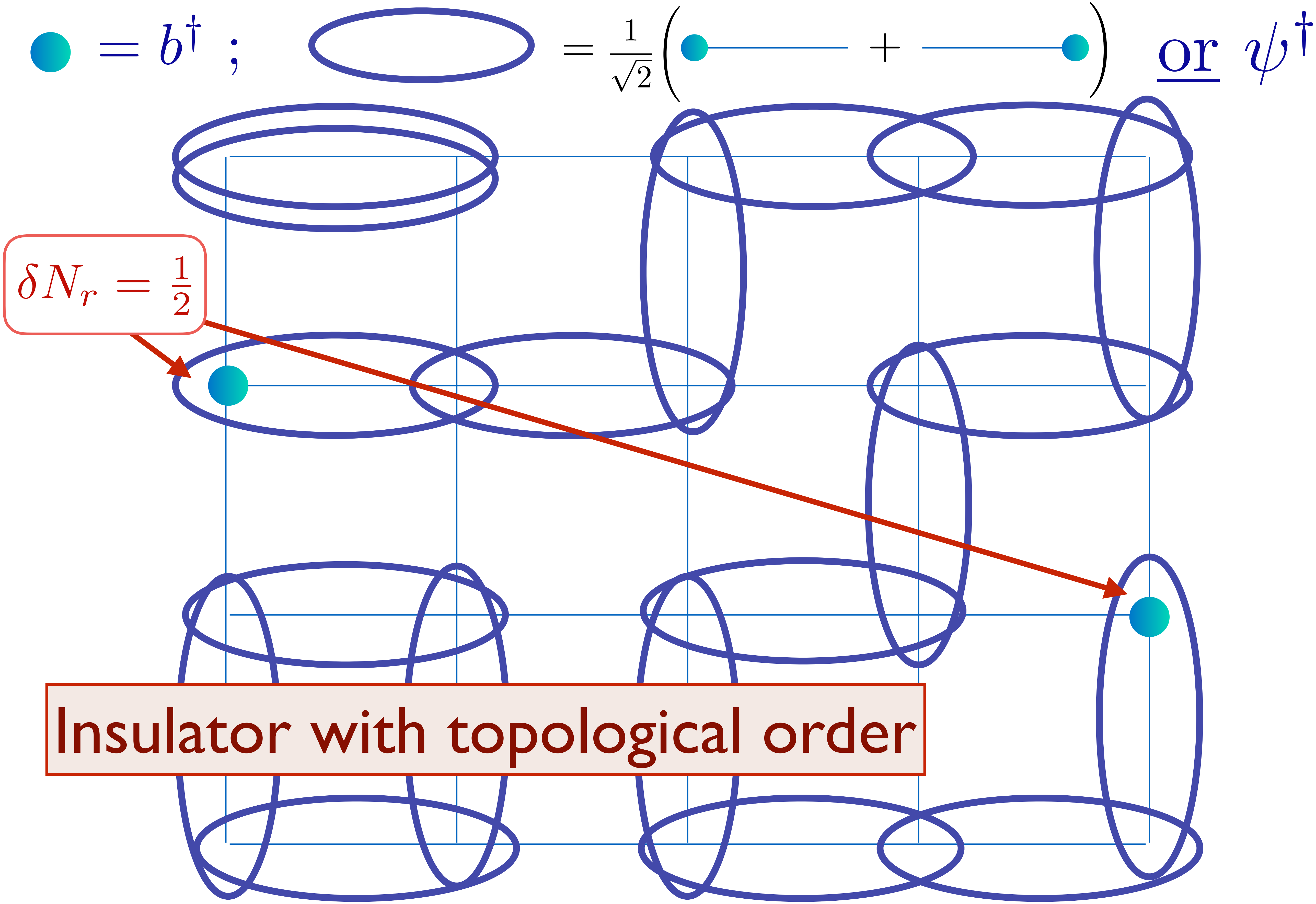
The half charge bosons can then move freely through the lattice



The half charge bosons can then move freely through the lattice



The half charge bosons can then move freely through the lattice



The half charge bosons can then move freely through the lattice

We can move beyond the $N_r = 1$ subspace, and account for these half-charged states, by introducing an operator $e^{i\bar{\theta}_r}$ which has U(1) gauge charge 1, and global boson number charge $\varepsilon_r/2$. Then gauge-invariant boson operators for the site and bond bosons now become

$$b_r = e^{i\varepsilon_r(\phi_r - 2\bar{\theta}_r)} \quad , \quad \psi_{r\alpha} = e^{i\varepsilon_r(a_{r\alpha} - \Delta_\alpha \bar{\theta}_r)}$$

It is easy to verify that these new representations leave the J and K terms in the previous spacetime lattice action for $N_r = 1$ independent of $\bar{\theta}_r$. We can also write the half-charge hopping terms illustrated in the previous slides in this formulation. The final form of the action so obtained is simplest in terms of a new field θ_r in the mapping

$$\bar{\theta}_r = \theta_r \text{ when } \varepsilon_r = 1 \quad , \quad \bar{\theta}_r = -\theta_r + \phi_r \text{ when } \varepsilon_r = -1$$

Note that $e^{i\theta_r}$ which has U(1) gauge charge 1, and global boson number charge 1/2

Bosons at unit density on the square lattice

Collecting these transformations, we obtain the complete action for the full phase diagram is

$$
\begin{aligned}
S &= -t \sum_r \cos(\Delta_\mu \theta_r - a_{r\mu} - A_{r\mu}/2) \\
&\quad - J \sum_r \cos(\Delta_\mu \phi_r - 2a_{r\mu}) \\
&\quad - K \sum_{\Box} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda)
\end{aligned}
$$

In this form, $e^{i\theta_r}$ has U(1) gauge charge 1, and boson number charge 1/2; $e^{i\phi_r}$ has U(1) gauge charge 2, and boson number charge 0. We have also included an external (fixed) gauge field A_μ, which couples to the boson number. The gauge-invariant boson operator is

$$
b_r \sim e^{-i2\theta_r + i\phi_r}
$$

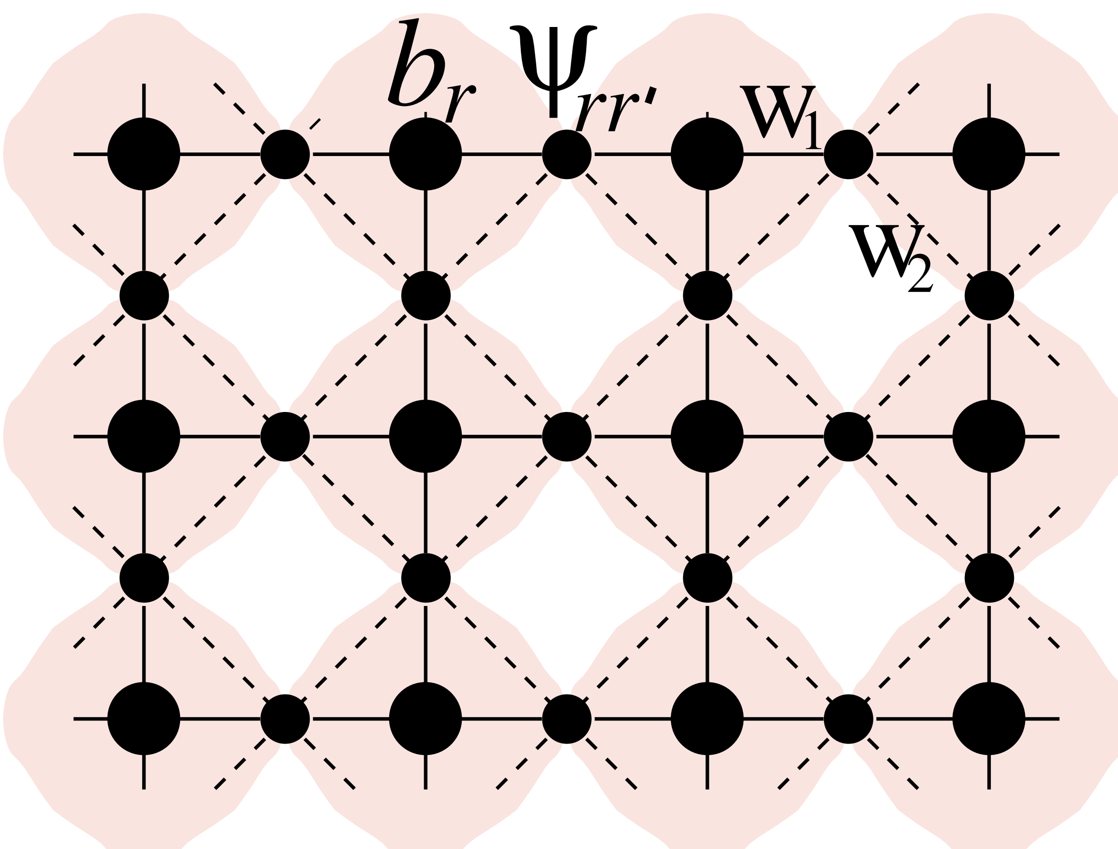
and this only has A_μ charge 1/2.

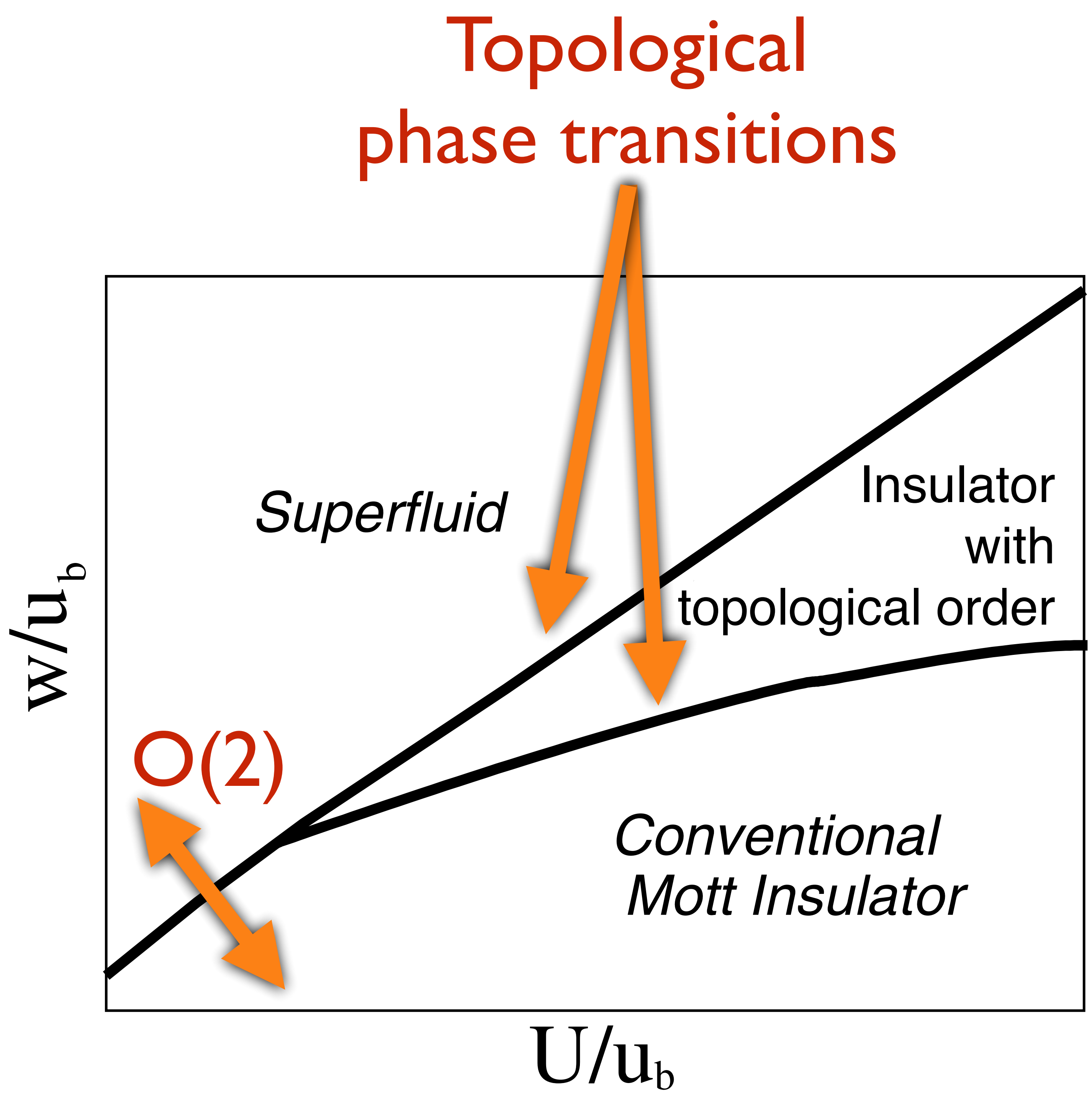
Bosons at unit density on the square lattice

An alternative formulation in two dimensions preemptively accounts for the strong effects of monopoles. We take the strong-coupling limit, in which the Higgs field is locally condensed, to a $\mathbb{Z}_2$ gauge field where $e^{i\phi_r} = 1$ and $e^{ia_{r\mu}} = \pm 1$. Then we have the Hamiltonian of a $\mathcal{Z}_2$ gauge field coupled to a half-charged boson $e^{i\theta_r}$:

$$
\begin{aligned}
H &= -t \sum_{r,\alpha=x,y} \tau^z_{r\alpha} \cos(\Delta_\alpha \theta_r - A_{r\alpha}/2) \\
&\quad - K \sum_{\square} \tau^z \tau^z \tau^z \tau^z \\
&\quad - g \sum_{r,\alpha=x,y} \tau^x_{r\alpha}
\end{aligned}
$$

Bosons at unit density on the square lattice

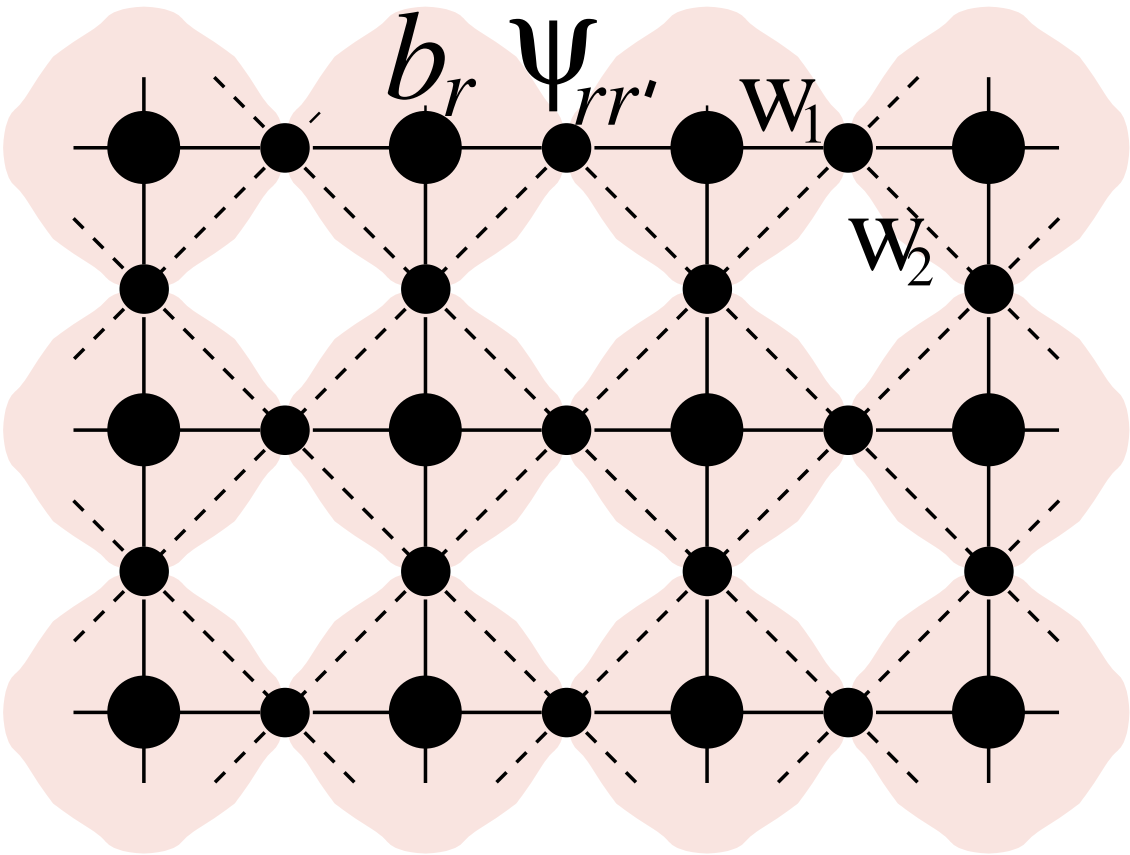


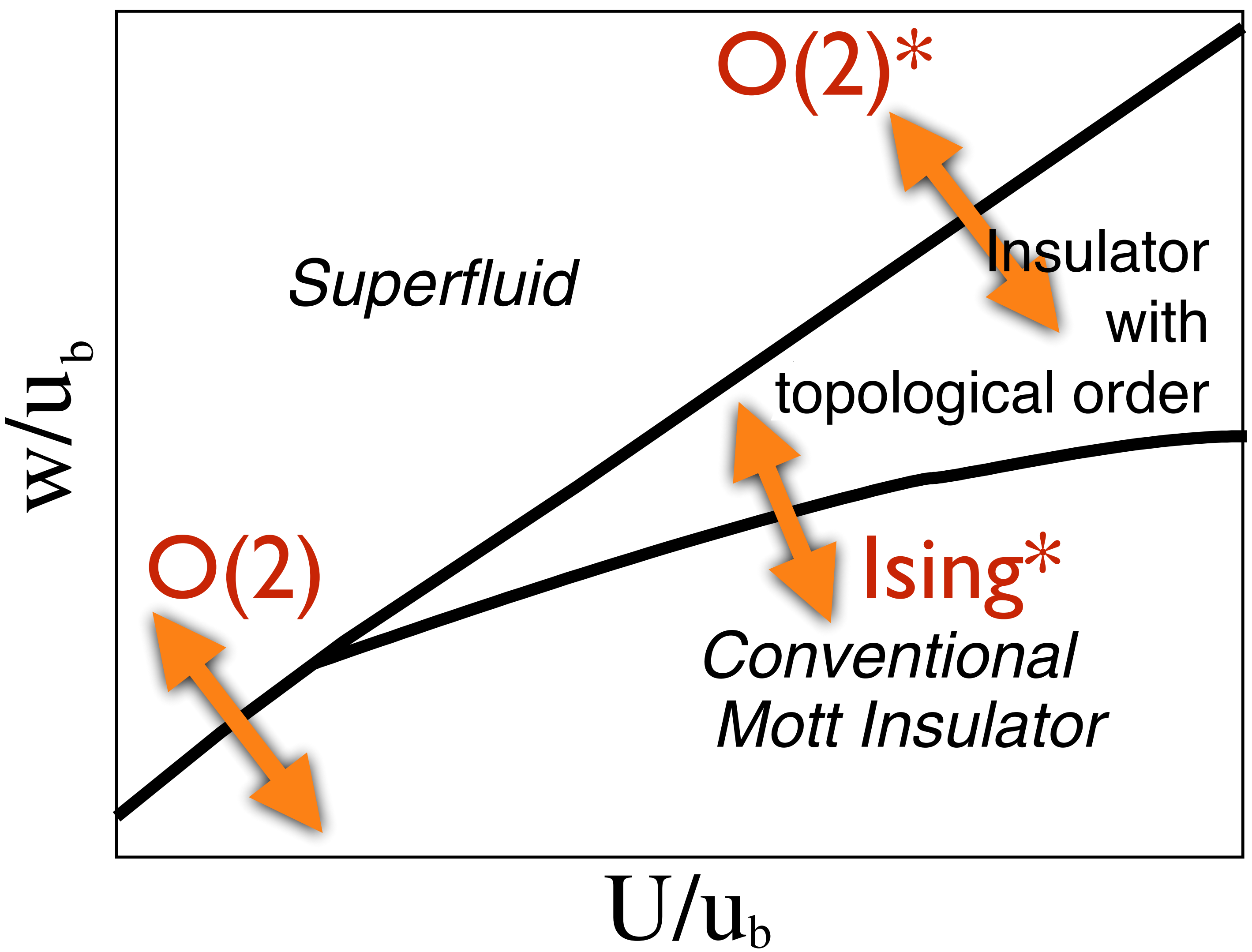


Average of one boson per site: $\langle N_r \rangle = 1$

R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); O.I. Motrunich and T. Senthil, PRL **89**, 277004 (2002).

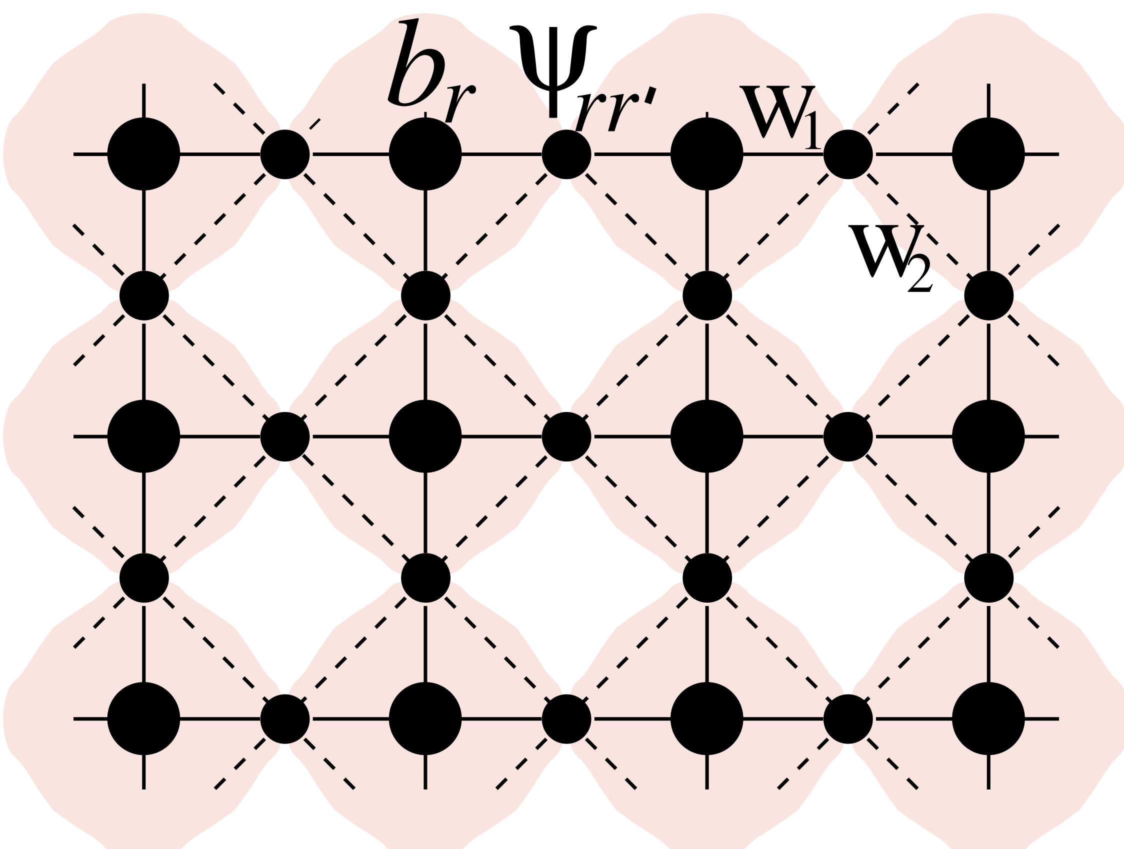
R. Jalabert and S. Sachdev PRB 44, 686 (1991); A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); O.I. Motrunich and T. Senthil, PRL **89**, 277004 (2002).

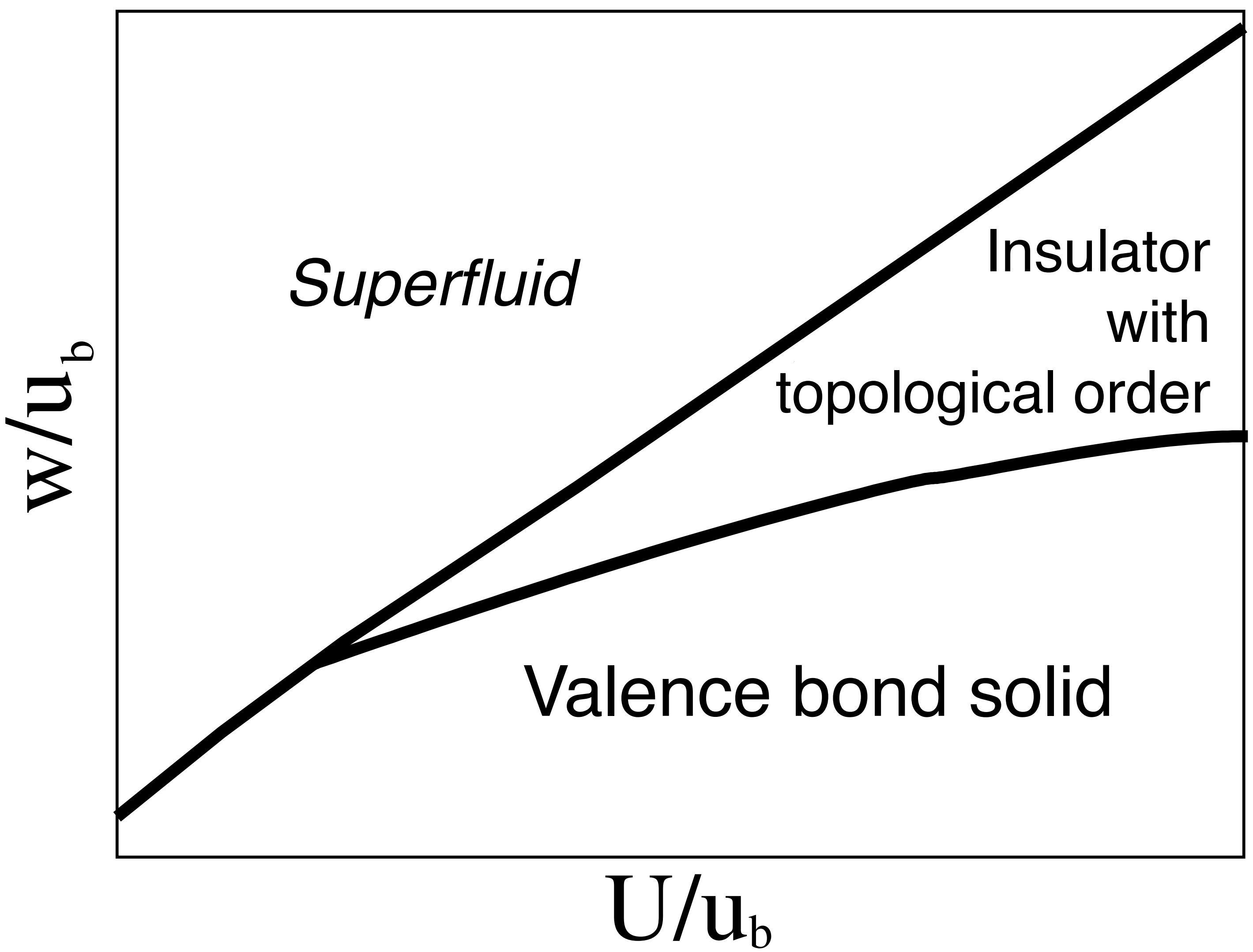




Bosons at unit density on the square lattice

S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000).

