Equilibrium dynamics of entangled states near quantum critical points

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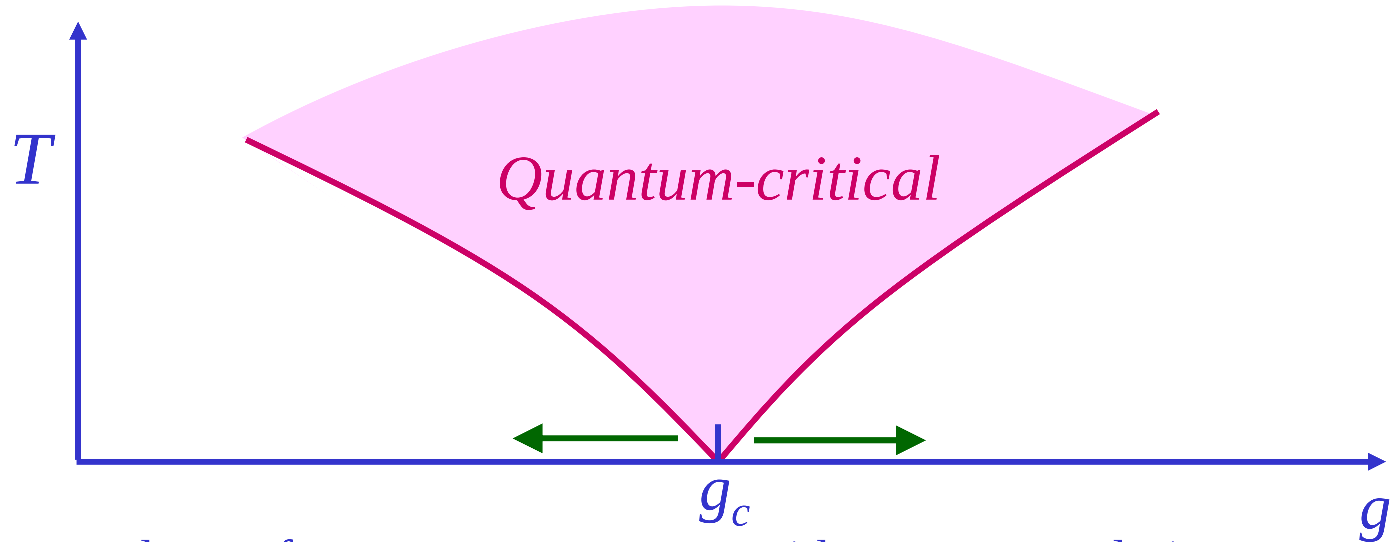
Kedar Damle (TIFR, Mumbai)
Subir Sachdev (Harvard)
Peter Young (UCSC)

Talk online at http://sachdev.physics.harvard.edu

Why study quantum phase transitions ?



• Theory for a quantum system with strong correlations: describe phases on either side of g_c by expanding in deviation from the quantum critical point.

• Critical point is (often) a novel (entangled) state of matter without quasiparticle excitations

• Critical excitations control dynamics in the wide *quantum-critical* region at non-zero temperatures.

Important property of ground state at $g=g_c$:
temporal and spatial scale invariance;
characteristic energy scale at other values of g: $\Delta \sim \left| g - g_c \right|^{z\nu}$

I. Quantum Ising Chain

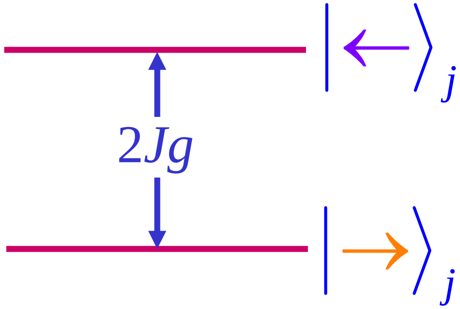
Degrees of freedom: $j = 1 \ldots N$ qubits, N "large"

$$\left|\uparrow\right\rangle_j , \left|\downarrow\right\rangle_j$$

or $$\left|\rightarrow\right\rangle_j = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_j + \left|\downarrow\right\rangle_j\right) , \quad \left|\leftarrow\right\rangle_j = \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_j - \left|\downarrow\right\rangle_j\right)$$