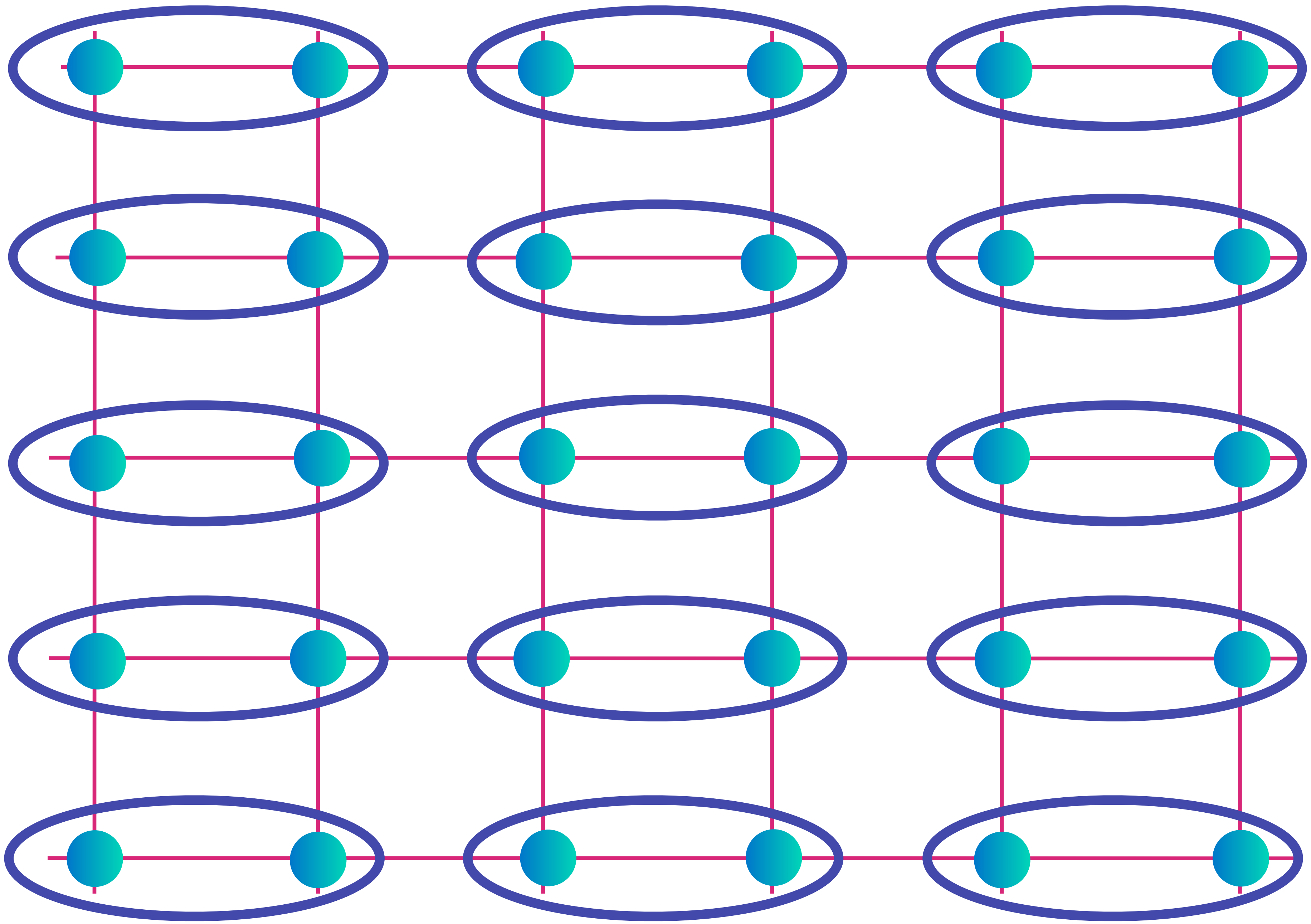




Bosons at half-integer density on the square lattice

VBS states on the square lattice

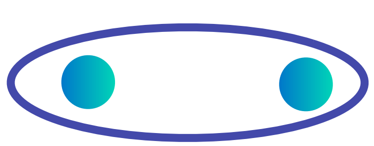
Columnar VBS



Equivalently, this can be interpreted as a model of bosons at 1/2 filling, with the mapping

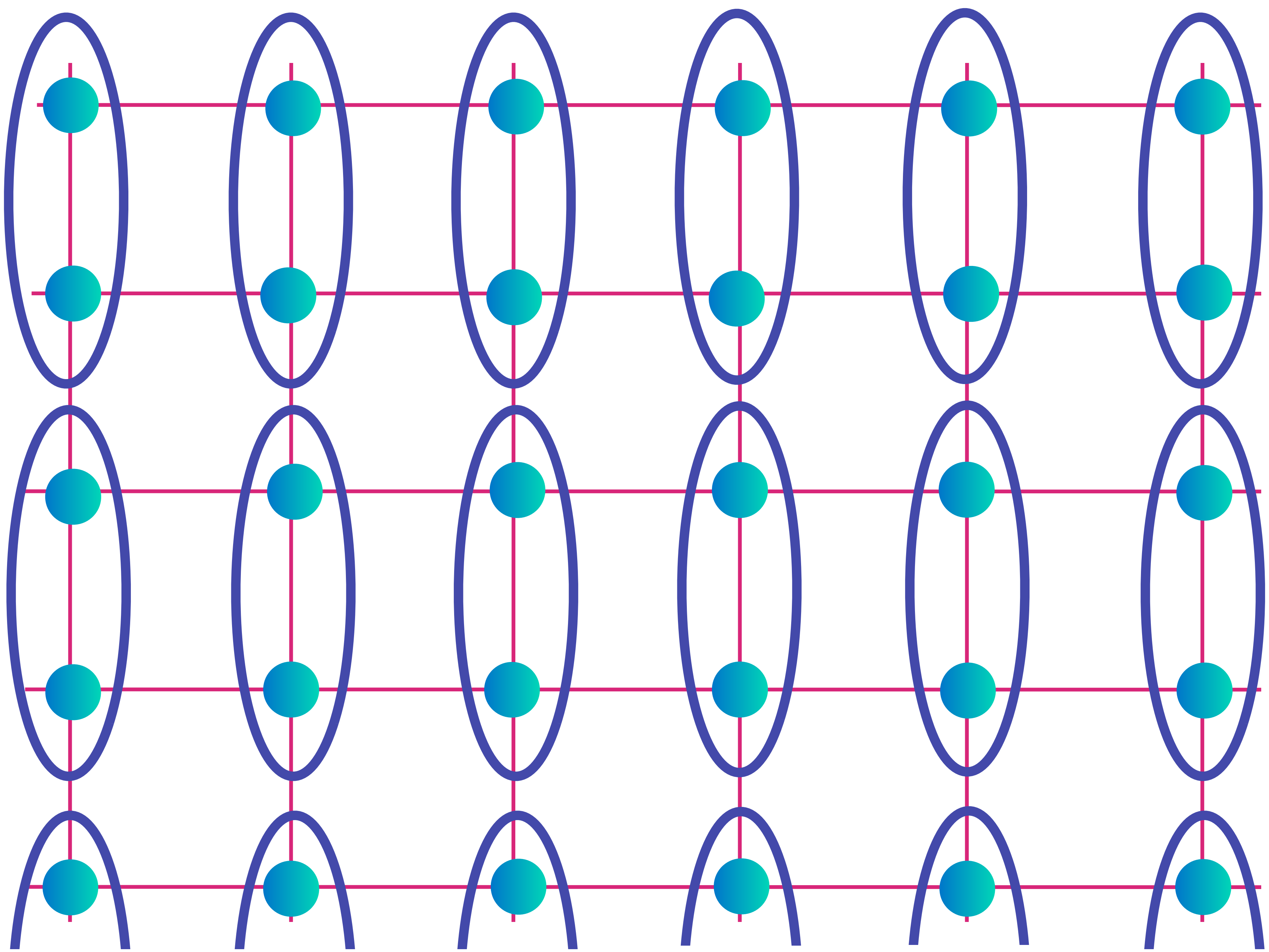
$$|\uparrow\rangle \Rightarrow |0\rangle$$

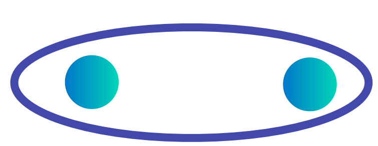
$$|\downarrow\rangle \Rightarrow b^\dagger\,|0\rangle$$

 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}$

VBS states on the square lattice

Columnar VBS



 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) / \sqrt{2}$

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VBS states on the square lattice

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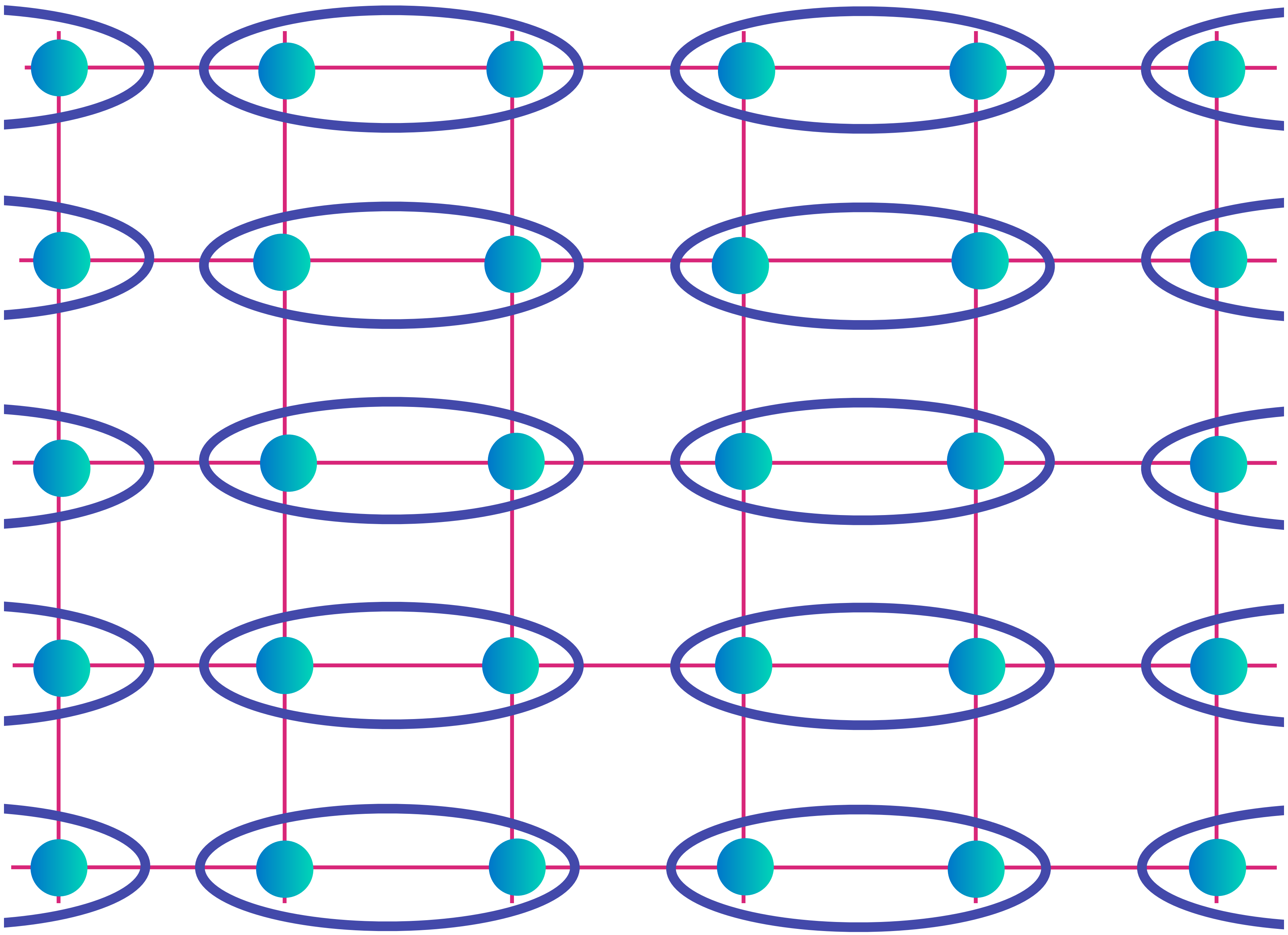

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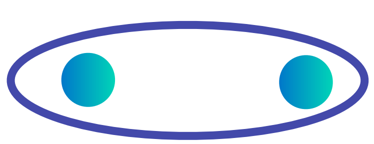
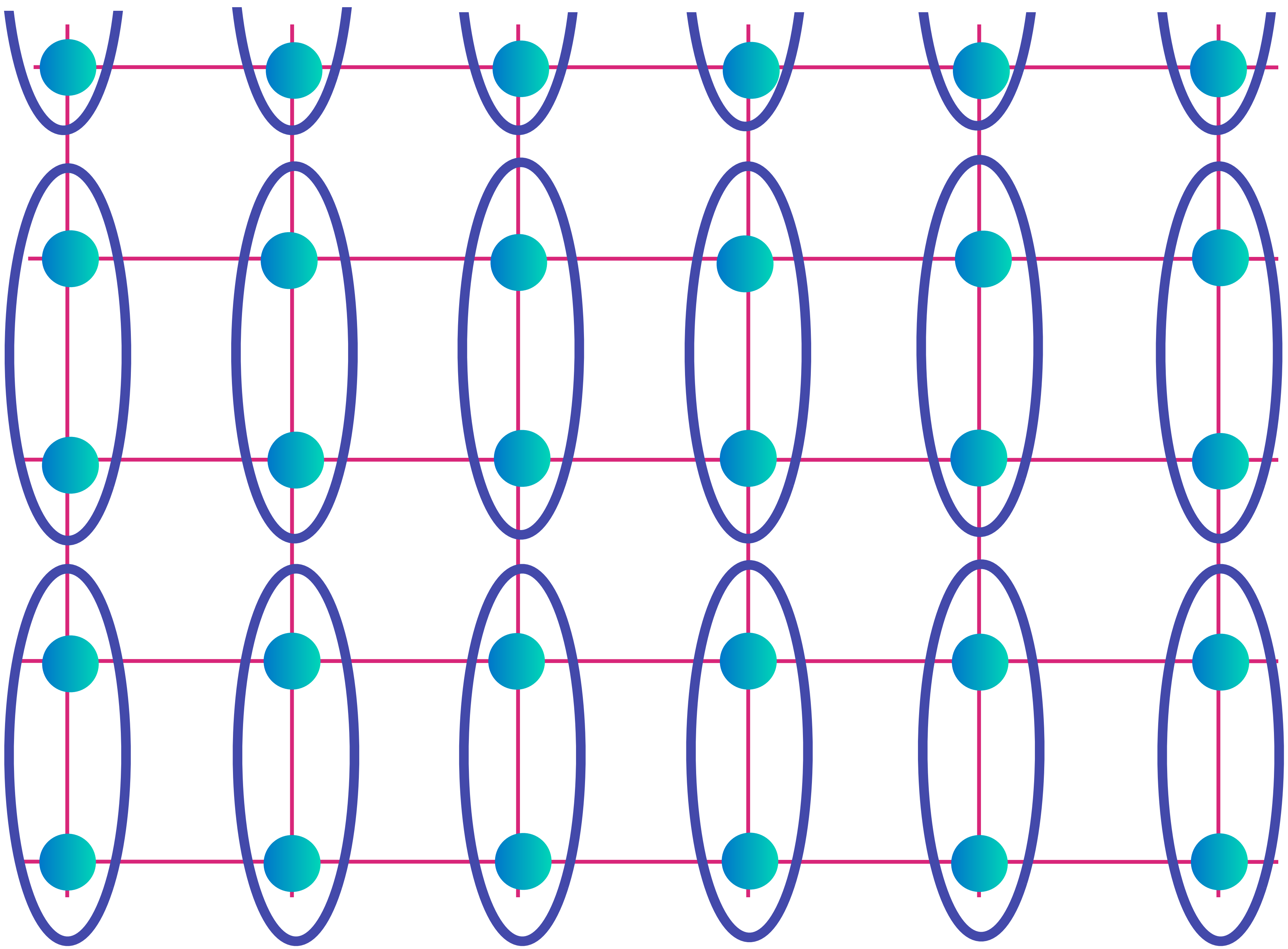
 $= \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) / \sqrt{2}$

VBS states on the square lattice

Columnar VBS



$$= (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) / \sqrt{2}$$

Equivalently, this can be interpreted as a model of bosons at 1/2 filling, with the mapping

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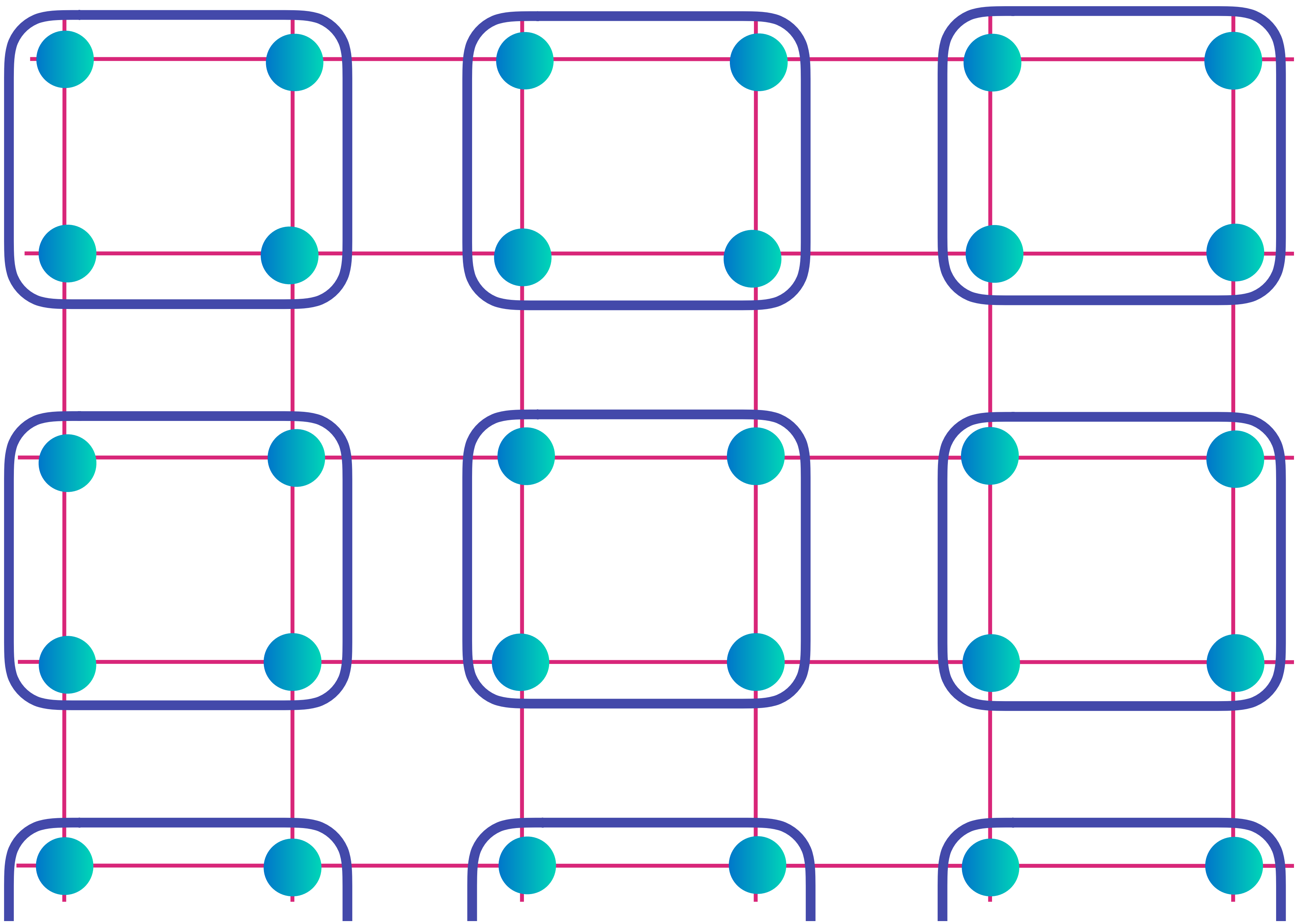
VBS states on the square lattice

Plaquette VBS



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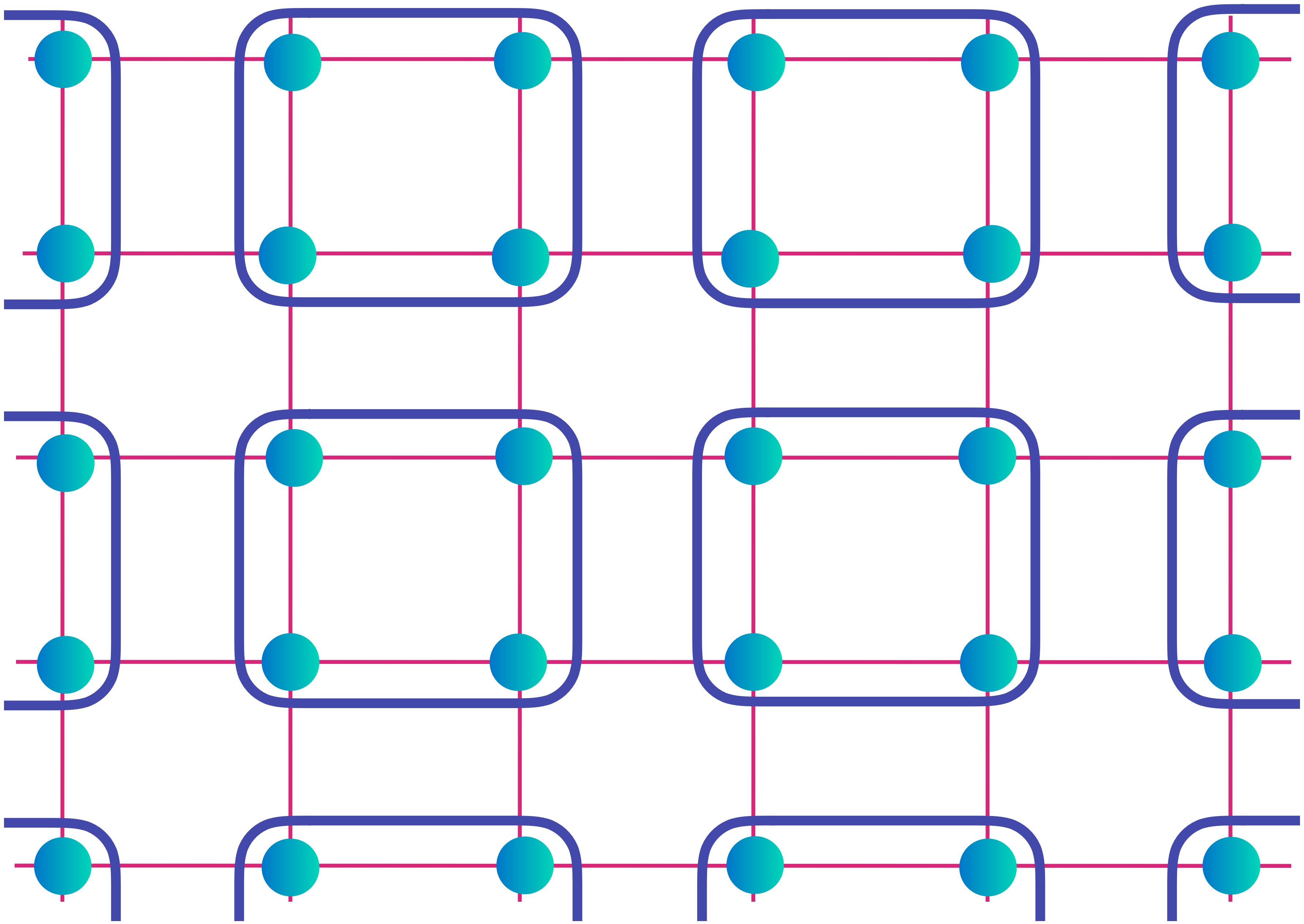
$|\uparrow\rangle \Rightarrow |0\rangle$
$|\downarrow\rangle \Rightarrow b^\dagger |0\rangle$

VBS states on the square lattice

Plaquette VBS



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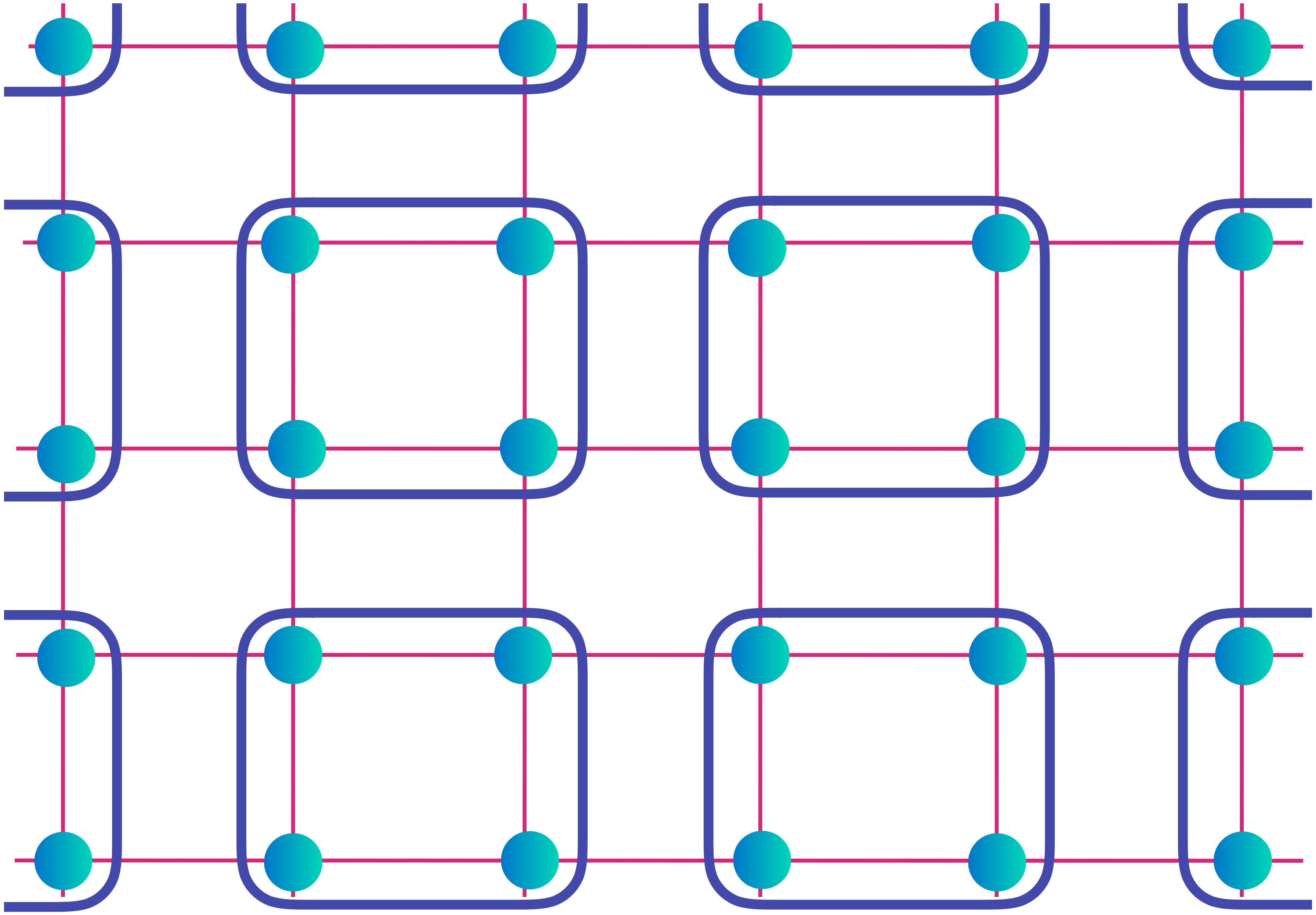
VBS states on the square lattice

Plaquette VBS



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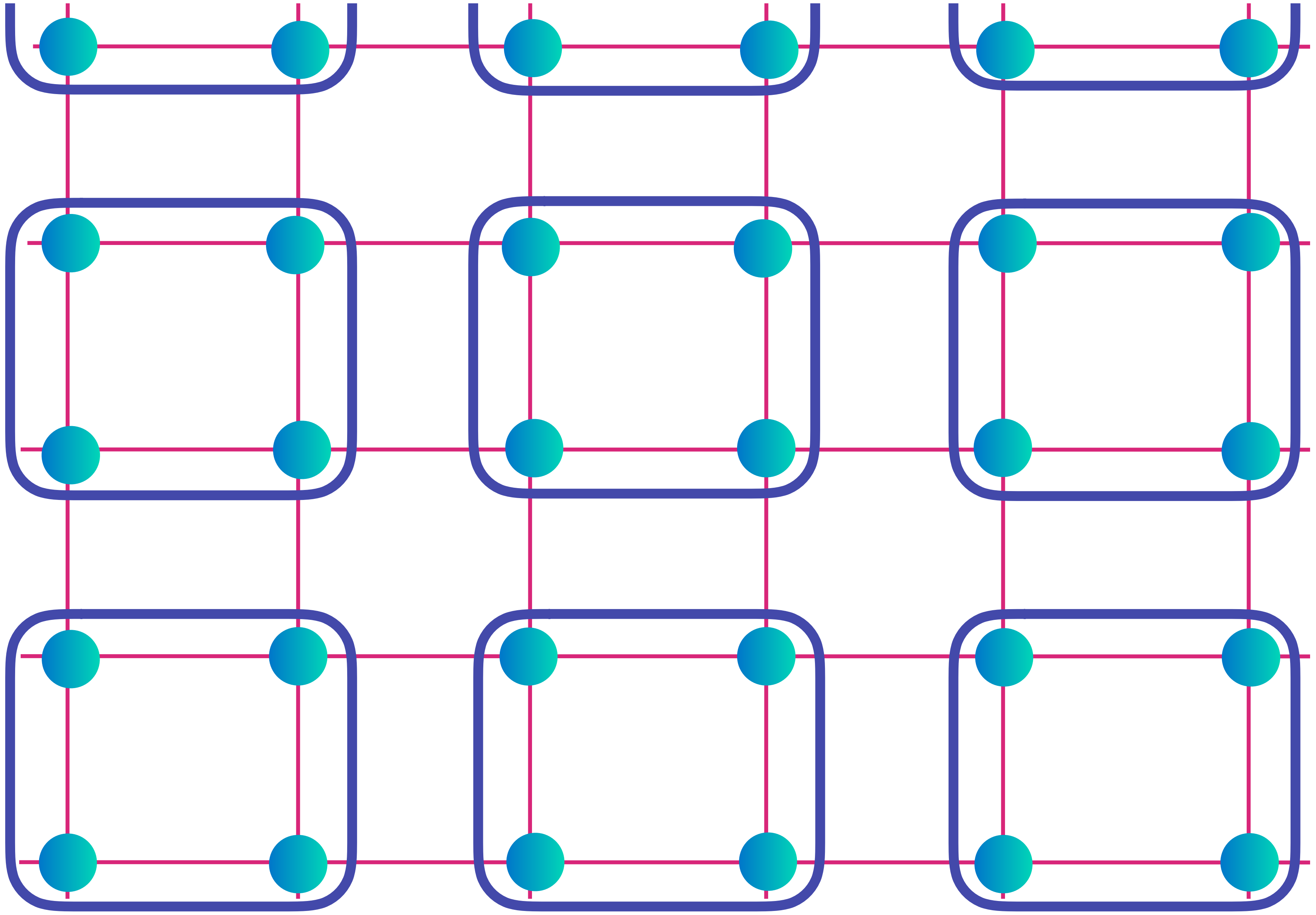
VBS states on the square lattice

Plaquette VBS



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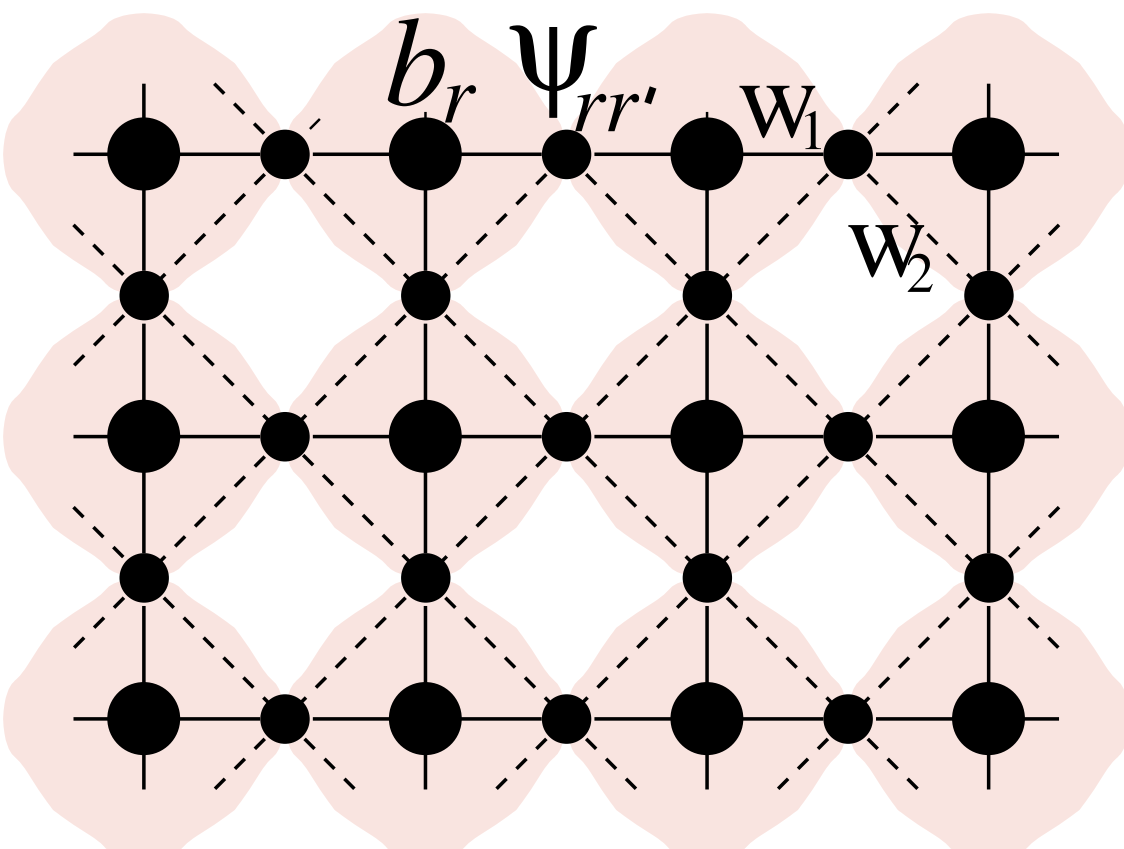
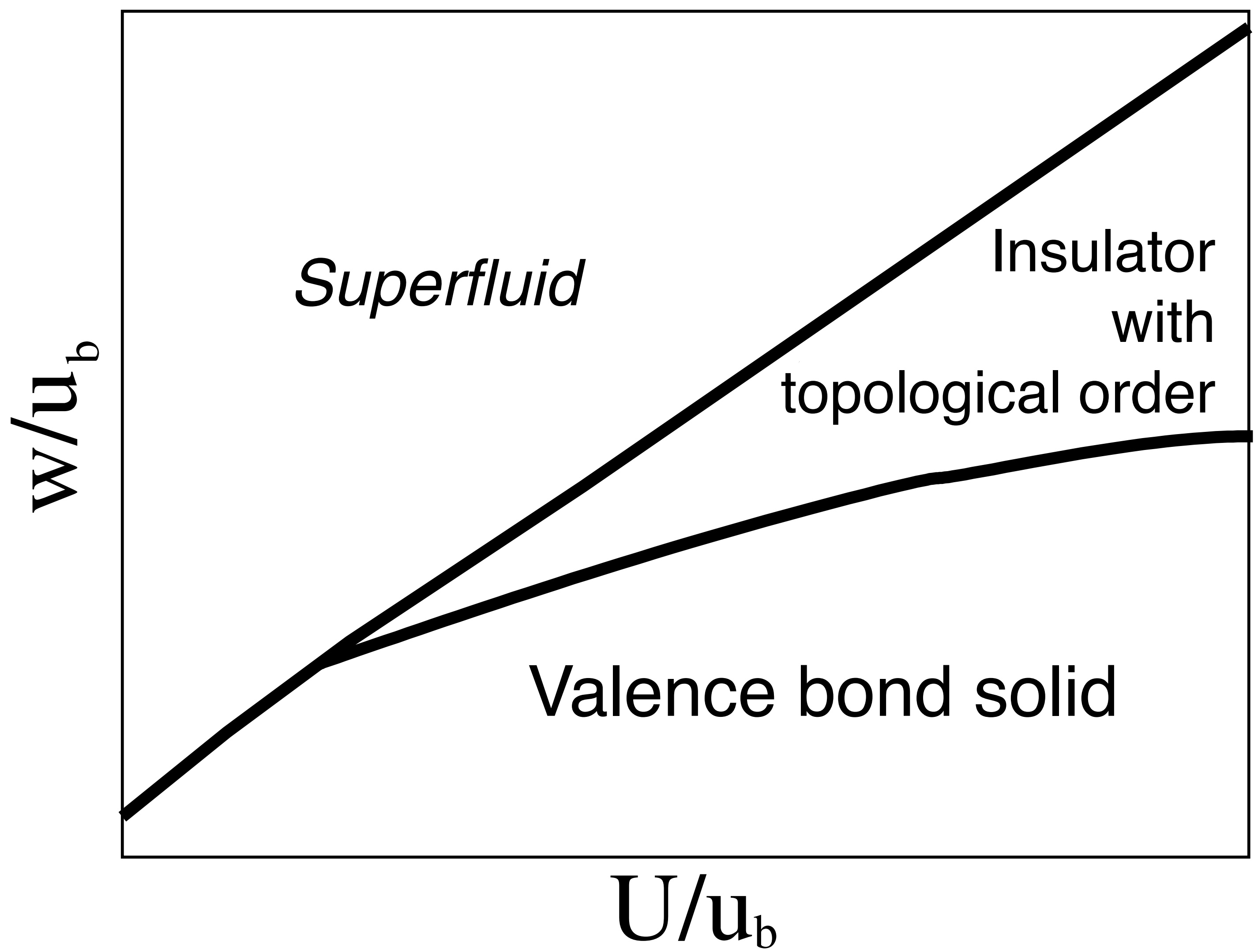
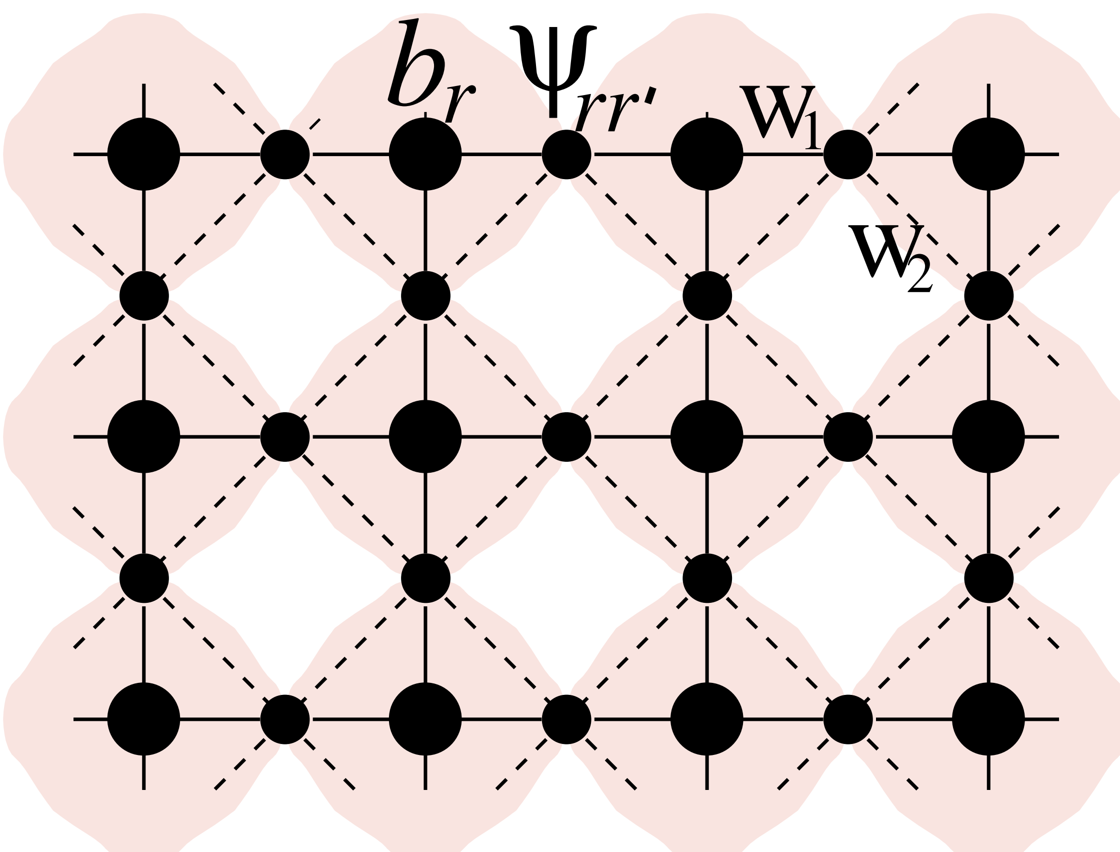
S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000).

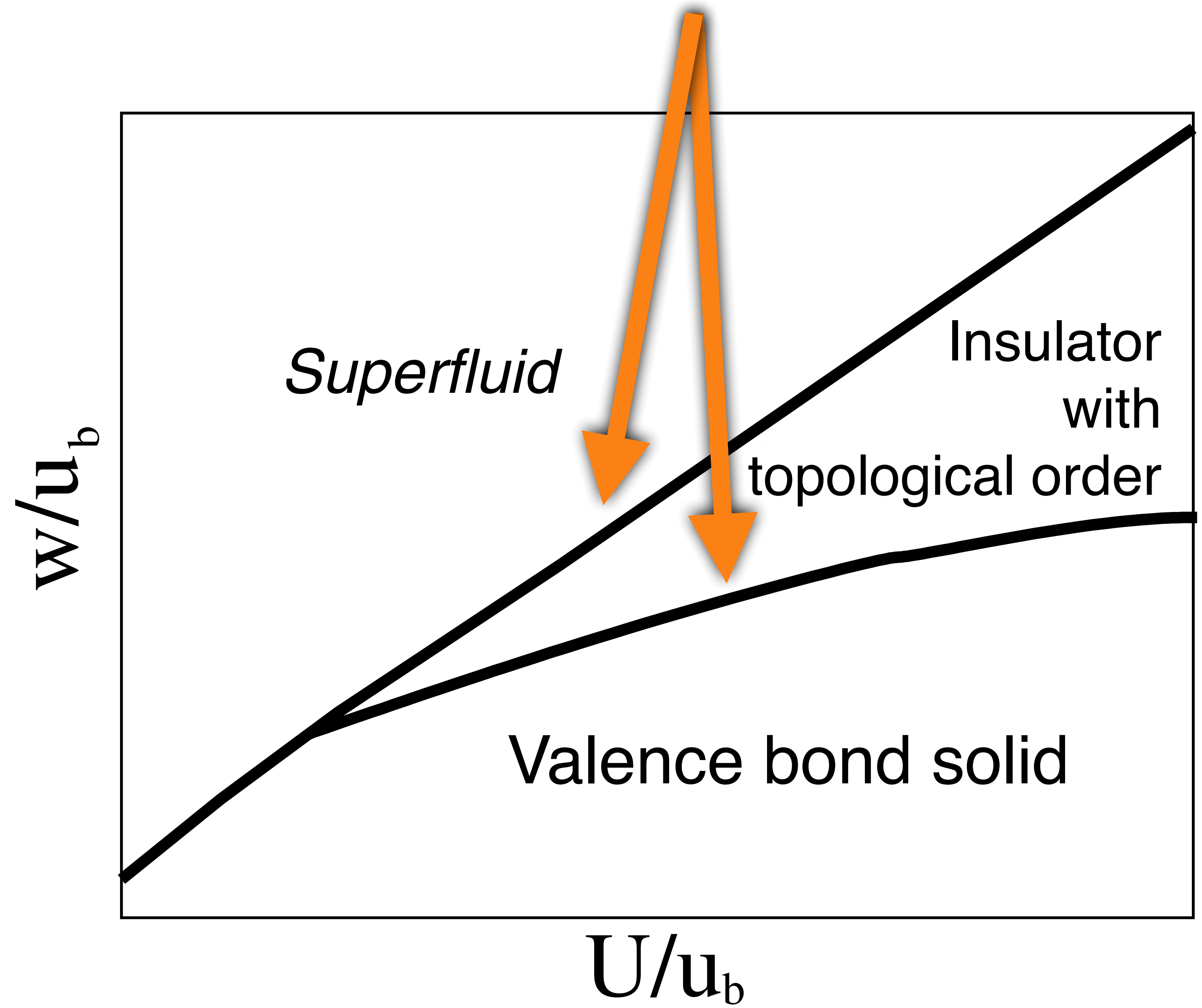




Bosons at half-integer density on the square lattice

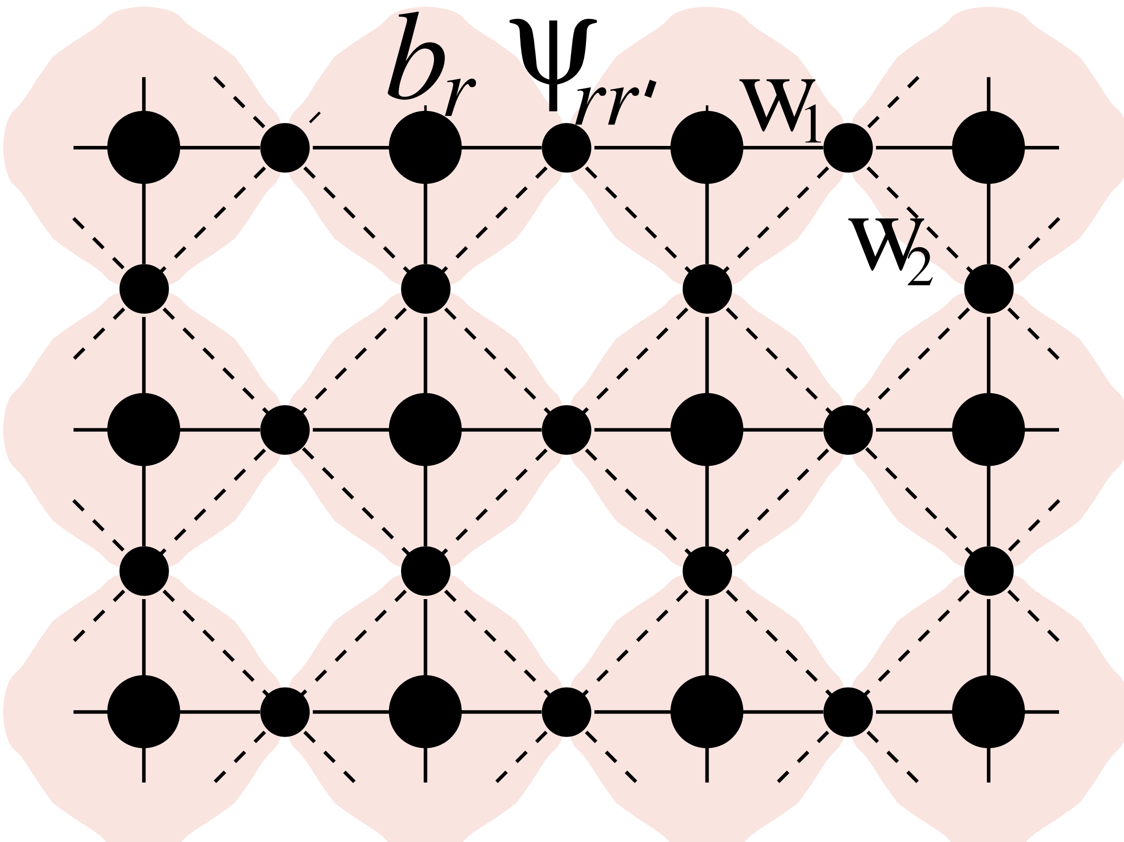
Topological phase transitions

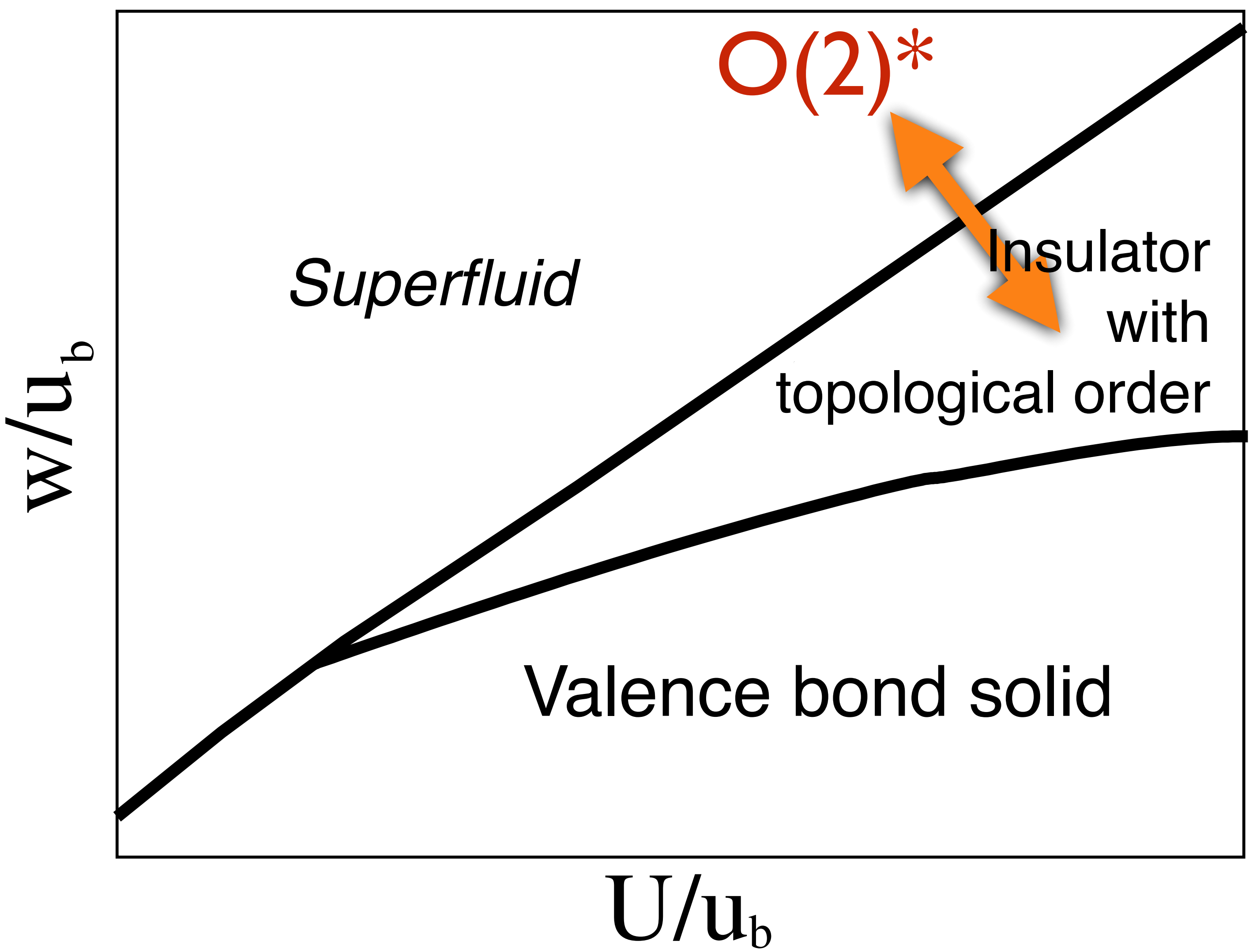




Bosons at half-integer density on the square lattice

A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994)