Hamiltonian of decoupled qubits:

$$H_0 = -Jg\sum_j \sigma_j^x$$



Coupling between qubits:

$$H_1 = -J \sum_j \underbrace{\sigma_j^z \sigma_{j+1}^z}$$

$$\Big(|\rightarrow\rangle_j \langle\leftarrow| + |\leftarrow\rangle_j \langle\rightarrow| \Big) \Big(|\rightarrow\rangle_{j+1} \langle\leftarrow| + |\leftarrow\rangle_{j+1} \langle\rightarrow| \Big)$$

Prefers neighboring qubits

are *either* $|\uparrow\rangle_j |\uparrow\rangle_{j+1}$ *or* $|\downarrow\rangle_j |\downarrow\rangle_{j+1}$

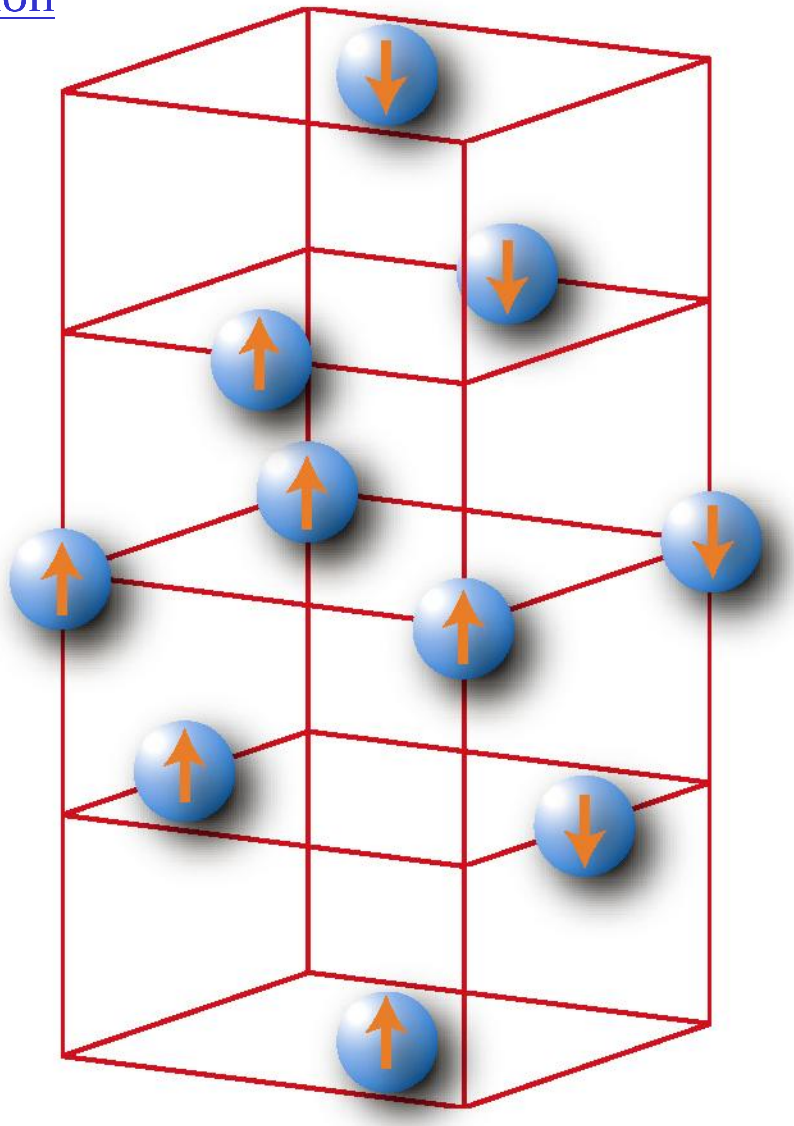
(not entangled)