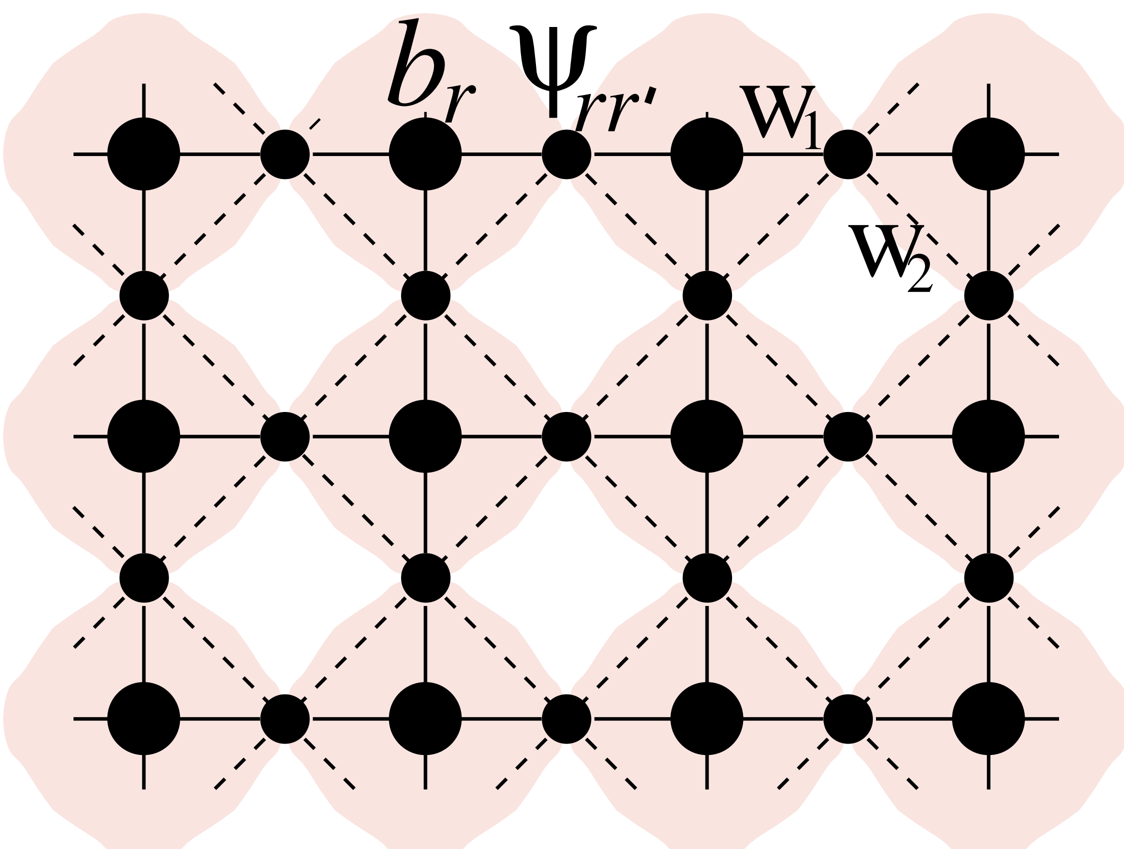


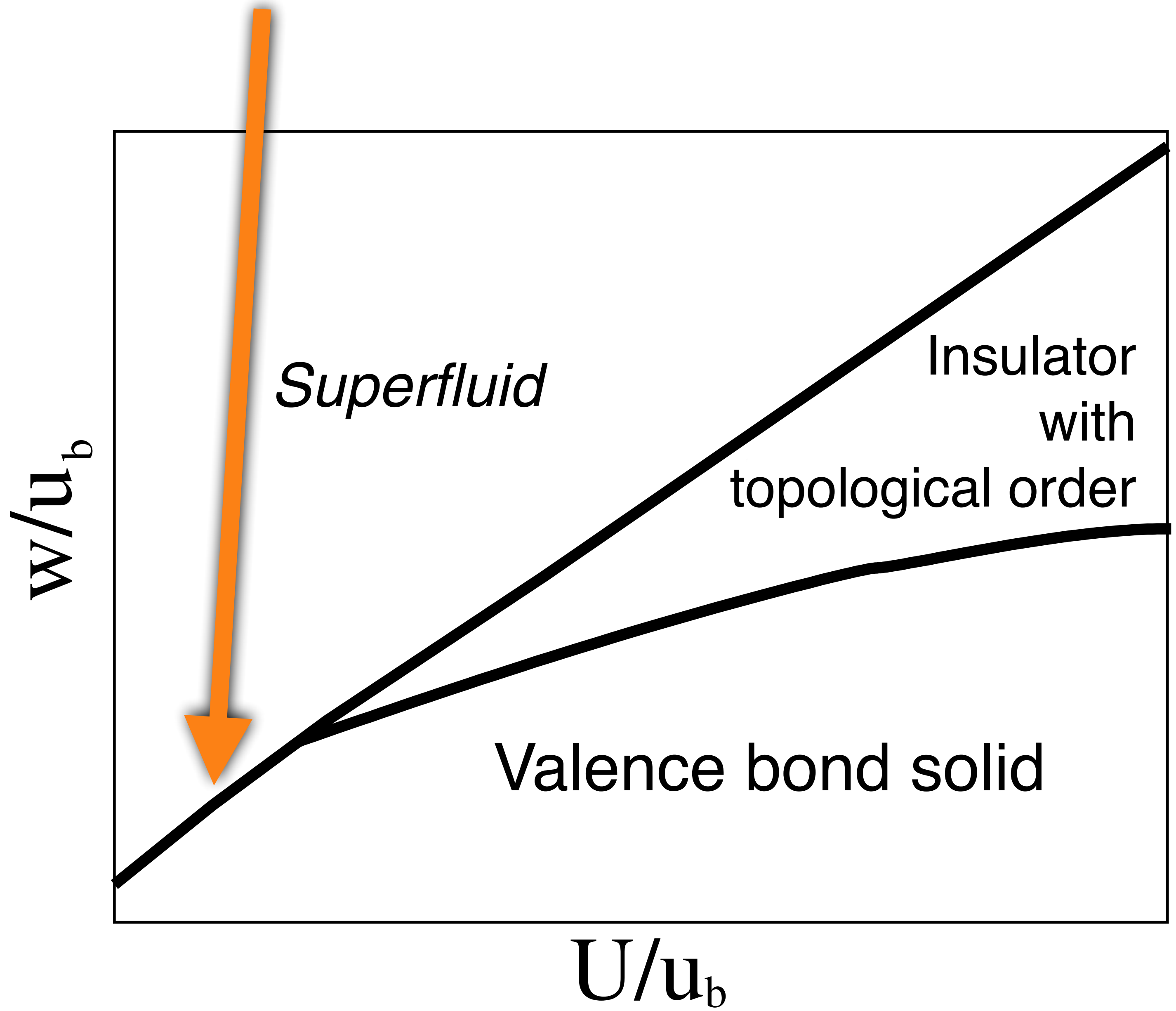


Bosons at half-integer density on the square lattice

T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P.A. Fisher, Science **303**, 1490 (2004)

Deconfined criticality





Bosons at half-integer density on the square lattice

S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000).

At half-integer density, we obtain an additional Berry phase term ($\varepsilon_r = \pm 1$ on the two sub lattices)

$$
\begin{aligned}
S \;=\; & i \sum_r \varepsilon_r a_{r\tau} \\
& - t \sum_r \cos(\Delta_\mu \theta_r - a_{r\mu} - A_{r\mu}/2) \\
& - J \sum_r \cos(\Delta_\mu \phi_r - 2a_{r\mu}) \\
& - K \sum_{\Box} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda)
\end{aligned}
$$

The Berry phases prohibit a "trivial" phase with no broken symmetry and no topological order: instead we obtain a phase with valence bond solid order and broken translational symmetry. Also, the $\mathbb{Z}_2$ spin liquid is now a "symmetry enriched topological" (SET) state.

Bosons at half-integer density on the square lattice

“Anomaly” constraints on phase diagram

Consider a general lattice Hamiltonian with 2 global symmetries: translation by a lattice spacing, $\hat{T}_x$, and a global U(1) symmetry, $\mathcal{U}$, associated in our case with conservation of boson number. The global U(1) symmetry is generated by

$$\mathcal{U} = \exp\left(i\alpha \sum_i \hat{n}_i\right)$$

where α is the rotation angle, and $\hat{n}_i$ is the boson number on site i. There two symmetries clearly commute

$$\hat{T}_x \mathcal{U} = \mathcal{U} \hat{T}_x$$

Now place the system on a $L_x \times L_y$ torus, let us consider a spatially-dependent rotation angle ($i.e.$ we gauge the U(1) symmetry)

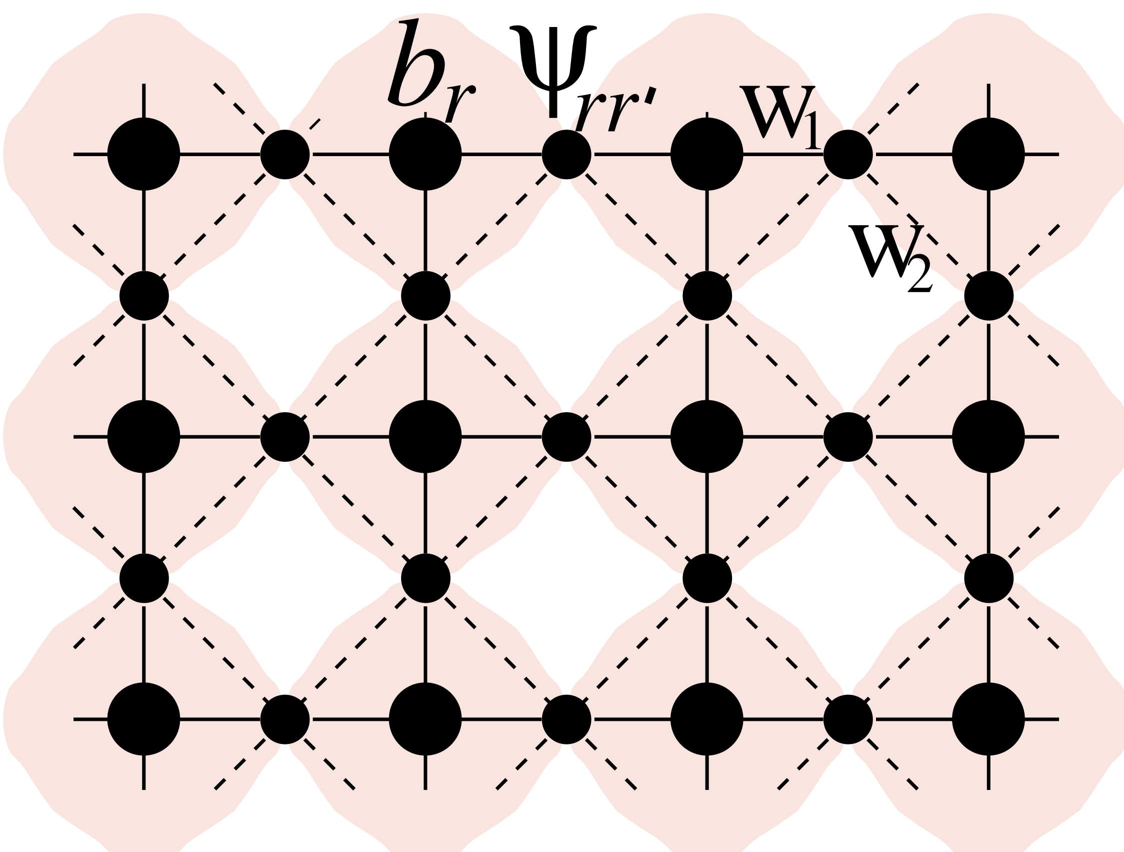
$$\mathcal{U}_G = \exp\left(i\frac{2\pi}{L_x}\sum_i x_i \hat{n}_i\right).$$

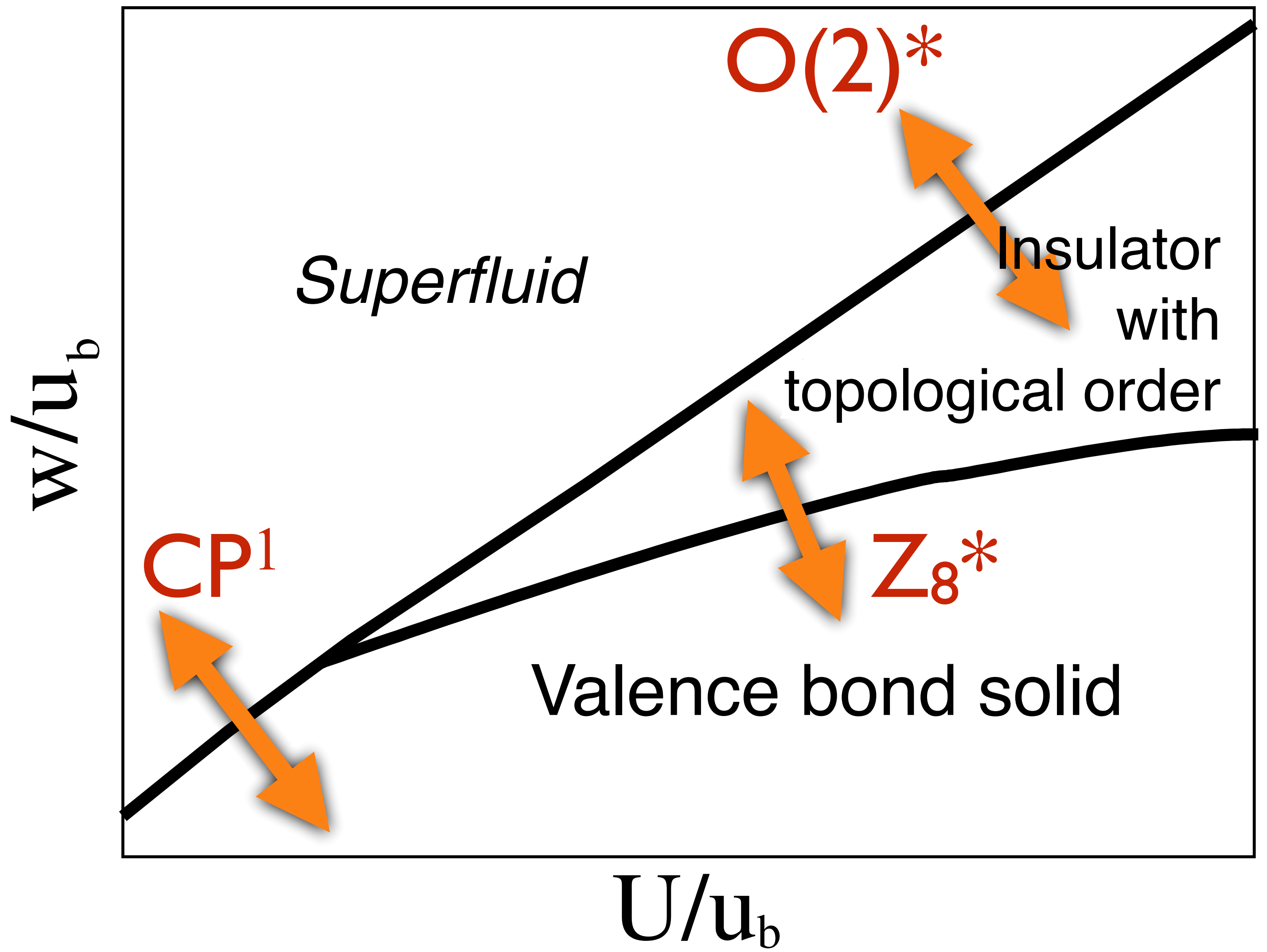
It is now easy to show that $\hat{T}_x$ and $\mathcal{U}_G$ do not commute, and

$$\hat{T}_x \mathcal{U}_G = \exp\left(-i2\pi\frac{N}{L_x}\right) \mathcal{U}_G \hat{T}_x$$

where N is the total number of bosons. This anomaly is an obstruction to gauging the $\mathrm{U}(1)\times\hat{T}_x$ global symmetry.

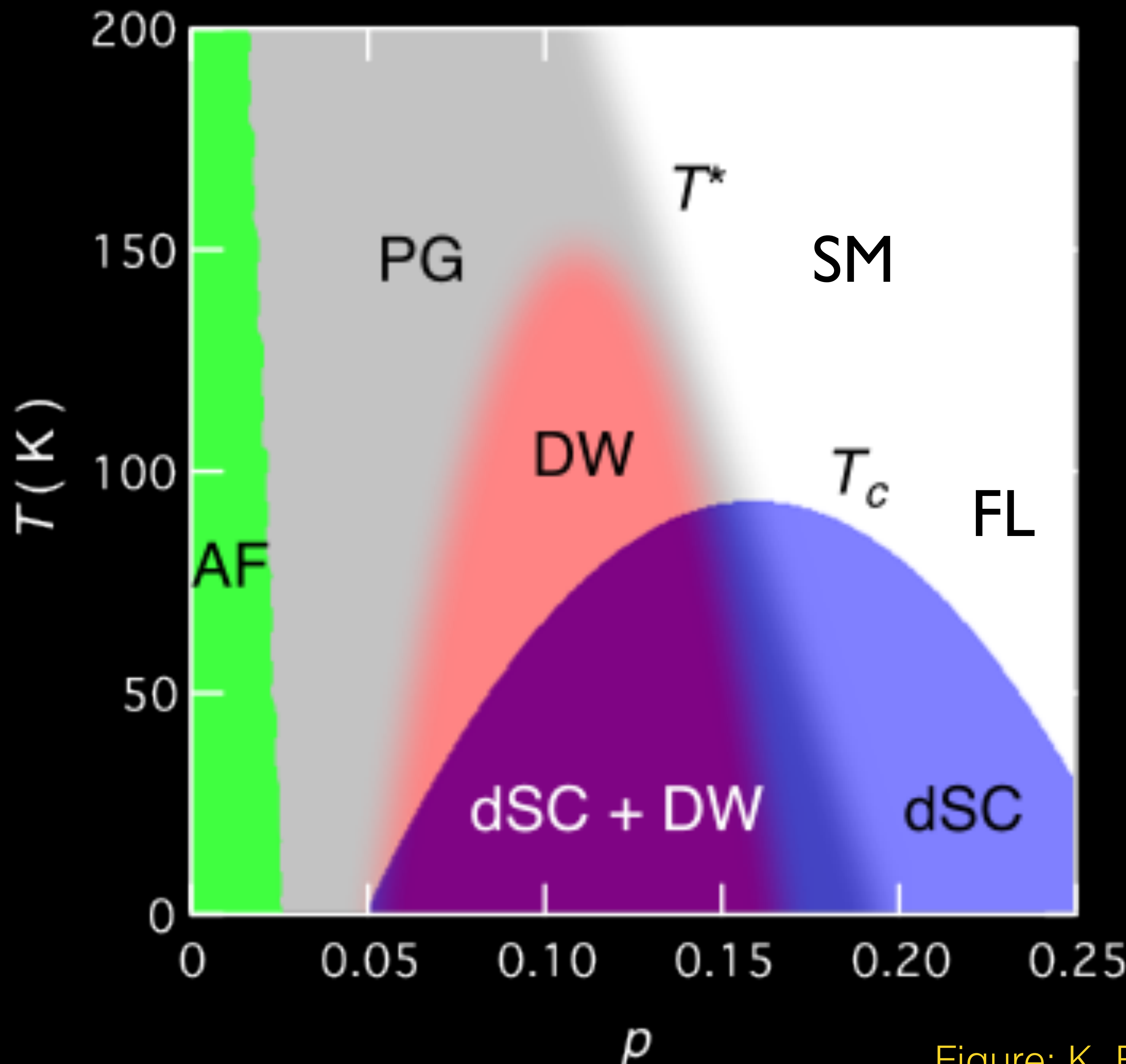
S. Sachdev and R. Jalabert, Modern Physics Letters B **4**, 1043 (1990); R. Jalabert and S. Sachdev Phys. Rev. B 44, 686 (1991); S. Sachdev and M. Vojta, Journal of the Physical Society of Japan **69,** Suppl. B, 1 (2000); T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science **303**, 1490 (2004)

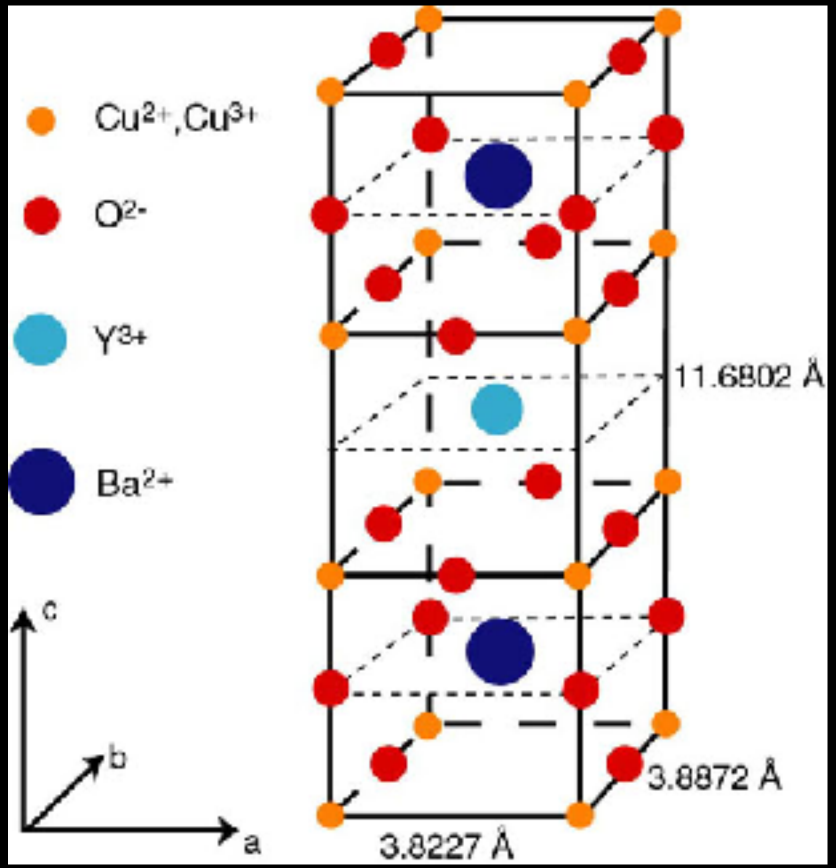




Bosons at half-integer density on the square lattice

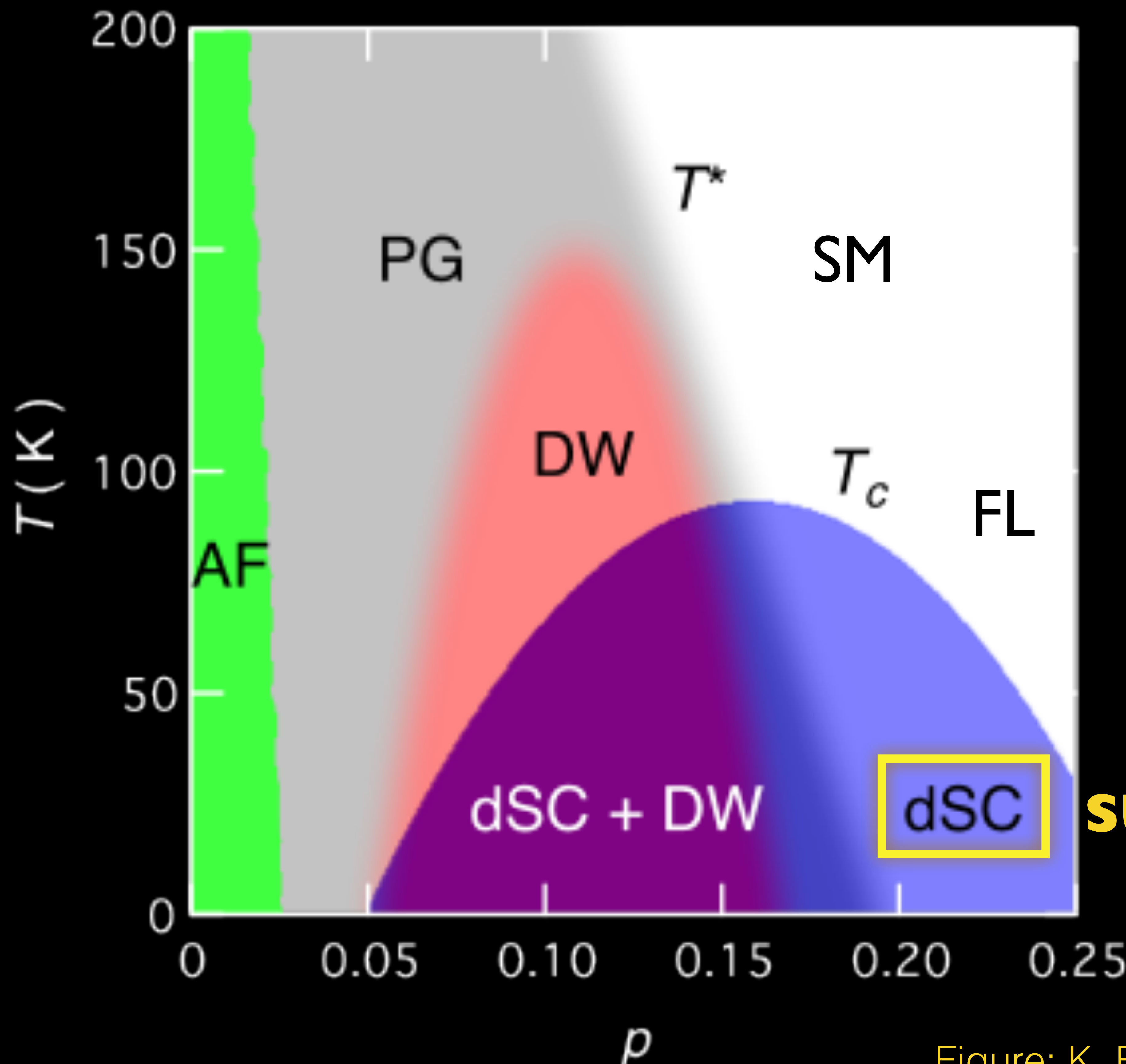
1. Introduction to spin liquids and topological order
2. Topological order and phase transitions in a model of bosons on the square lattice
3. Survey of recent experiments in the cuprates
4. Model of a topological phase transition for the cuprates