Full Hamiltonian

$$H = H_0 + H_1 = -J \sum_j \left(g\sigma_j^x + \sigma_j^z \sigma_{j+1}^z \right)$$

leads to entangled states at *g* of order unity

Experimental realization



$LiHoF_4$

Weakly-coupled qubits $(g \gg 1)$

Ground state:

$$|G\rangle = |\cdots \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \cdots\rangle$$

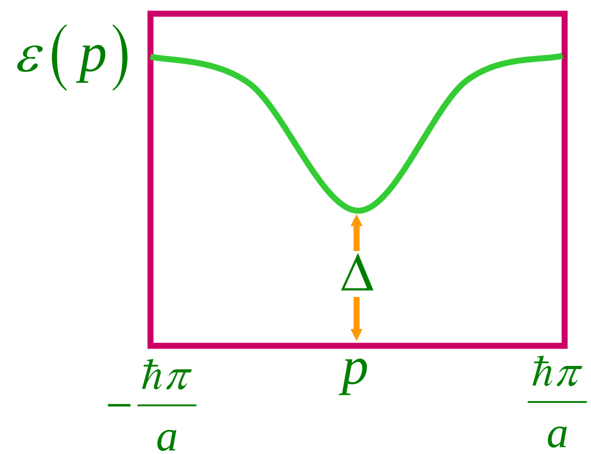
$$-\frac{1}{2g}|\cdots \rightarrow\rightarrow\rightarrow\rightarrow\leftarrow\leftarrow\rightarrow\rightarrow\rightarrow\rightarrow \cdots\rangle - \cdots$$

Lowest excited states:

$$|\ell_j\rangle = |\cdots \rightarrow\rightarrow\rightarrow\rightarrow\leftarrow_j\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \cdots\rangle + \cdots$$

Coupling between qubits creates "flipped-spin" *quasiparticle* states at momentum p

$$|p\rangle = \sum_j e^{ipx_j/\hbar}|\ell_j\rangle$$



Excitation energy $\varepsilon(p) = \Delta + 4J\sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^{-1}\right)$

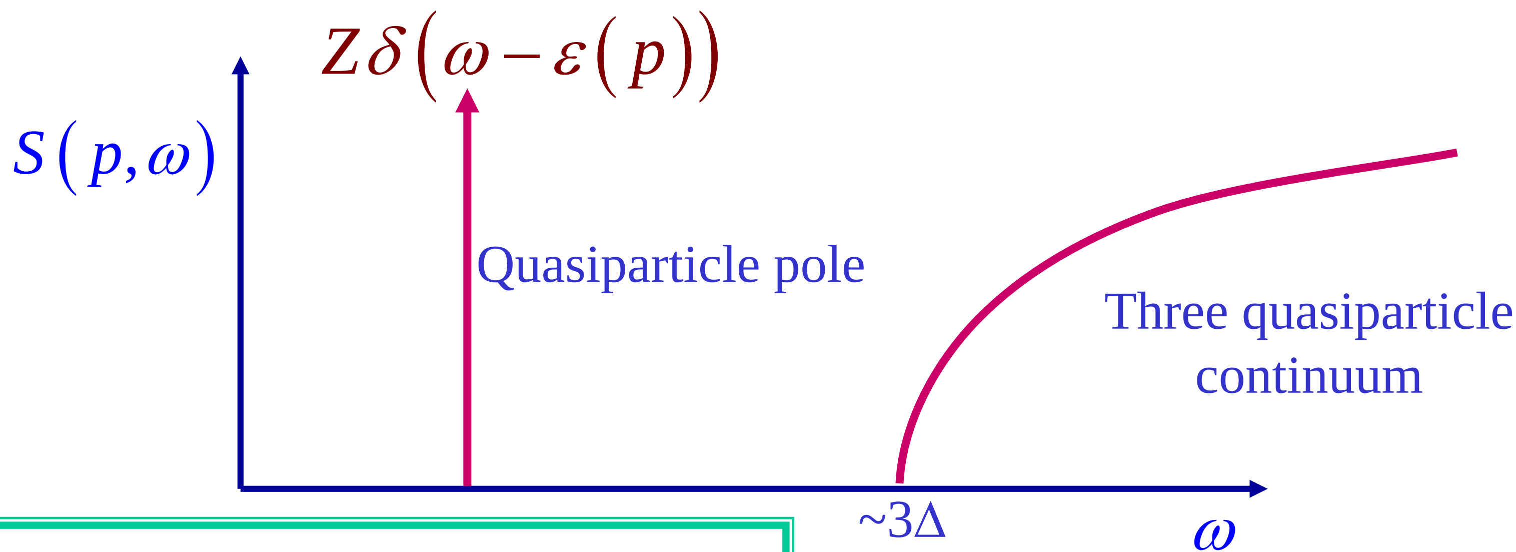
Excitation gap $\Delta = 2gJ - 2J + O\left(g^{-1}\right)$