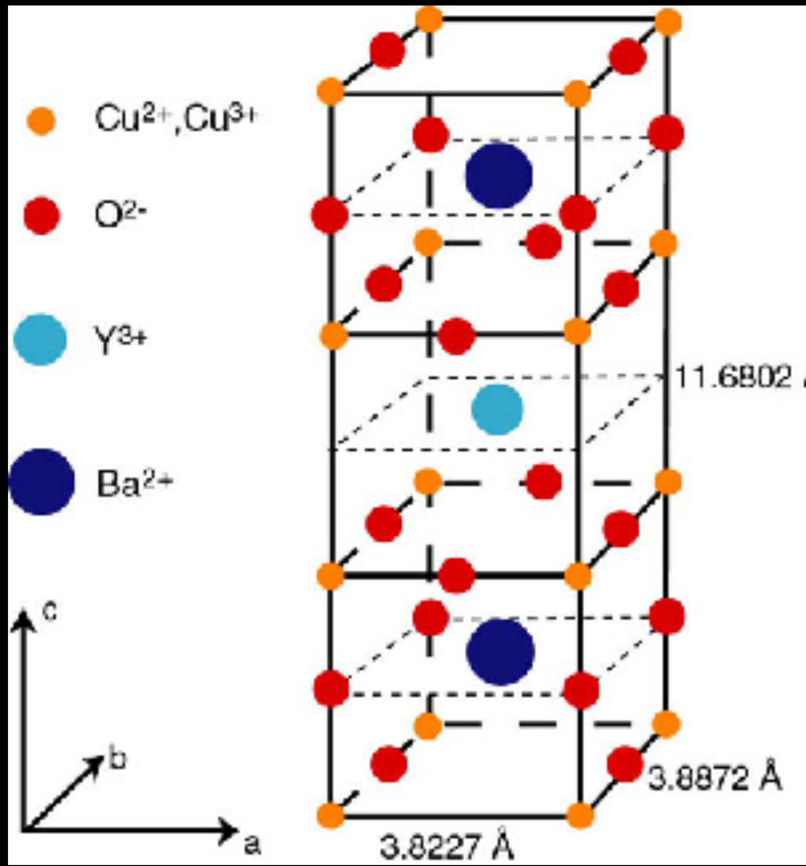




$YBa_2Cu_3O_{6+x}$

Figure: K. Fujita and J. C. Seamus Davis





$YBa_2Cu_3O_{6+x}$

d-wave superconductor

Figure: K. Fujita and J. C. Seamus Davis

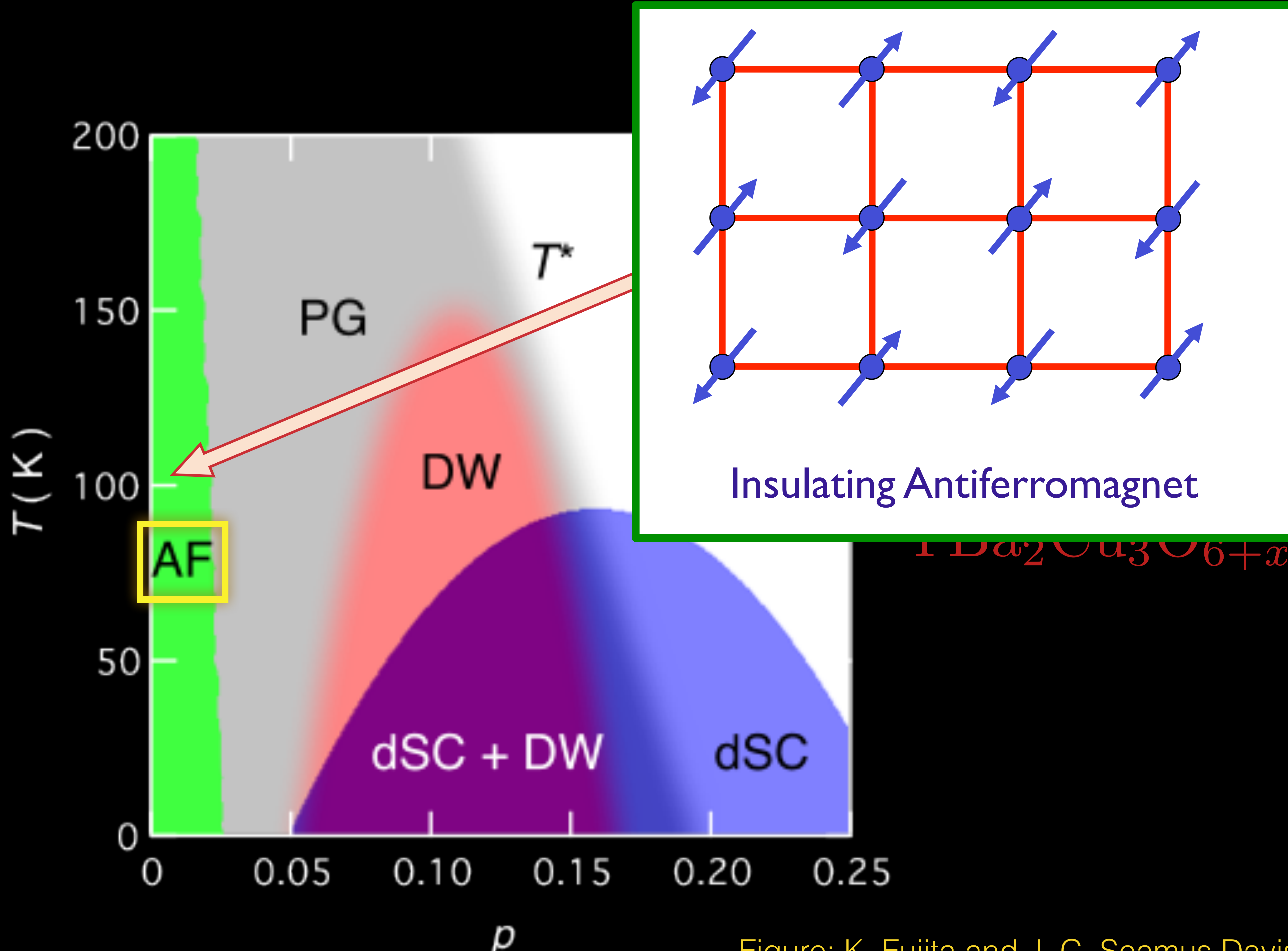
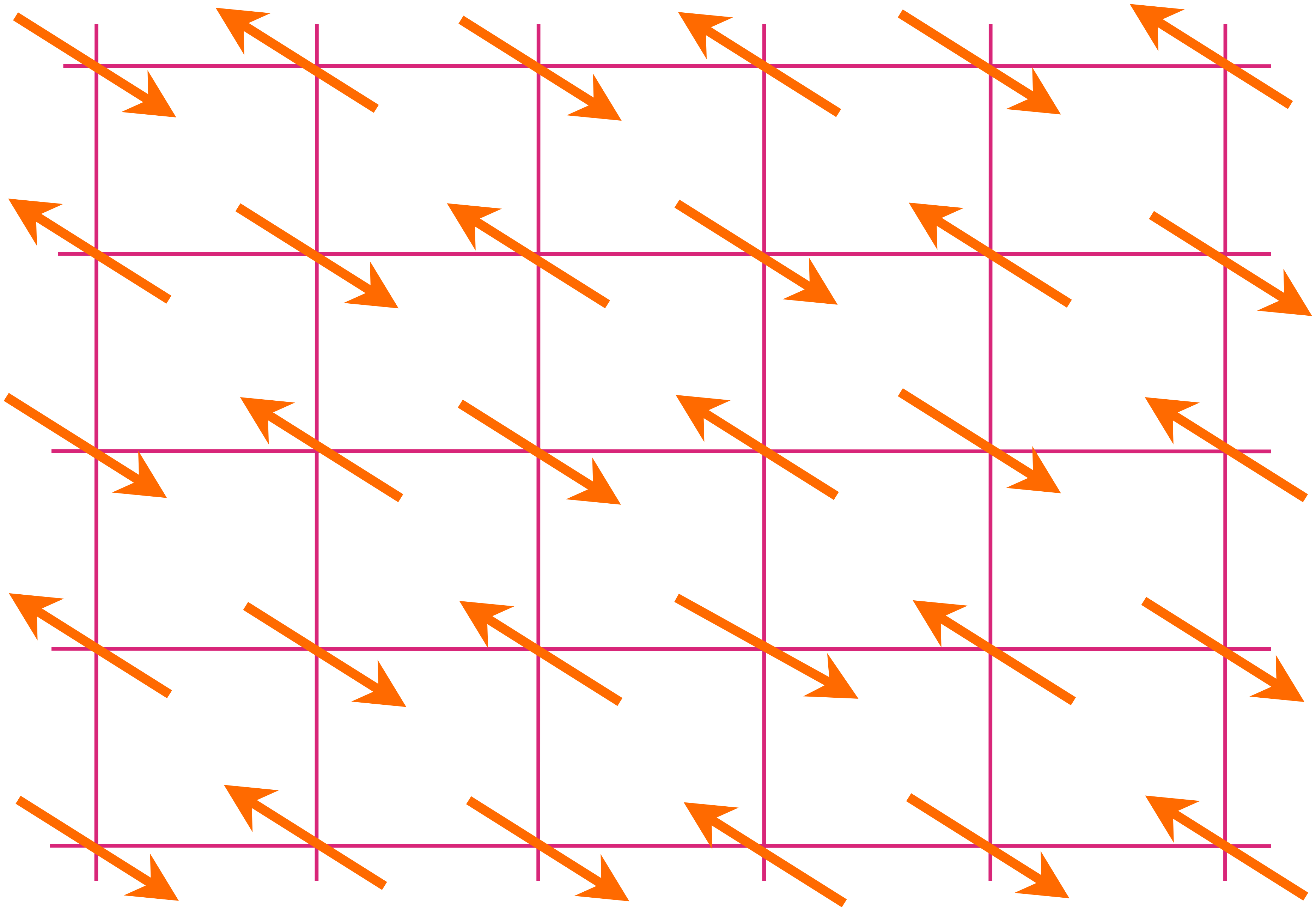
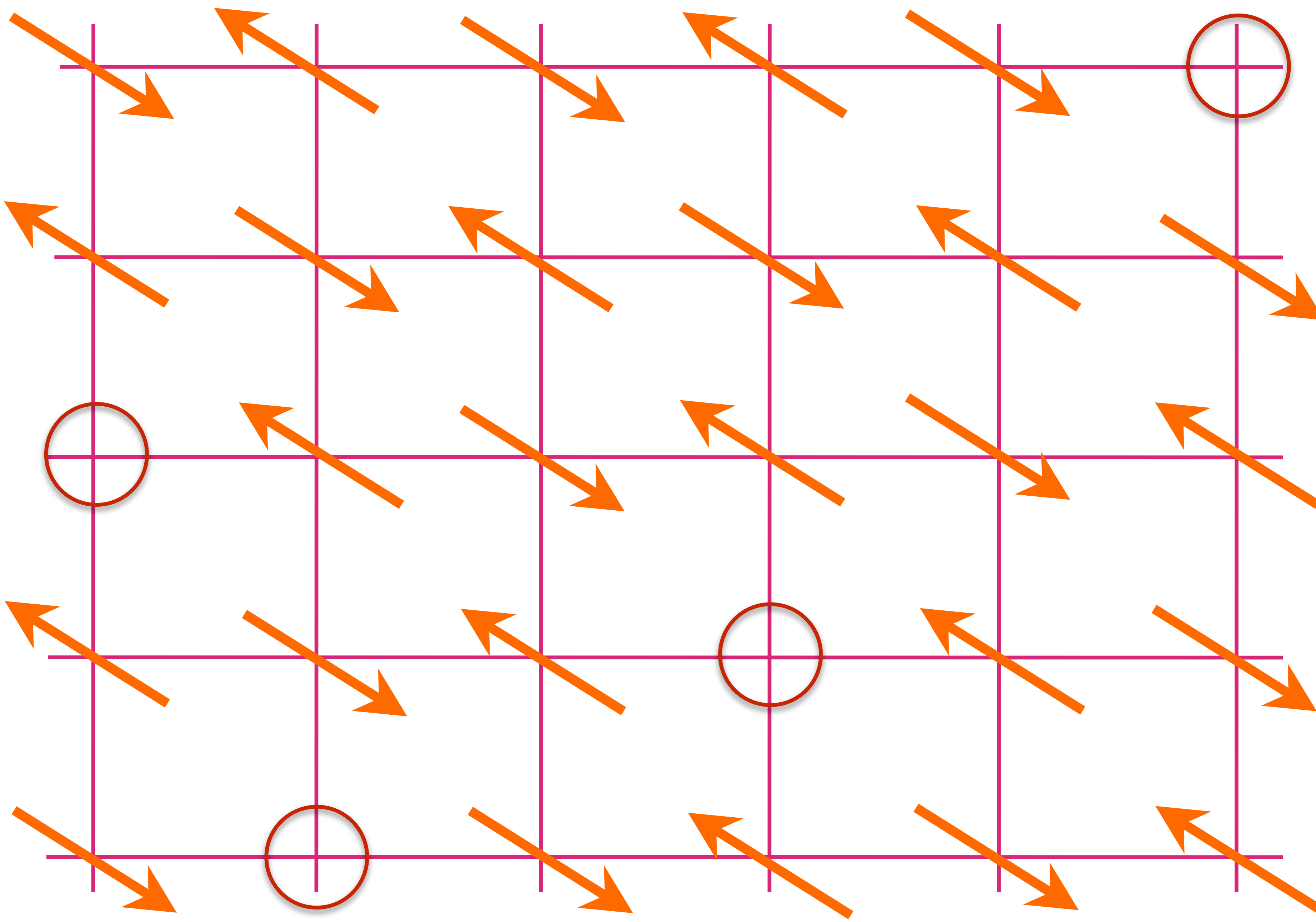


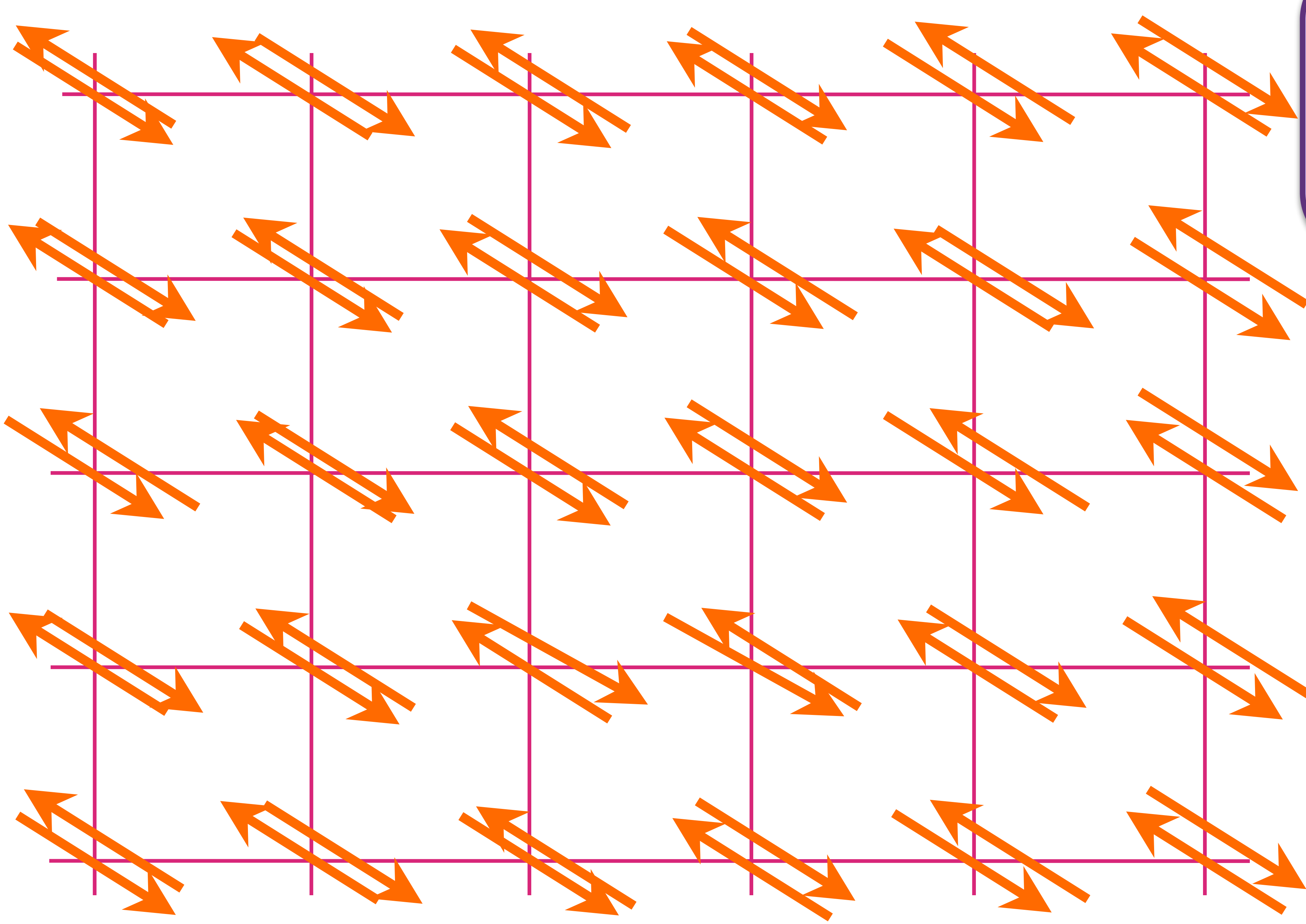
Figure: K. Fujita and J. C. Seamus Davis