Entire spectrum can be constructed out of multi-quasiparticle states

Dynamic Structure Factor $S(p,\omega)$:

Weakly-coupled qubits $(g \gg 1)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum p



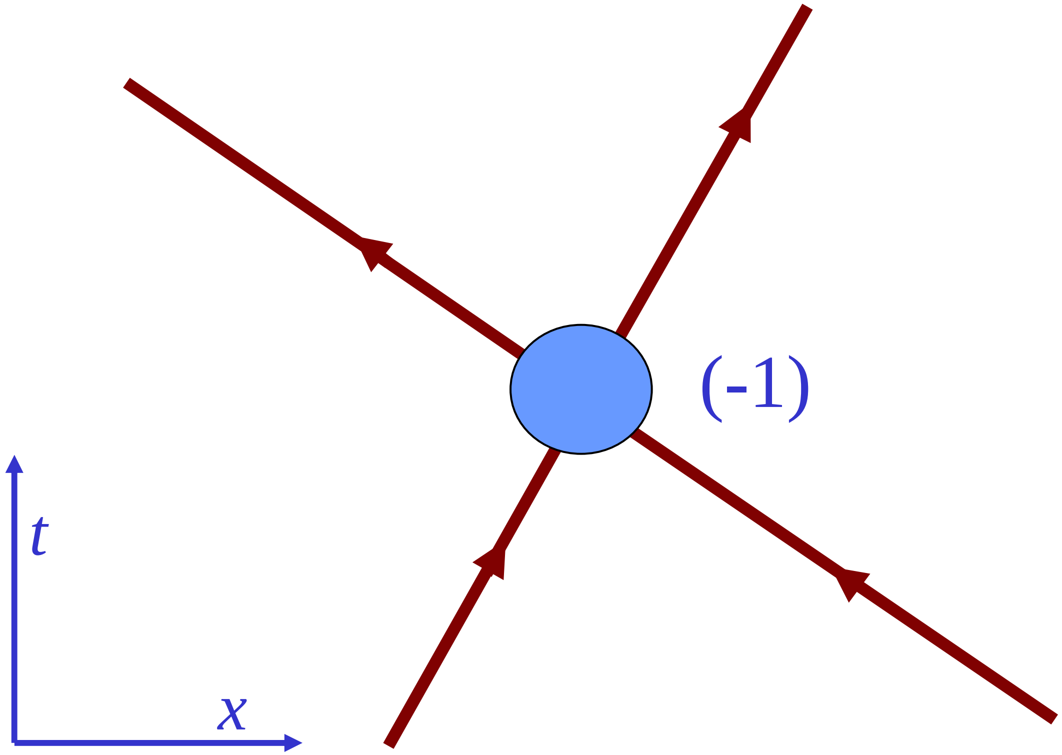
Structure holds to all orders in $1/g$

At $T > 0$, collisions between quasiparticles broaden pole to a Lorentzian of width $1/\tau_\varphi$ where the ***phase coherence time*** τ_φ is given by

$$\frac{1}{\tau_\varphi} = \frac{2k_B T}{\pi\hbar} e^{-\Delta/k_B T}$$

S. Sachdev and A.P. Young, *Phys. Rev. Lett.* **78**, 2220 (1997)

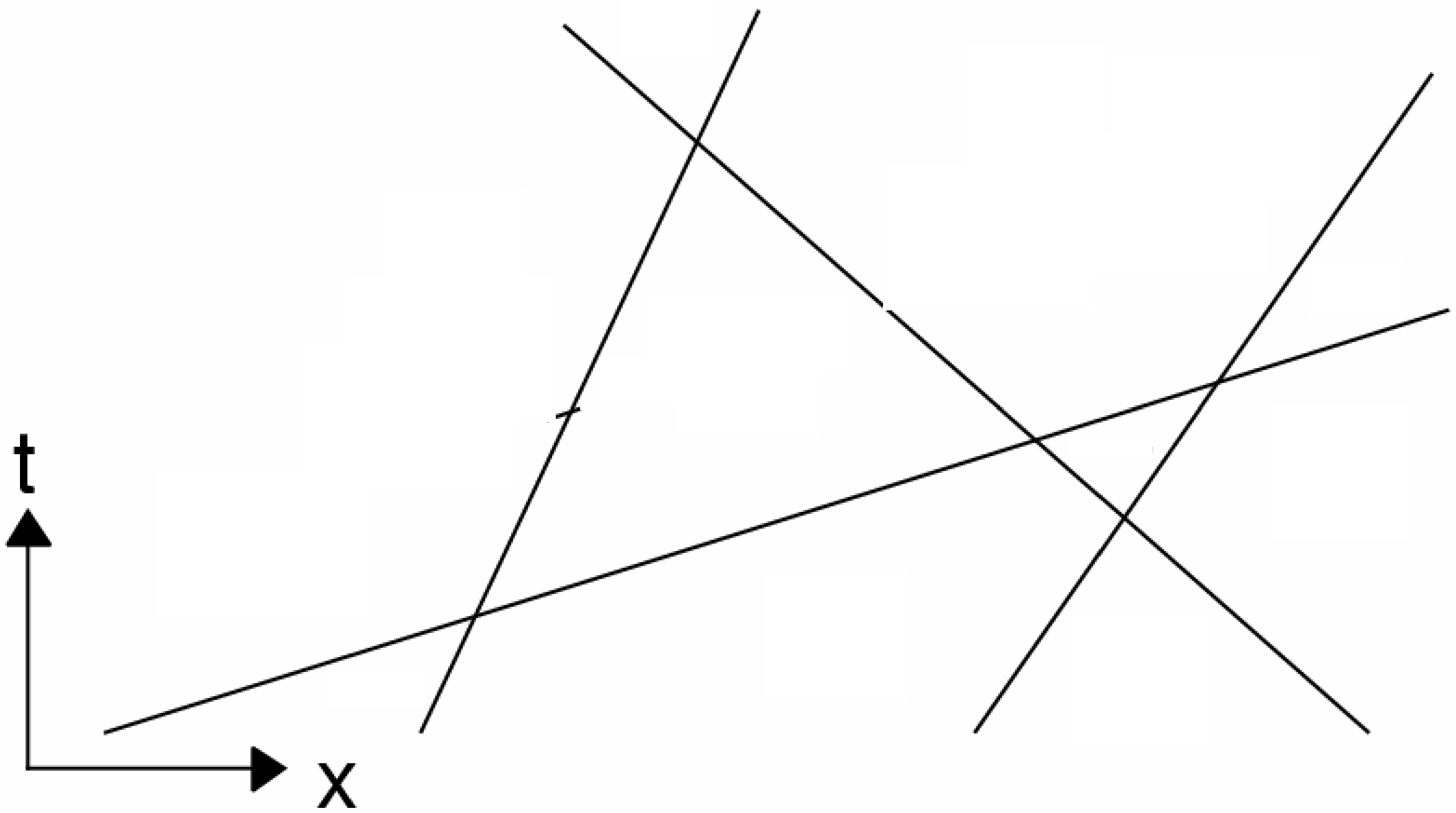
Quantum mechanical S-matrix
has a universal form at low momenta
(in one dimension)



Semiclassical theory for $C(x,t) = \text{Tr}\left(e^{-H/T} e^{iHt/\hbar} \sigma_x^z e^{-iHt/\hbar} \sigma_0^z\right)$