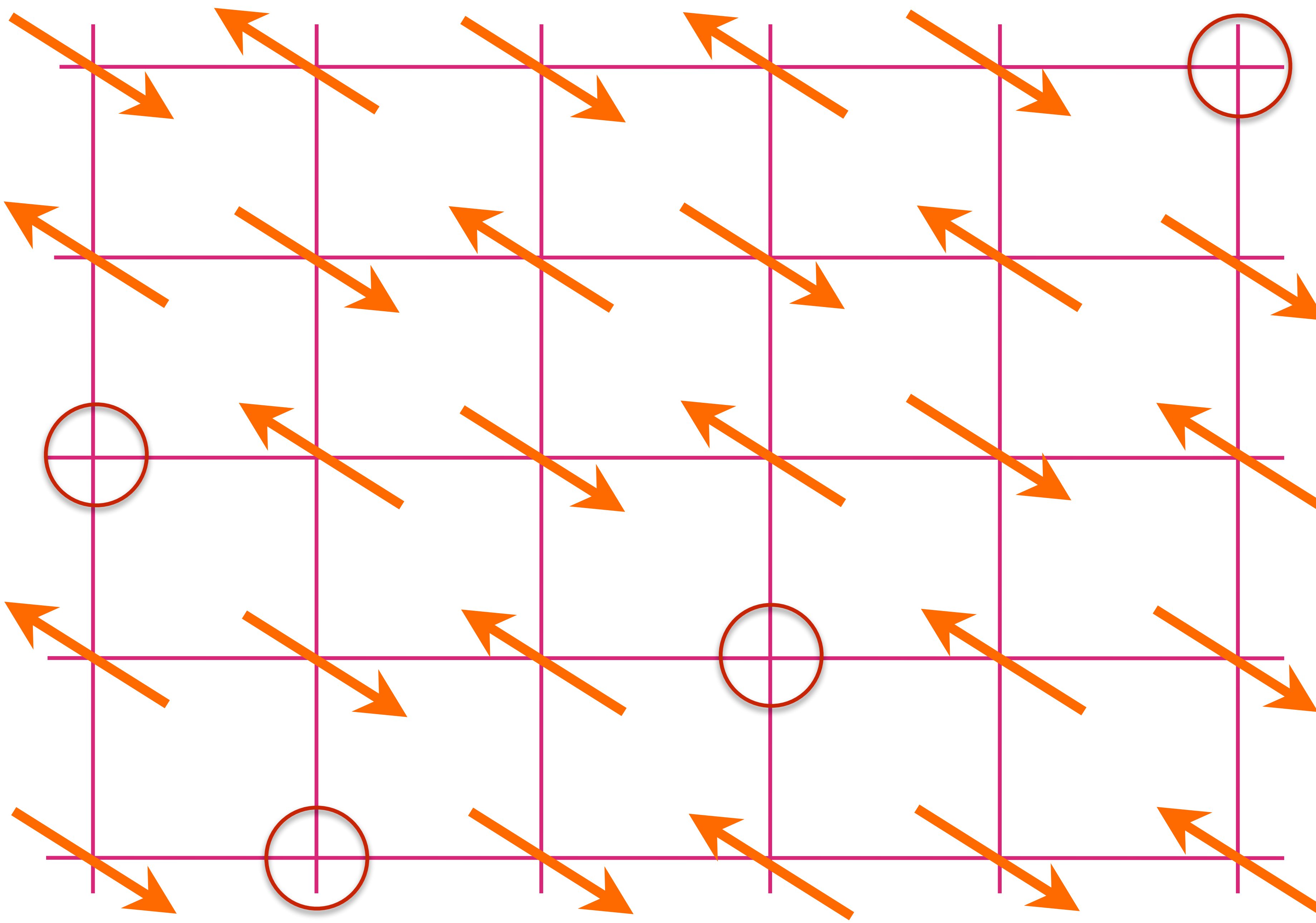
"Undoped" insulating anti-ferromagnet



Anti-
ferromagnet
with p mobile
holes
per square

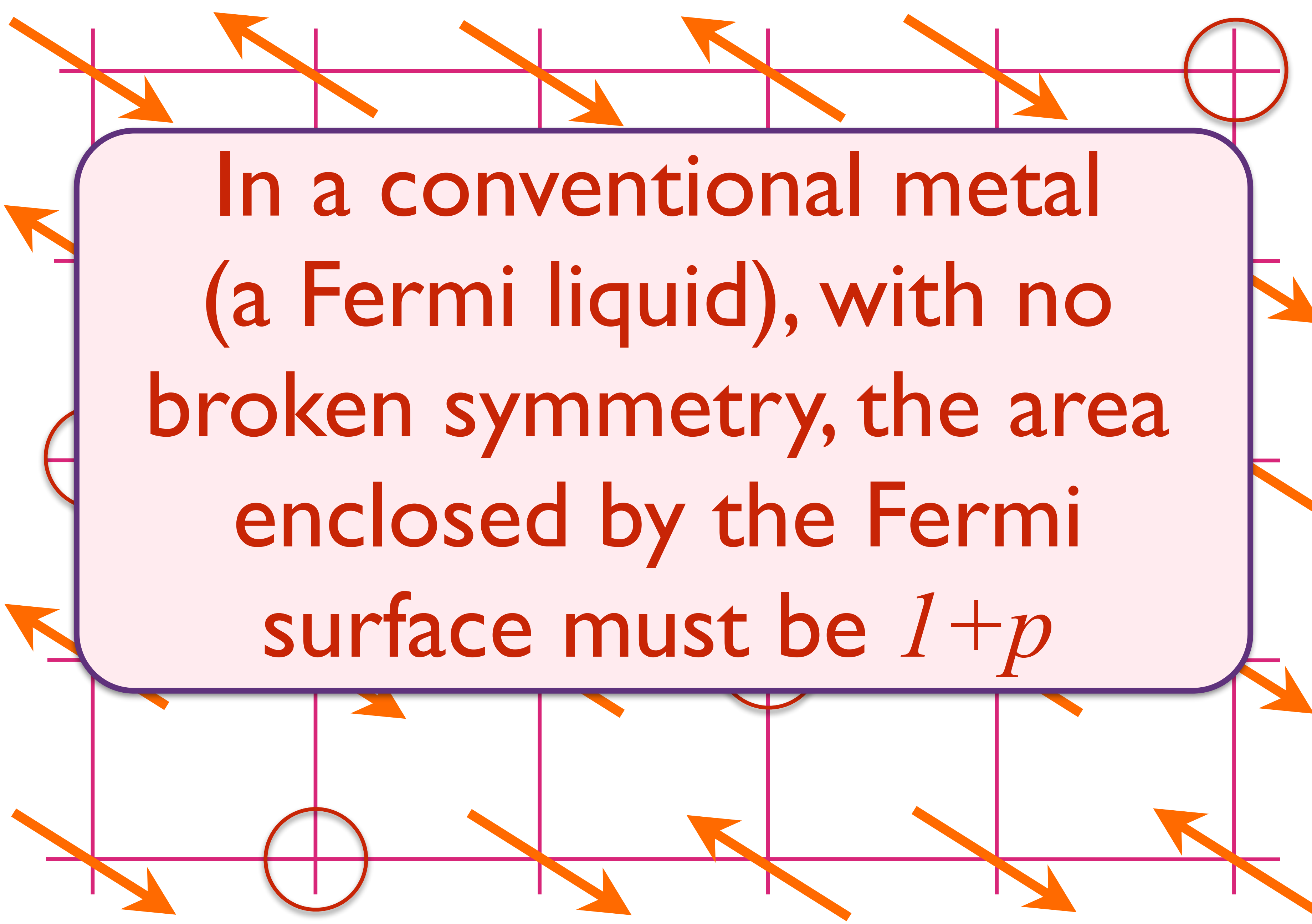


Filled Band



Anti-ferromagnet with p mobile holes per square

But relative to the band insulator, there are $1+p$ holes per square

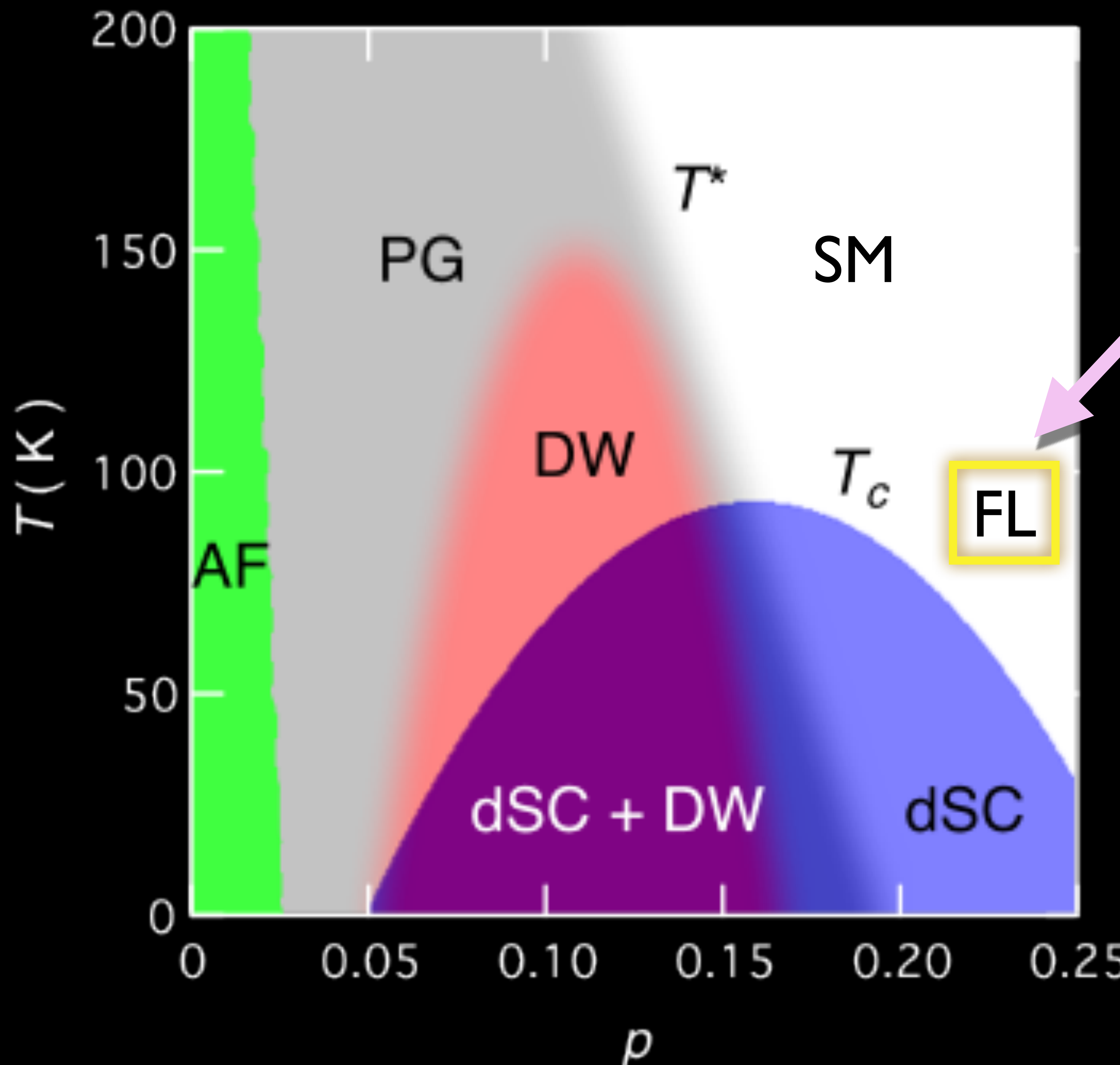


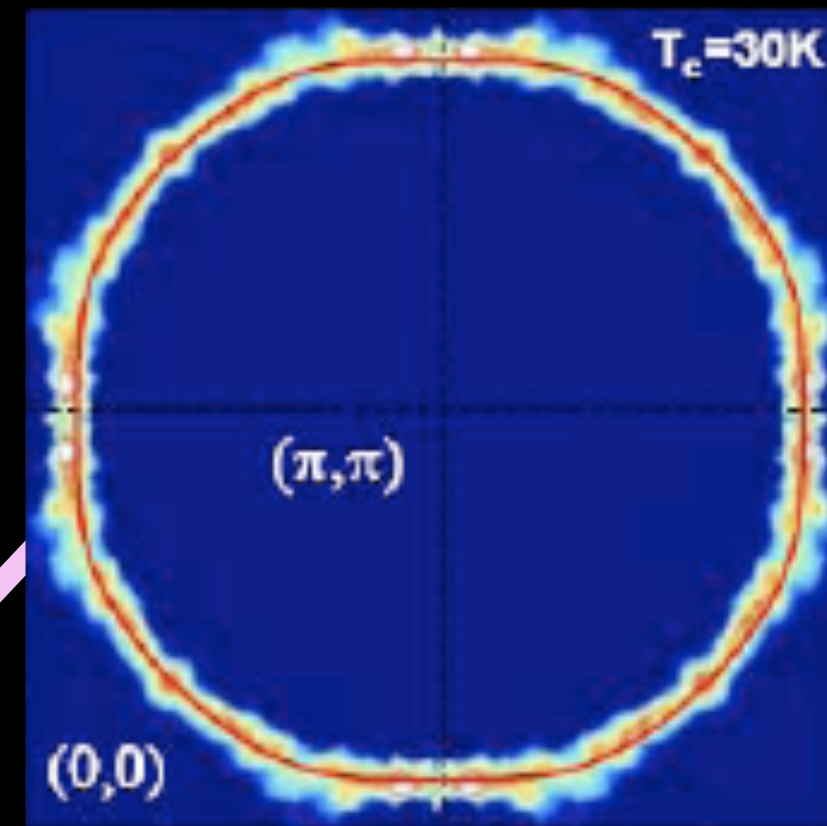
In a conventional metal (a Fermi liquid), with no broken symmetry, the area enclosed by the Fermi surface must be $1+p$

Anti-ferromagnet with p mobile holes per square

But relative to the band insulator, there are $1+p$ holes per square

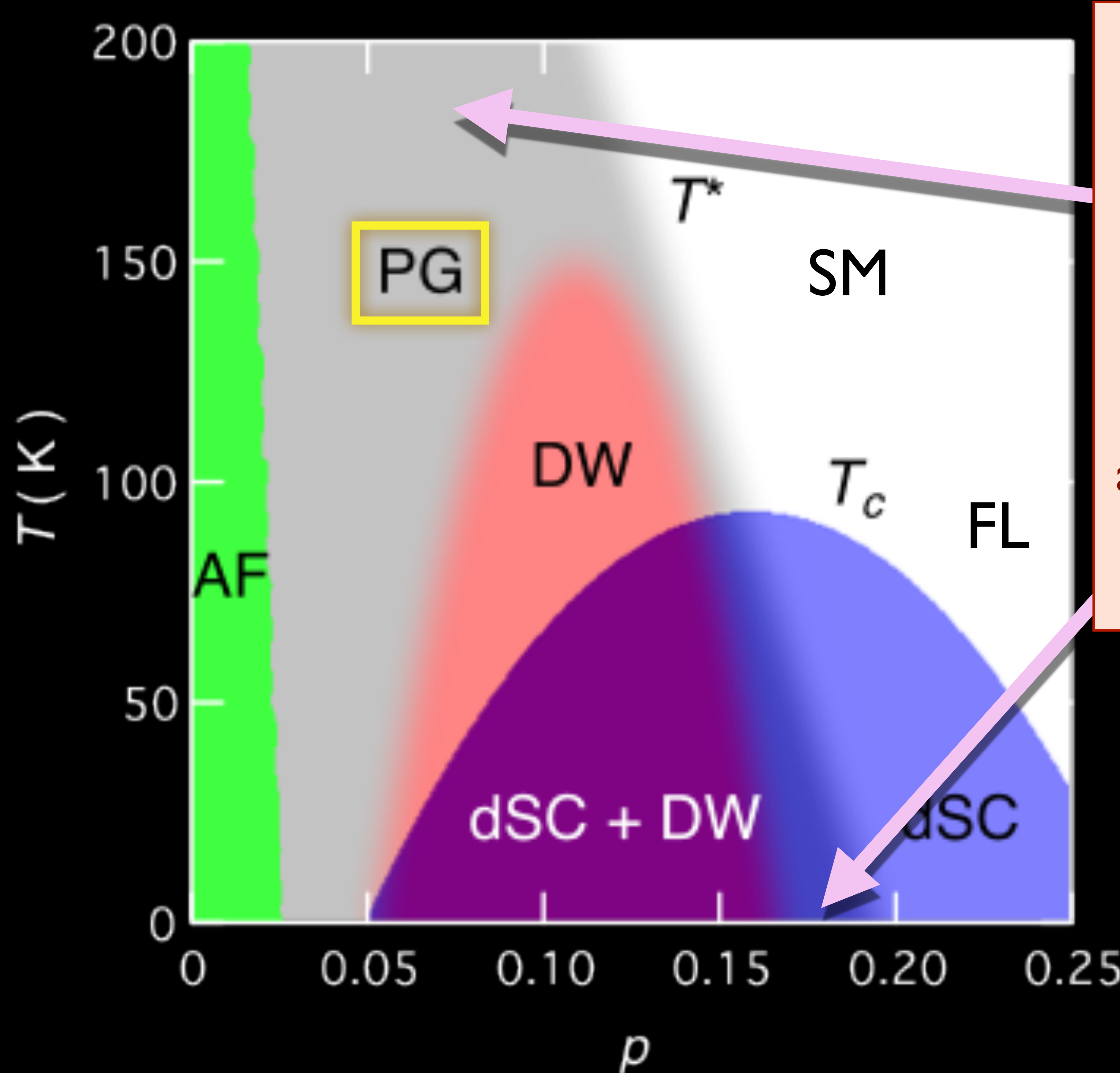
M. Platé, J. D. F. Mottershead, I. S. Elfimov, D. C. Peets, Ruixing Liang, D. A. Bonn, W. N. Hardy, S. Chiuzbaian, M. Falub, M. Shi, L. Patthey, and A. Damascelli, Phys. Rev. Lett. **95**, 077001 (2005)





A conventional metal:
the Fermi liquid with Fermi surface of size *1+p*

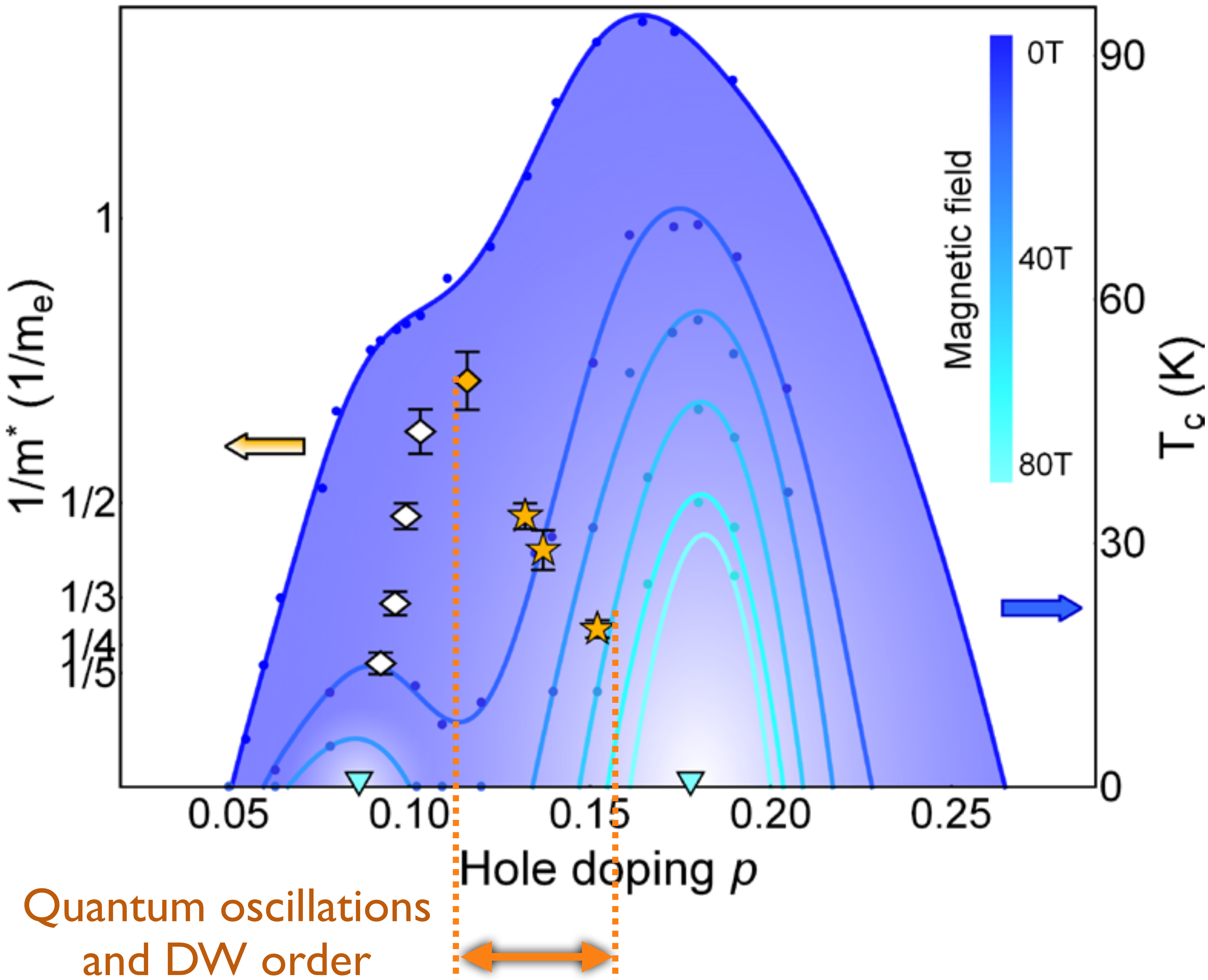
S. Badoux, W. Tabis, F. Laliberté, G. Grissonnanche, B. Vignolle, D. Vignolles, J. Béard, D.A. Bonn, W.N. Hardy, R. Liang, N. Doiron-Leyraud, L. Taillefer, and C. Proust, Nature **531**, 210 (2016).



Pseudogap metal at low p

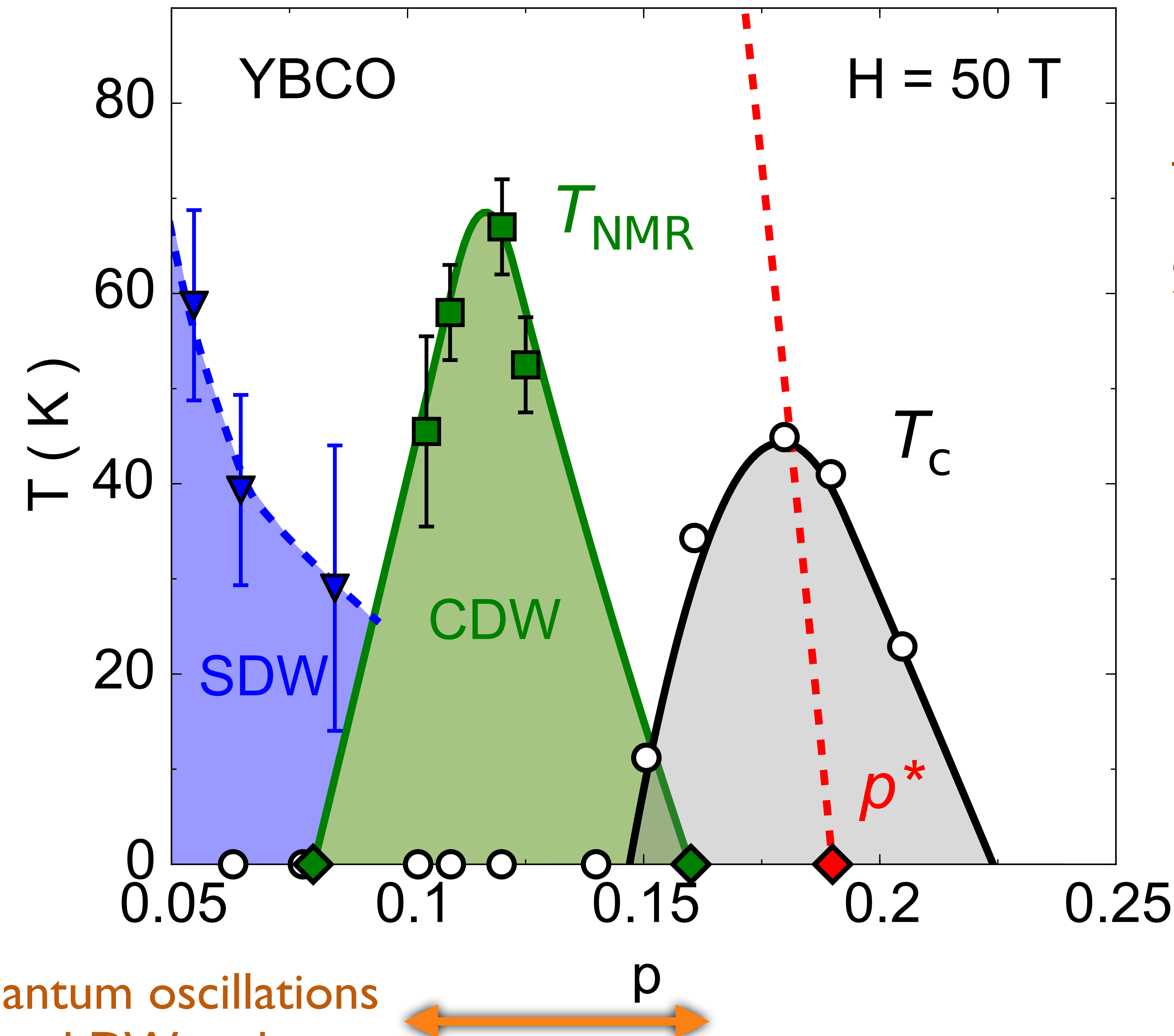
Many indications that this metal behaves like a Fermi liquid, but with Fermi surface size p and *not* $1+p$.

Phase diagram in a high magnetic field



B. J. Ramshaw, S. E. Sebastian, R. D. McDonald, James Day, B. S. Tan, Z. Zhu, J. B. Betts, Ruixing Liang, D. A. Bonn, W. N. Hardy, N. Harrison, Science **348**, 6232 (2105).

Phase diagram in a high magnetic field

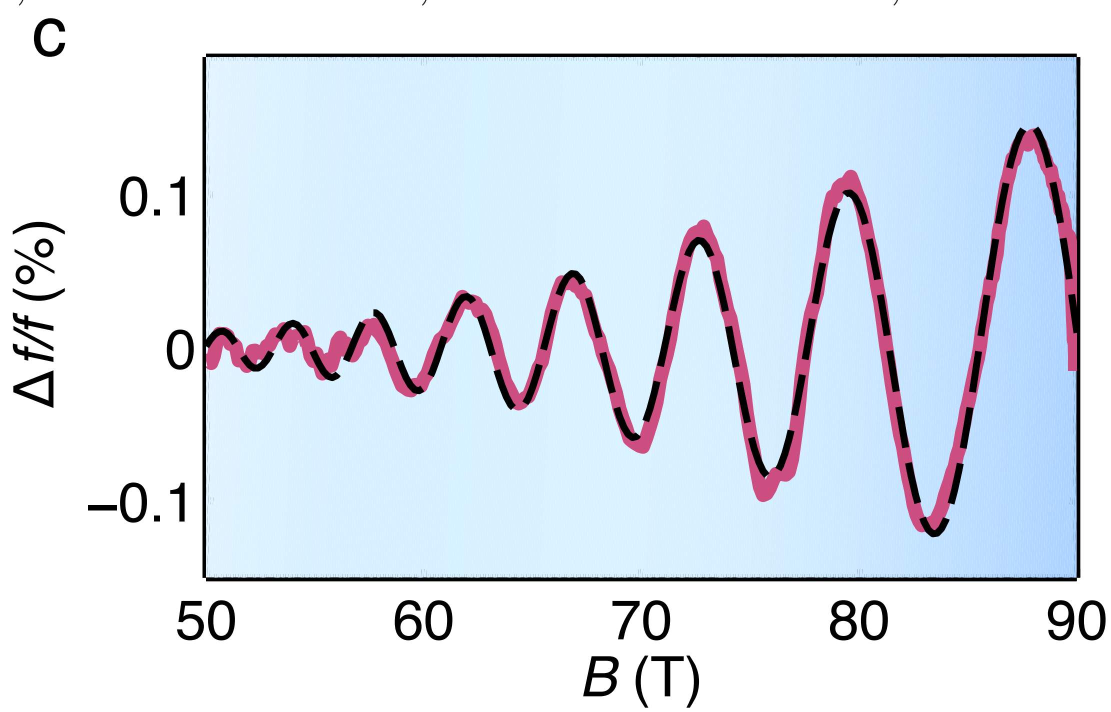


Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)

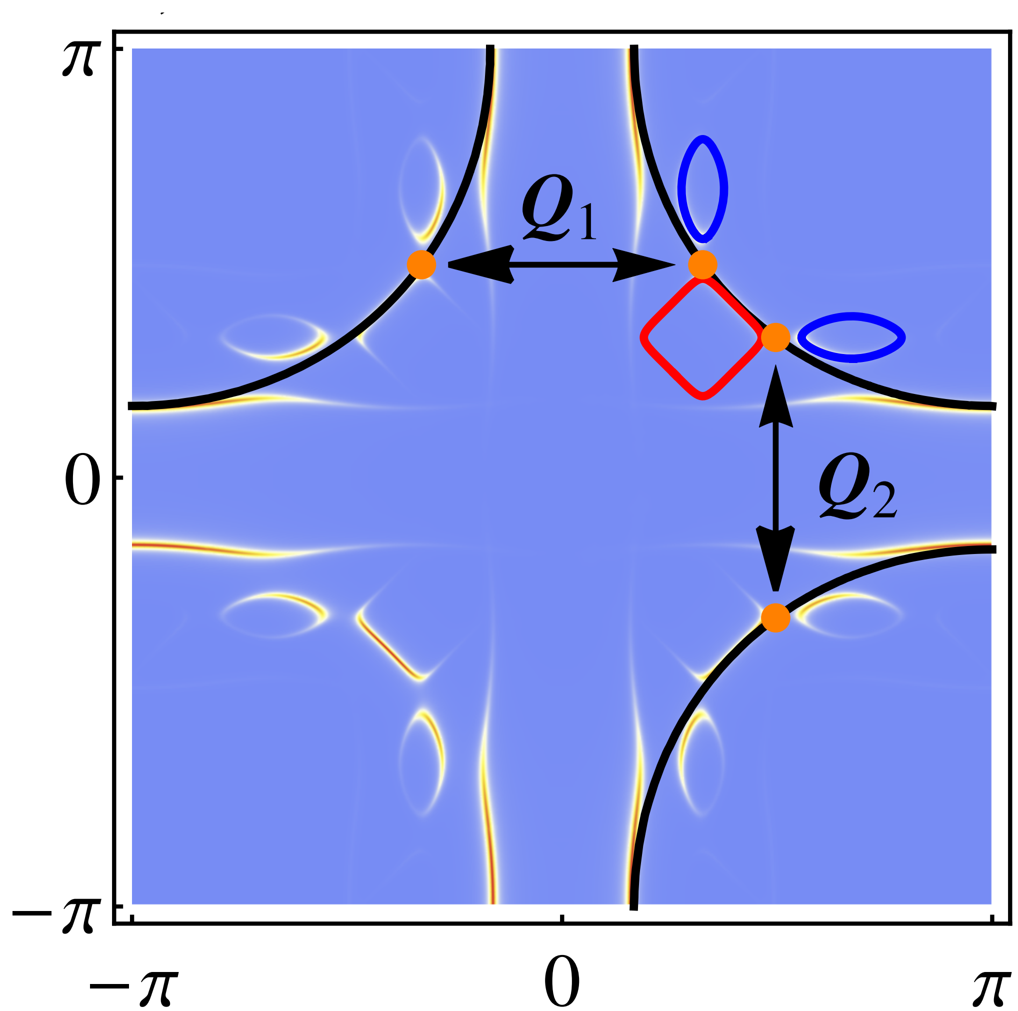
Quantum oscillations and DW order

Single reconstructed Fermi surface pocket in an underdoped single layer cuprate superconductor

M. K. Chan,[1,2,*] N. Harrison,[1,*] R. D. McDonald,[1] B. J. Ramshaw,[1] K. A. Modic,[1] N. Barišić,[3,2] and M. Greven[2]



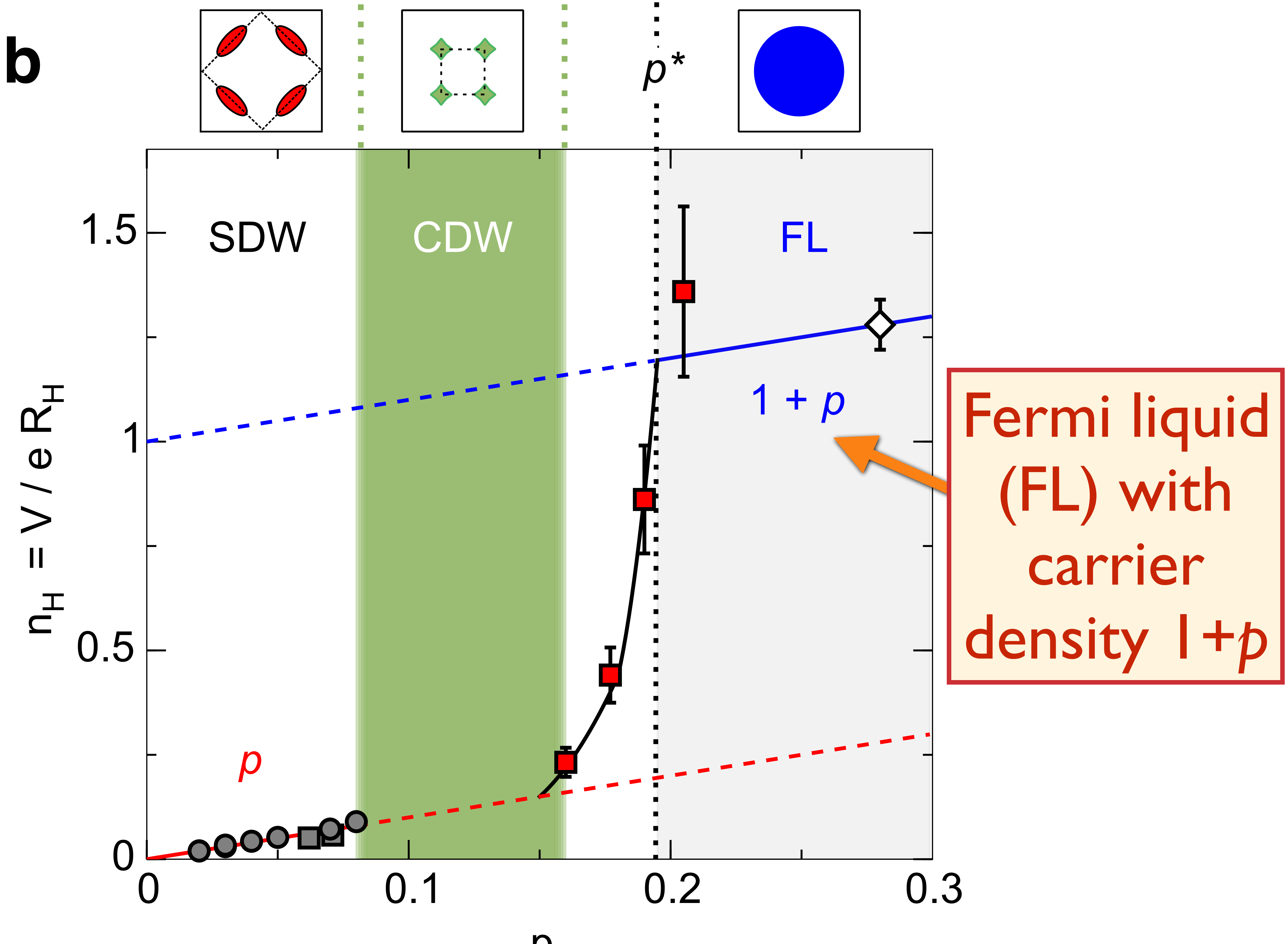
Nature Communications 7, 12244 (2016)



Reconstructing the large Fermi surface with a CDW does not lead to a single electron pocket.

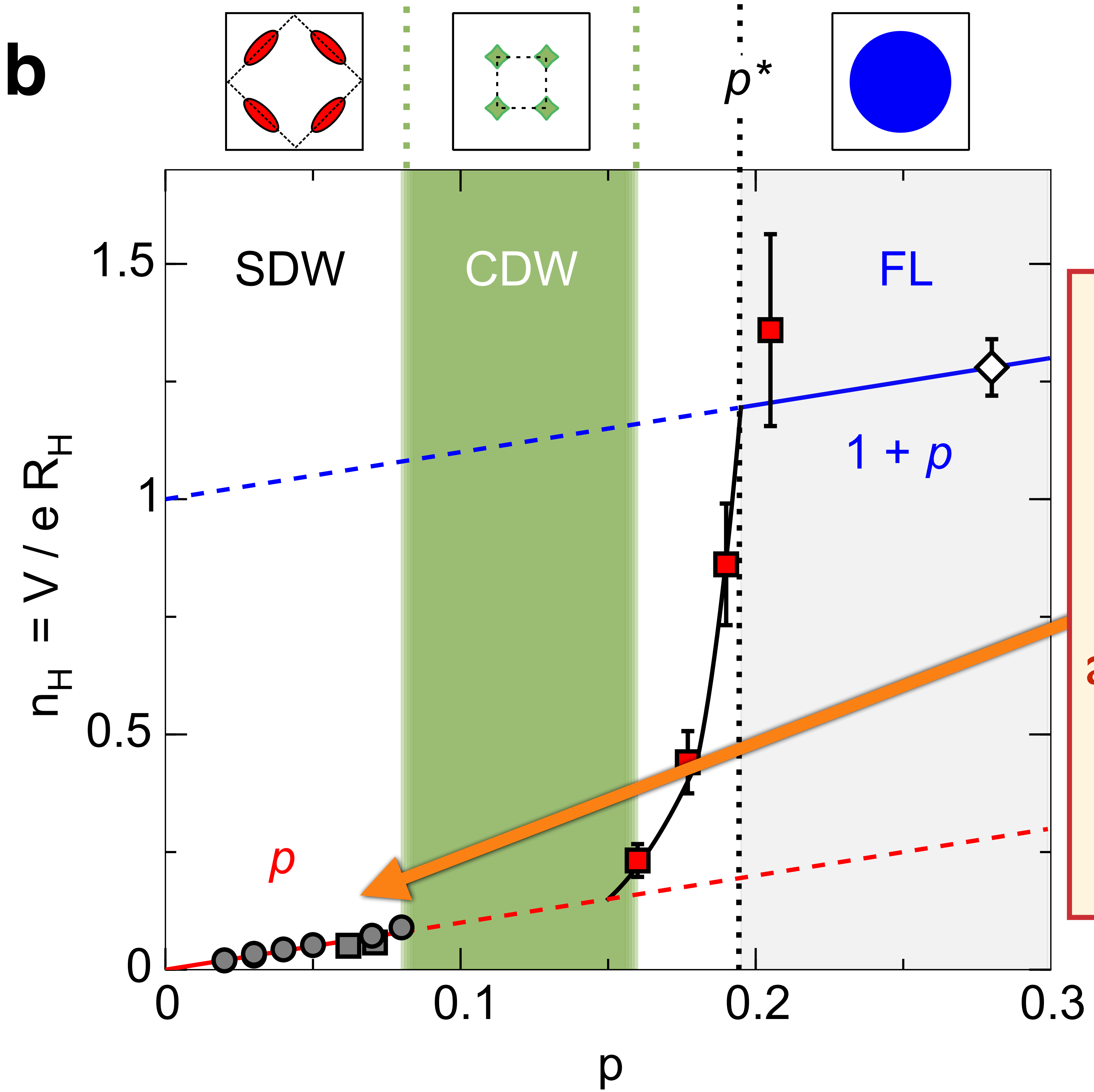
A. Allais, D. Chowdhury, and S. Sachdev, Nature Communications **5**, 5771 (2014)

Hall effect measurements in YBCO



Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)

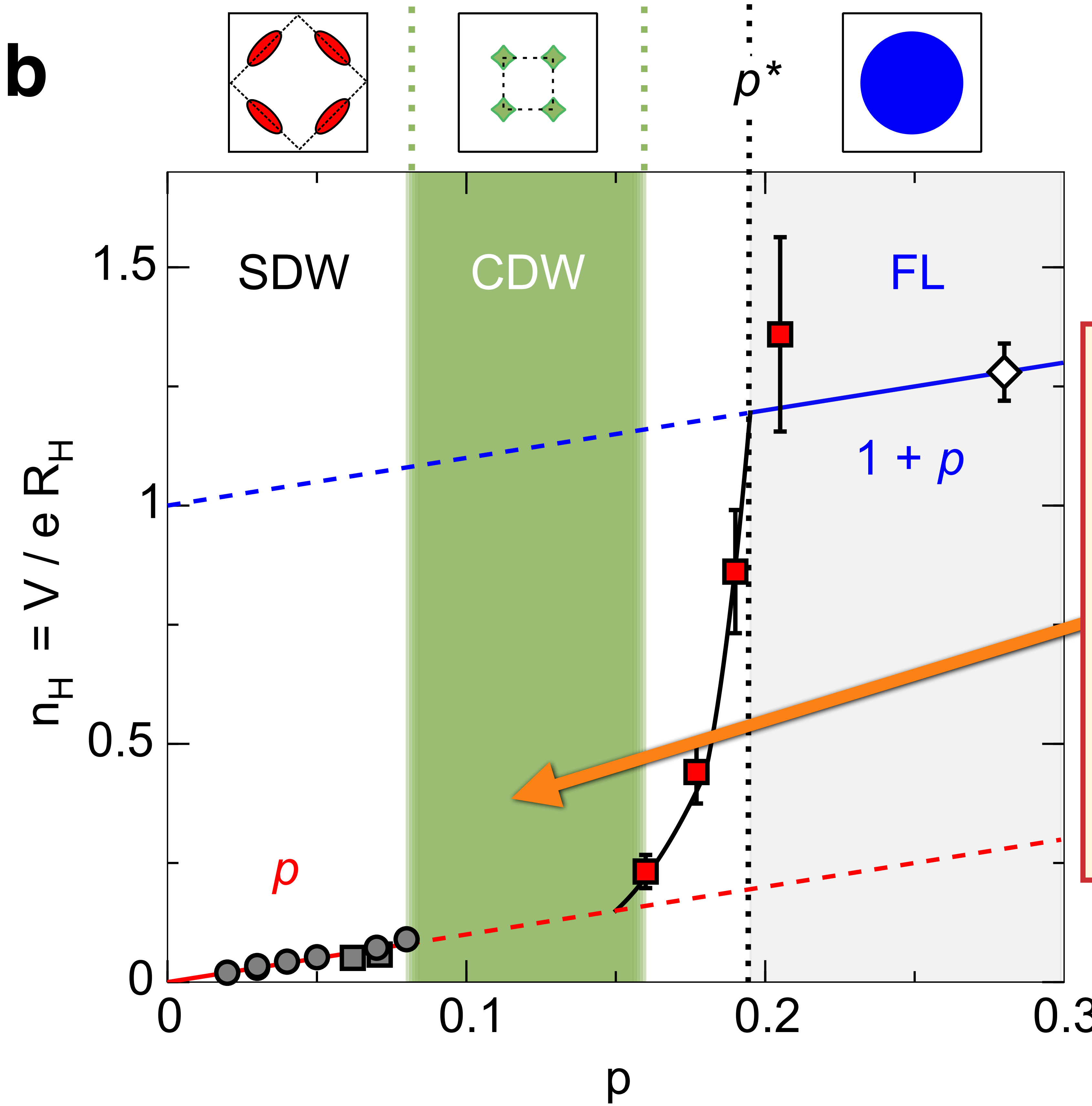
Hall effect measurements in YBCO



Spin density wave (SDW) breaks translational invariance, and the Fermi liquid then has carrier density *p*

Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)

Hall effect measurements in YBCO



Charge density wave (CDW) leads to complex Fermi surface reconstruction and negative Hall resistance

Badoux, Proust, Taillefer et al., Nature **531**, 210 (2016)

A topological phase transition at $p = p^*$?

- The high-field, shrinking superconducting dome around $p = p^*$ suggests the presence of a phase transition at $p = p^*$.

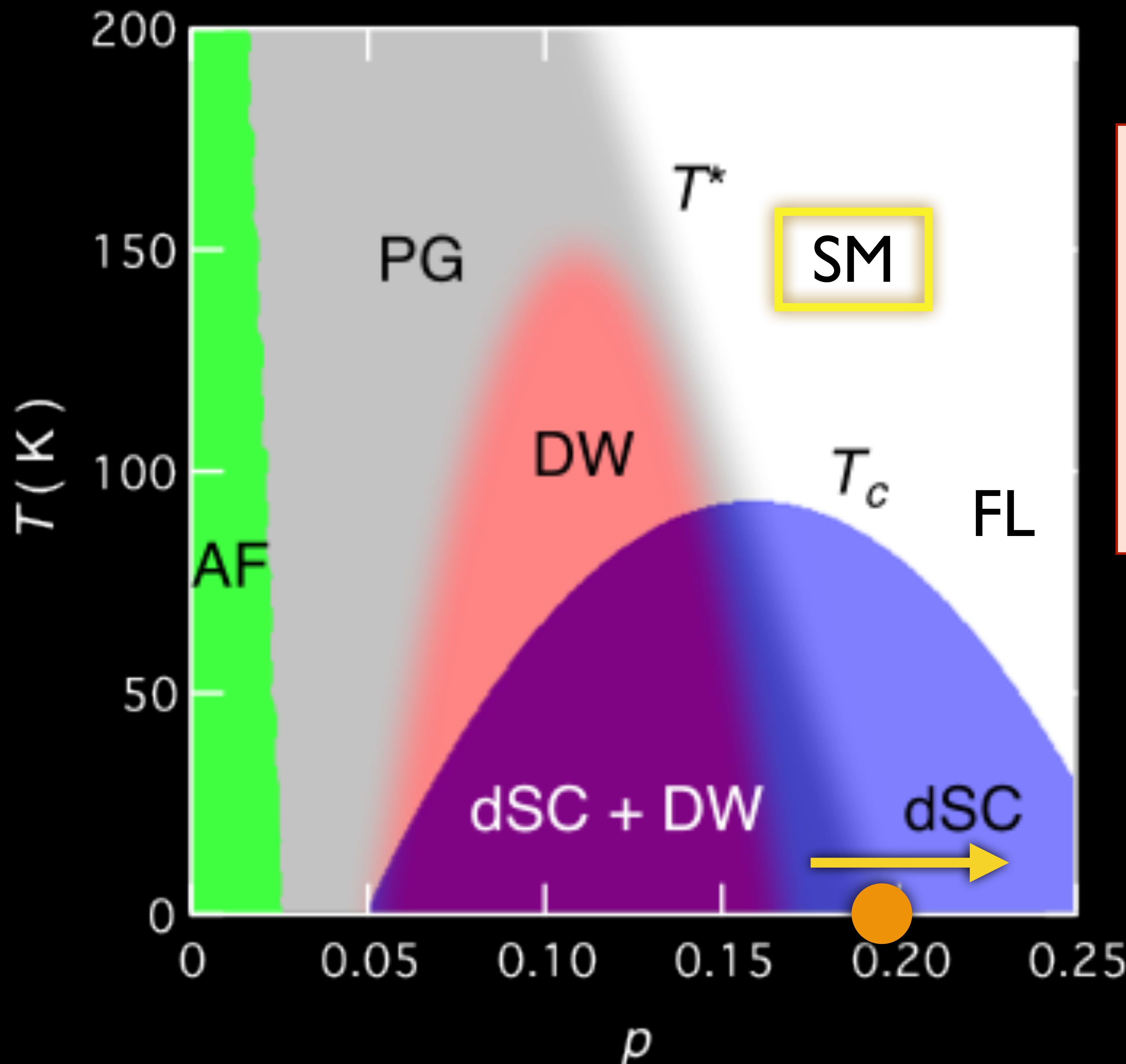
- The X-ray, NMR, and Hall effect data indicate that the CDW disappears at a $p = p_{\rm cdw}$ with $p_{\rm cdw} < p^*$.

- The single electron pocket cannot appear by CDW-induced Fermi surface reconstruction on a large Fermi surface of size $1 + p$. So the parent metallic state of the CDW must be a distinct metallic state.

- The Hall effect for $p_{\rm cdw} < p < p^*$ also supports the presence of a metallic state distinct from the large Fermi surface of size $1 + p$.

- No other translational symmetry breaking has been observed for $p_{\rm cdw} < p < p^*$.

- Taken together, the above facts are strong evidence for a "topological phase transition" between two distinct metallic states at $p = p^*$.



Topological
`deconfined’
phase
transition

1. Introduction to spin liquids and topological order
2. Topological order and phase transitions in a model of bosons on the square lattice
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4. Model of a topological phase transition for the cuprates

The Hubbard Model

$$H = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha}$$

$t_{ij} \rightarrow$ "hopping". $U \rightarrow$ local repulsion, $\mu \rightarrow$ chemical potential

Spin index $\alpha = \uparrow, \downarrow$

$$n_{i\alpha} = c_{i\alpha}^\dagger c_{i\alpha}$$

$$c_{i\alpha}^\dagger c_{j\beta} + c_{j\beta} c_{i\alpha}^\dagger = \delta_{ij}\delta_{\alpha\beta}$$
$$c_{i\alpha} c_{j\beta} + c_{j\beta} c_{i\alpha} = 0$$

Will study on the square lattice

Fermi surface+antiferromagnetism

We use the operator equation (valid on each site i):

$$U\left(n_\uparrow - \frac{1}{2}\right)\left(n_\downarrow - \frac{1}{2}\right) = -\frac{2U}{3}\vec{S}^2 + \frac{U}{4} \qquad (1)$$

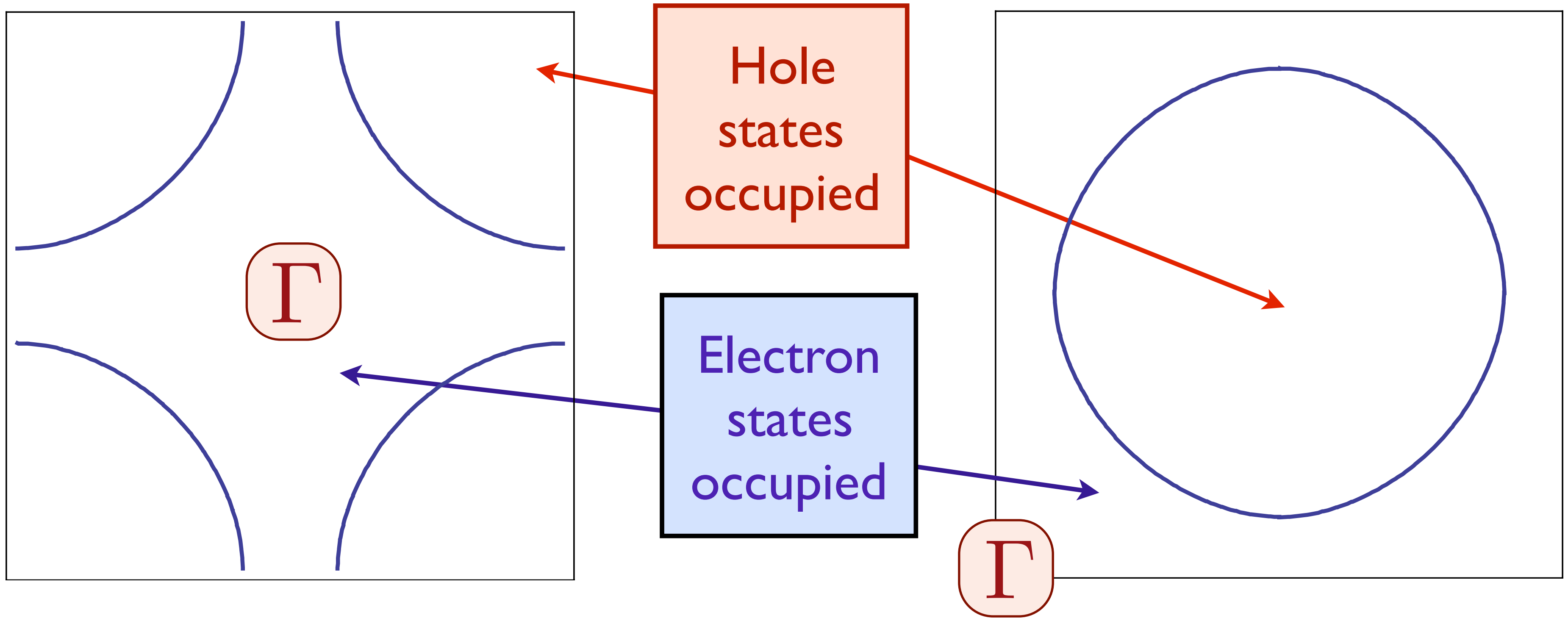
Then we decouple the interaction via

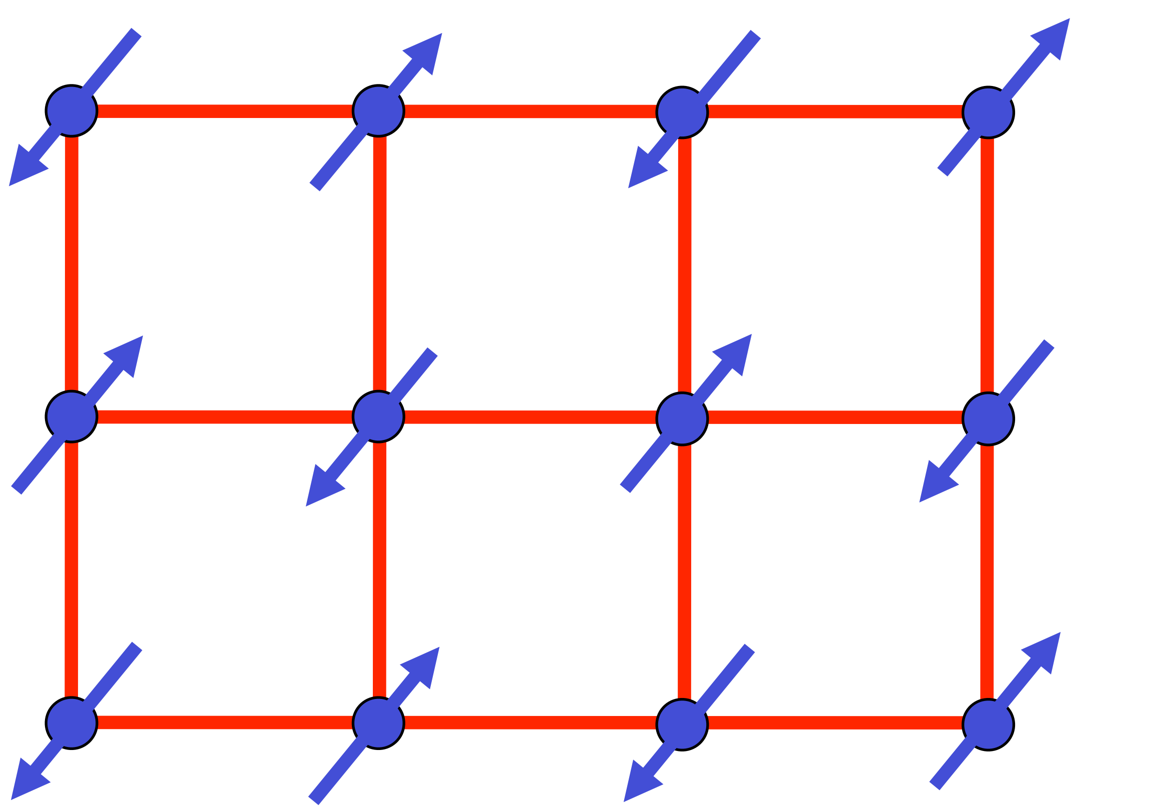
$$\exp\left(\frac{2U}{3}\sum_i \int d\tau \vec{S}_i^2\right) = \int \mathcal{D}\vec{J}_i(\tau)\exp\left(-\sum_i \int d\tau \left[\frac{3}{8U}\vec{J}_i^2 - \vec{J}_i\vec{S}_i\right]\right) \qquad (2)$$

We now integrate out the fermions, and look for the saddle point of the resulting effective action for $\vec{J}_i$. At the saddle-point we find that the lowest energy is achieved when the vector has opposite orientations on the A and B sublattices. Anticipating this, we look for a continuum limit in terms of a field $\vec{\varphi}_i$ where

$$\vec{J}_i = \vec{\varphi}_i \, e^{i\mathbf{K}\cdot\mathbf{r}_i} \qquad (3)$$

Fermi surface+antiferromagnetism





The electron spin polarization obeys

$$\left\langle \vec{S}(\mathbf{r},\tau) \right\rangle = \vec{\varphi}(\mathbf{r},\tau) e^{i\mathbf{K}\cdot\mathbf{r}}$$

where $\mathbf{K}$ is the ordering wavevector.

Fermi surface+antiferromagnetism

In this manner, we obtain the "spin-fermion" model

$$\mathcal{Z} = \int \mathcal{D}c_\alpha \mathcal{D}\vec{\varphi} \exp\left(-\mathcal{S}\right)$$

$$\mathcal{S} = \int d\tau \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} \left(\frac{\partial}{\partial\tau} - \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

$$- \lambda \int d\tau \sum_i c^\dagger_{i\alpha} \vec{\varphi}_i \cdot \vec{\sigma}_{\alpha\beta} c_{i\beta} e^{i\mathbf{K}\cdot\mathbf{r}_i}$$

$$+ \int d\tau d^2 r \left[\frac{1}{2} \left(\boldsymbol{\nabla}_r \vec{\varphi} \right)^2 + \frac{1}{2} \left(\partial_\tau \vec{\varphi} \right)^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right]$$

Fermi surface+antiferromagnetism

In the Hamiltonian form (ignoring, for now, the time dependence of $\vec{\varphi}$), the coupling between $\vec{\varphi}$ and the electrons takes the form

$$H_{\rm sdw} = \lambda \sum_{\mathbf{k},\mathbf{q},\alpha,\beta} \vec{\varphi}_{\mathbf{q}} \cdot c^{\dagger}_{\mathbf{k}+\mathbf{q},\alpha} \vec{\sigma}_{\alpha\beta} c_{\mathbf{k}+\mathbf{K},\beta}$$

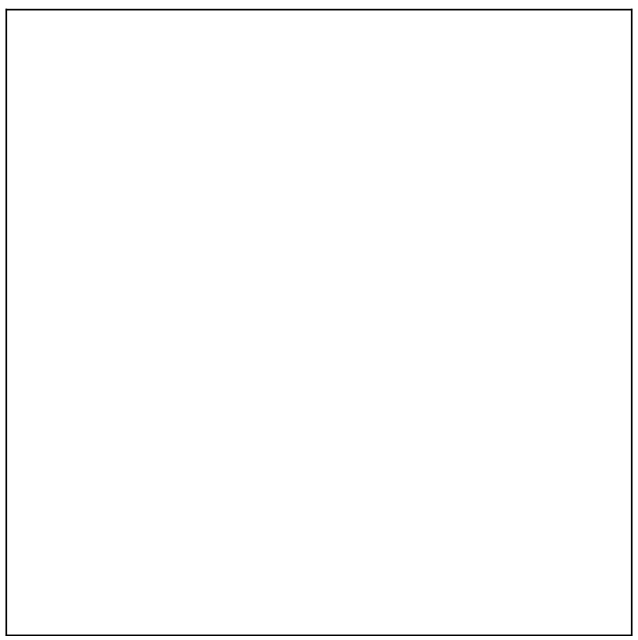
where $\vec{\sigma}$ are the Pauli matrices, the boson momentum $\mathbf{q}$ is small, while the fermion momenum $\mathbf{k}$ extends over the entire Brillouin zone. In the antiferromagnetically ordered state, we may take $\vec{\varphi} \propto (0,0,1)$, and the electron dispersions obtained by diagonalizing $H_0 + H_{\rm sdw}$ are

$$E_{\mathbf{k}\pm} = \frac{\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}+\mathbf{K}}}{2} \pm \sqrt{\left(\frac{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{K}}}{2}\right)^2 + \lambda^2 |\vec{\varphi}|^2}$$

This leads to the Fermi surfaces shown in the following slides as a function of increasing $|\vec{\varphi}|$.

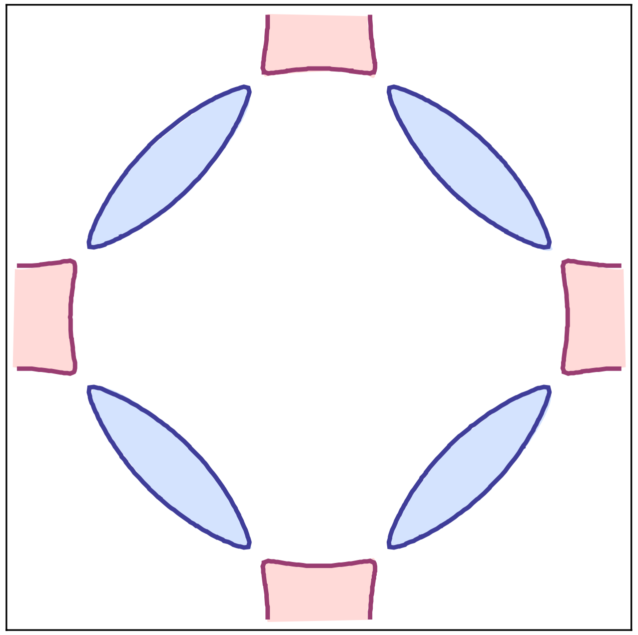
Square lattice Hubbard model at p=0

$\langle \vec{\varphi} \rangle \neq 0$ and large



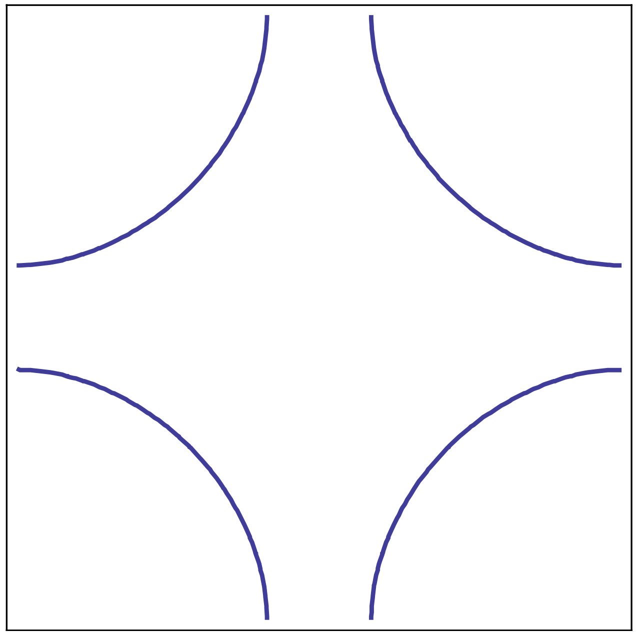
Insulator

$\langle \vec{\varphi} \rangle \neq 0$ and small



Metal with electron and hole pockets

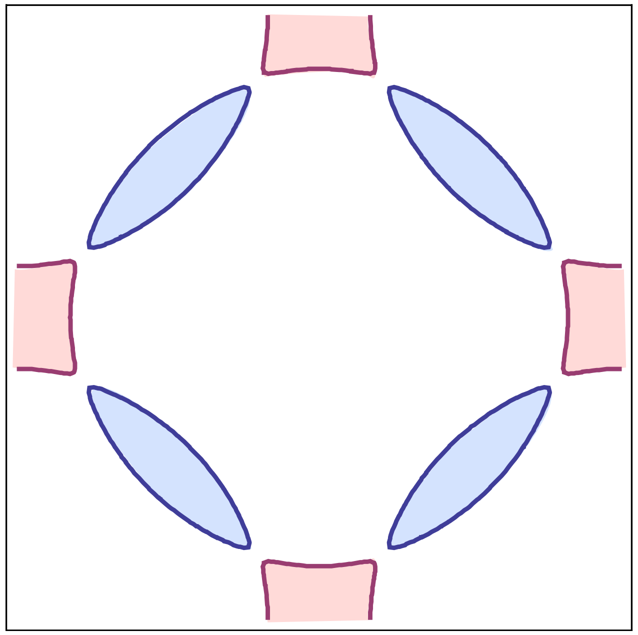
$\langle \vec{\varphi} \rangle = 0$



Metal with "large" Fermi surface

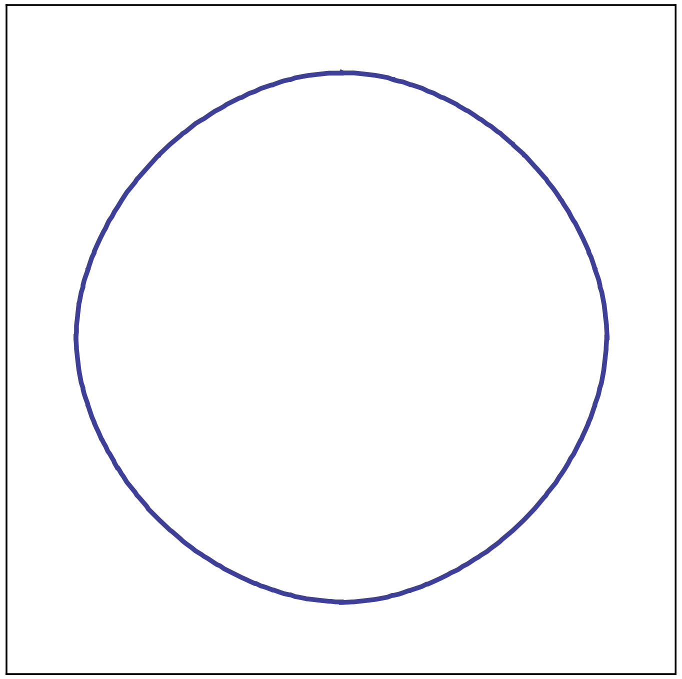
s

Separating onset of SDW order and Fermi surface reconstruction



$\langle \vec{\varphi} \rangle \neq 0$

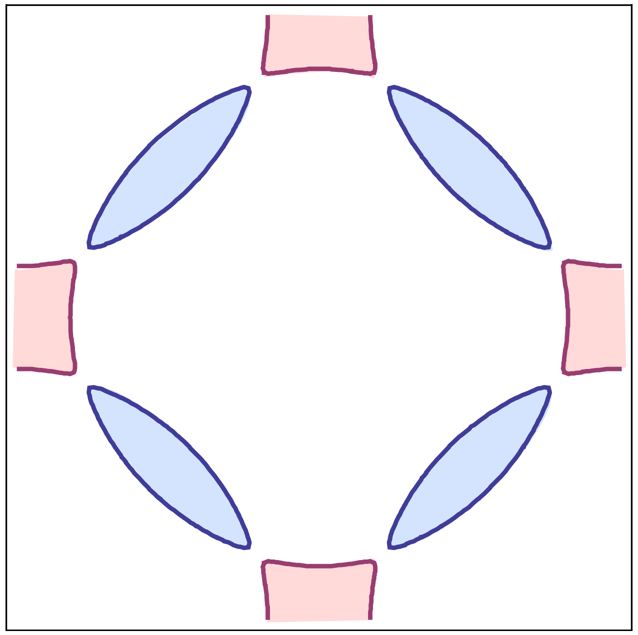
Metal with electron and hole pockets



$\langle \vec{\varphi} \rangle = 0$

Metal with "large" Fermi surface

Separating onset of SDW order and Fermi surface reconstruction

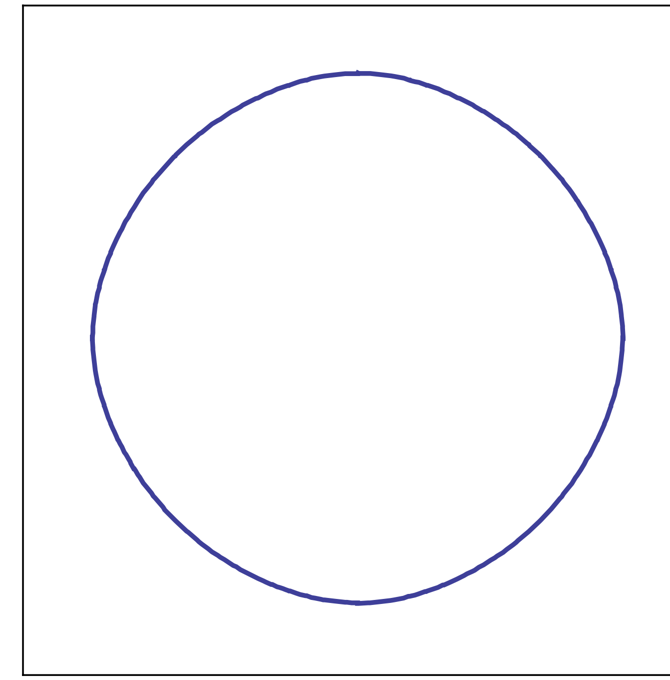


$\langle \vec{\varphi} \rangle \neq 0$

Metal with electron and hole pockets

Electron and/or hole Fermi pockets form in "local" SDW order, but quantum fluctuations destroy long-range SDW order. Such a phase must have topological order.

$\langle \vec{\varphi} \rangle = 0$



$\langle \vec{\varphi} \rangle = 0$

Metal with "large" Fermi surface

We replace the action for the SDW order parameter $\vec{\varphi}$

$$\int d\tau d^2r \left[\frac{1}{2} \left(\boldsymbol{\nabla}_r \vec{\varphi} \right)^2 + \frac{1}{2} \left(\partial_\tau \vec{\varphi} \right)^2 + \frac{s}{2} \vec{\varphi}^2 + \frac{u}{4} \vec{\varphi}^4 \right]$$

for the easy-plane case with $\varphi_x + i\varphi_y \sim e^{i\theta}$, by a new action which couples it to a $\mathbb{Z}_2$ gauge theory, just as in the previous boson model

$$\begin{aligned} H &= -t \sum_{r,\alpha=x,y} \tau^z_{r\alpha} \cos(\Delta_\alpha \theta_r) \\ &\quad - K \sum_{\square} \tau^z \tau^z \tau^z \tau^z \\ &\quad - g \sum_{r,\alpha=x,y} \tau^x_{r\alpha} \end{aligned}$$

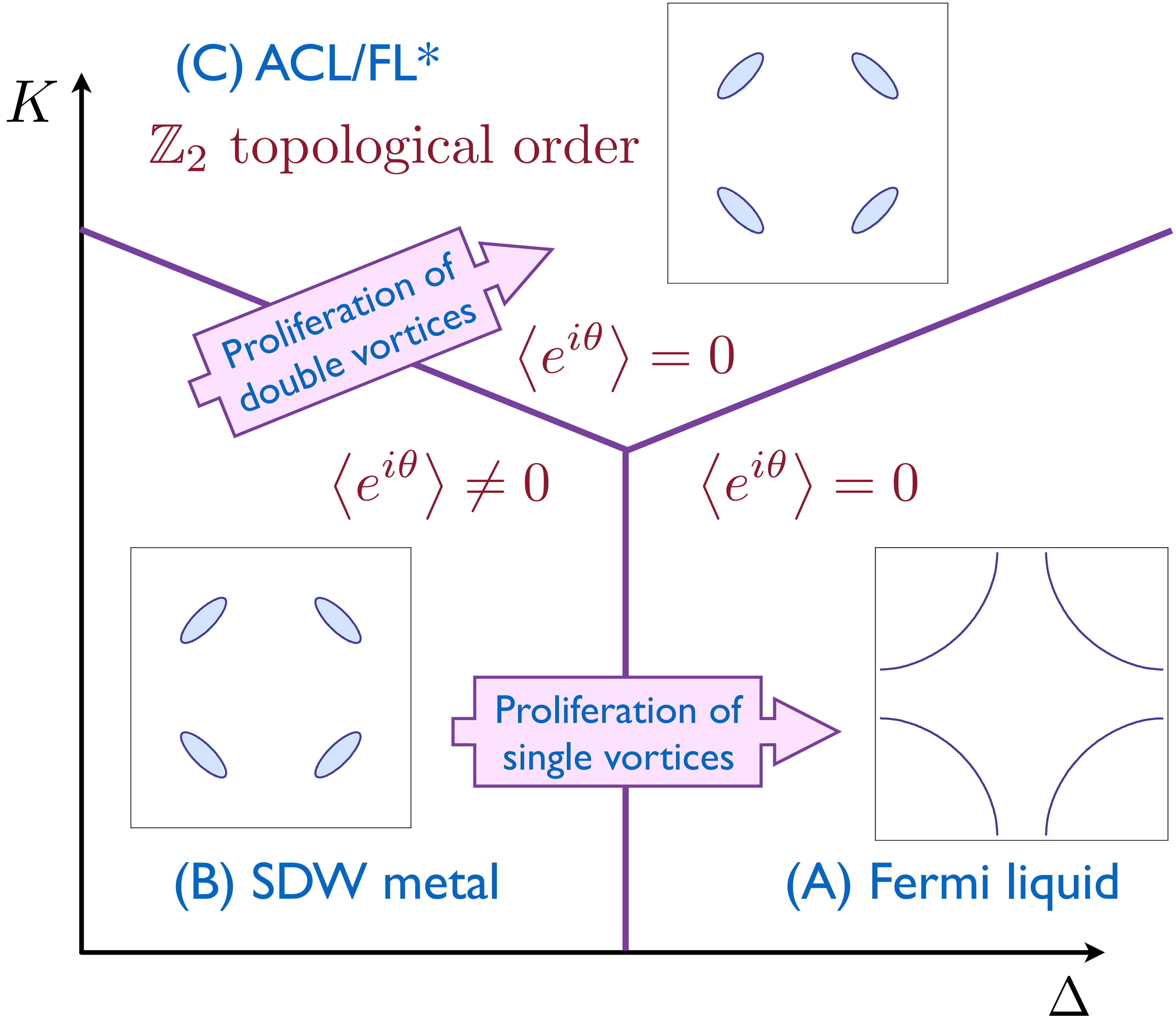
Consider the phase with $\mathbb{Z}_2$ topological order. In this state it is useful to perform a rotation about the z axis in spin space by introducing the fermion operators

$$\psi_+ = e^{i\theta/2} c_\uparrow \quad , \quad \psi_- = e^{-i\theta/2} c_\downarrow .$$

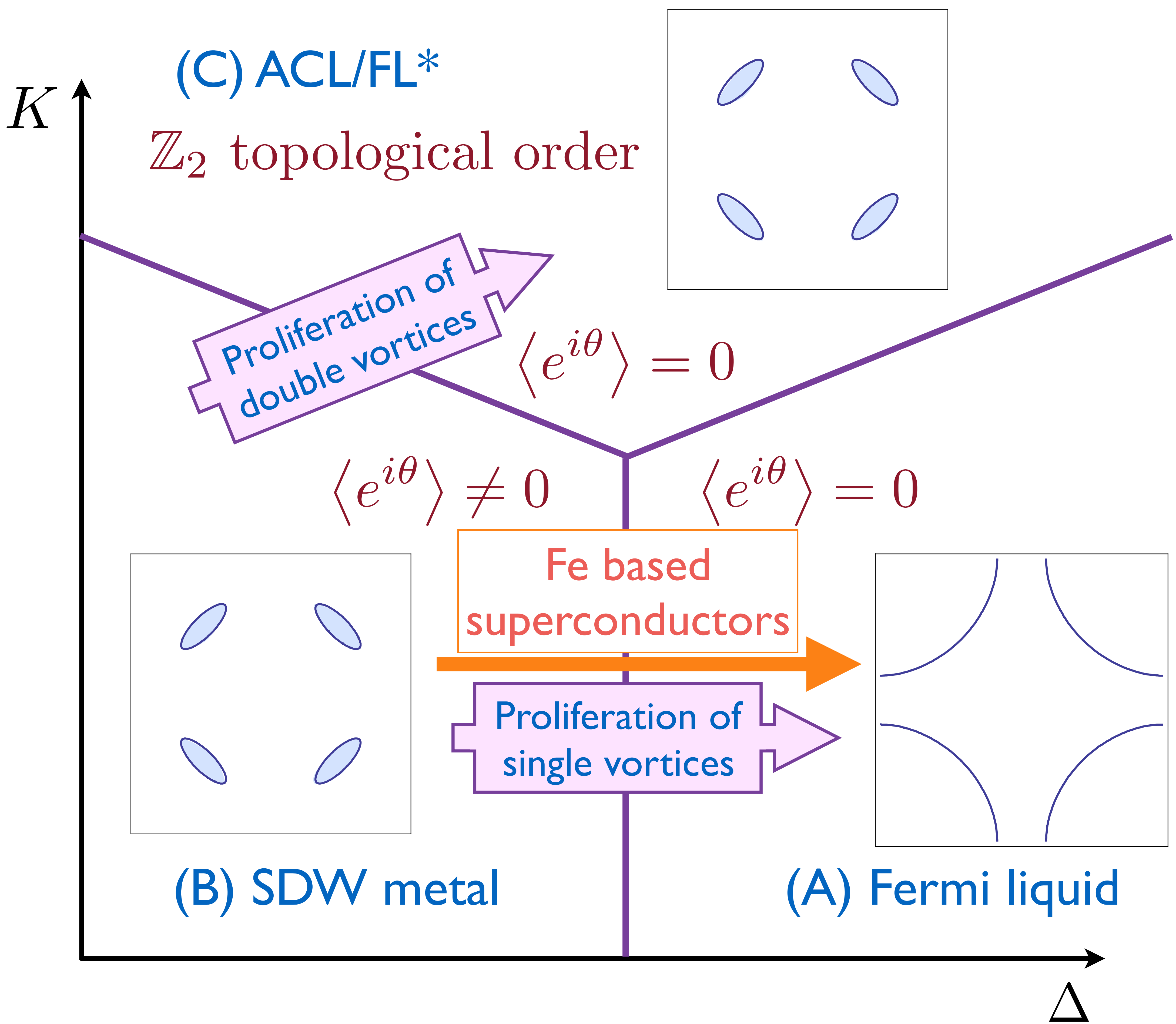
Then the Yukawa coupling, H_Y, takes a simple form independent of the orientation of the XY order:

$$H_Y = -\lambda \sum_i \eta_i \left[\psi_{i+}^\dagger \psi_{i-} + \psi_{i-}^\dagger \psi_{i+} \right] .$$

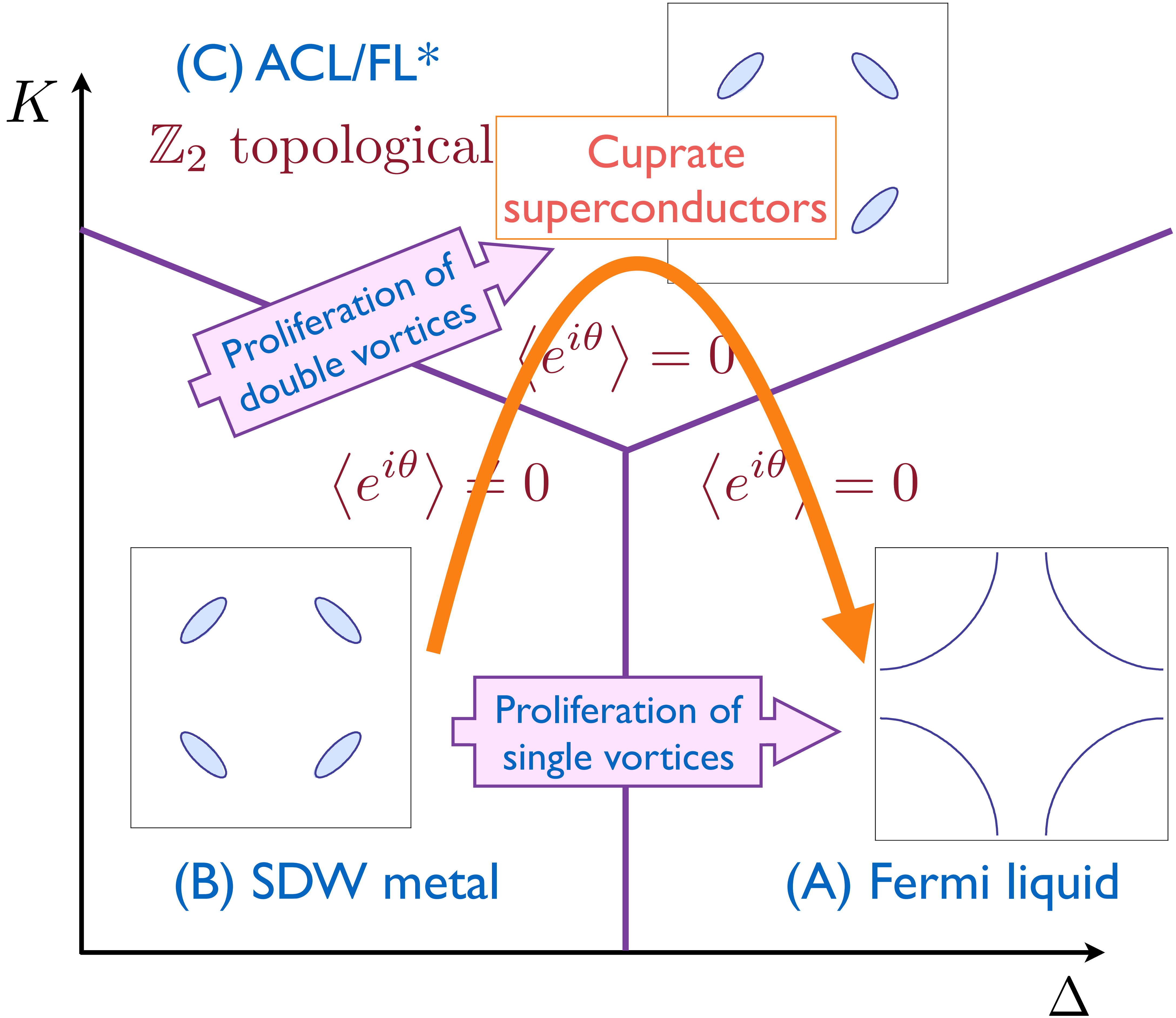
In other words, the $\psi_\pm$ fermions move in the presence of a spacetime-independent XY order, even though the actual orientation of the XY order rotates from point to point. Moreover, from the electron hopping term in H_c, we can obtain an effective hopping $Z_{ij} t_{ij} (\psi_{i+}^\dagger \psi_{j+} + \psi_{i-}^\dagger \psi_{j-})$ where $Z_{ij} = \langle e^{\pm i(\theta_i - \theta_j)/2} \rangle$ is a renormalization factor of order unity. So it appears we can realize a situation in which the $\psi_\pm$ fermions are approximately free, and their observation of constant XY order implies that they will form small pocket Fermi surfaces (or be fully gapped at $p = 0$).



(C) ACL/FL*
$\mathbb{Z}_2$ topological order
K
Proliferation of double vortices
$\langle e^{i\theta} \rangle = 0$
$\langle e^{i\theta} \rangle \neq 0$
$\langle e^{i\theta} \rangle = 0$
Proliferation of single vortices
(B) SDW metal
(A) Fermi liquid
Δ



(C) ACL/FL*
$\mathbb{Z}_2$ topological order
K
Proliferation of double vortices
$\langle e^{i\theta} \rangle = 0$
$\langle e^{i\theta} \rangle \neq 0$
$\langle e^{i\theta} \rangle = 0$
Fe based superconductors
Proliferation of single vortices
(B) SDW metal
(A) Fermi liquid
Δ


(C) ACL/FL*
$\mathbb{Z}_2$ topological
K
Cuprate
superconductors
Proliferation of
double vortices
$\langle e^{i\theta} \rangle = 0$
$\langle e^{i\theta} \rangle \neq 0$
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Proliferation of
single vortices
(B) SDW metal
(A) Fermi liquid
Δ