Valid for $T \ll \Delta$ because quasiparticle spacing $\sim e^{\Delta/T}$

$\gg$ de-Broglie wavelength $\sim T^{-1/2}$



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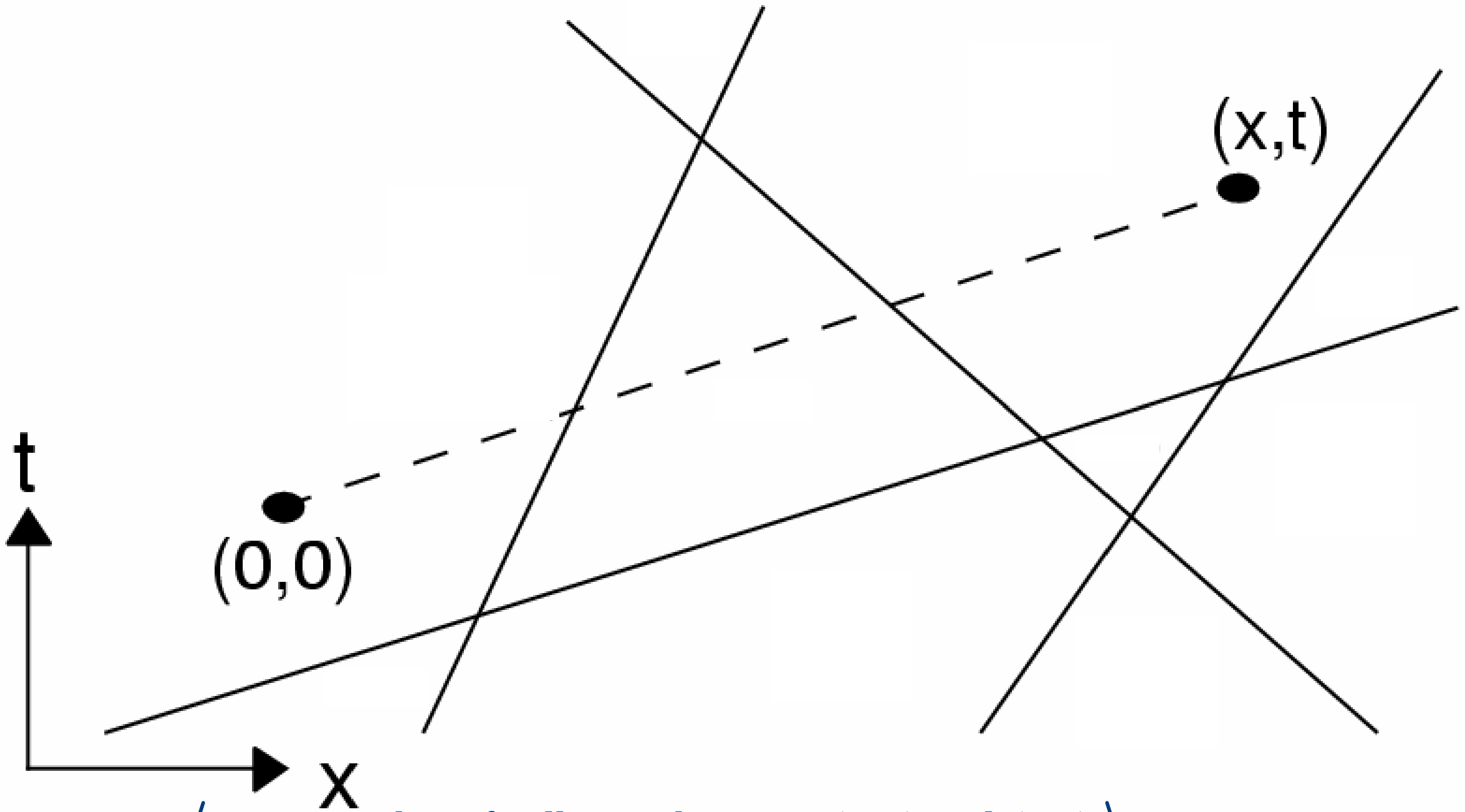
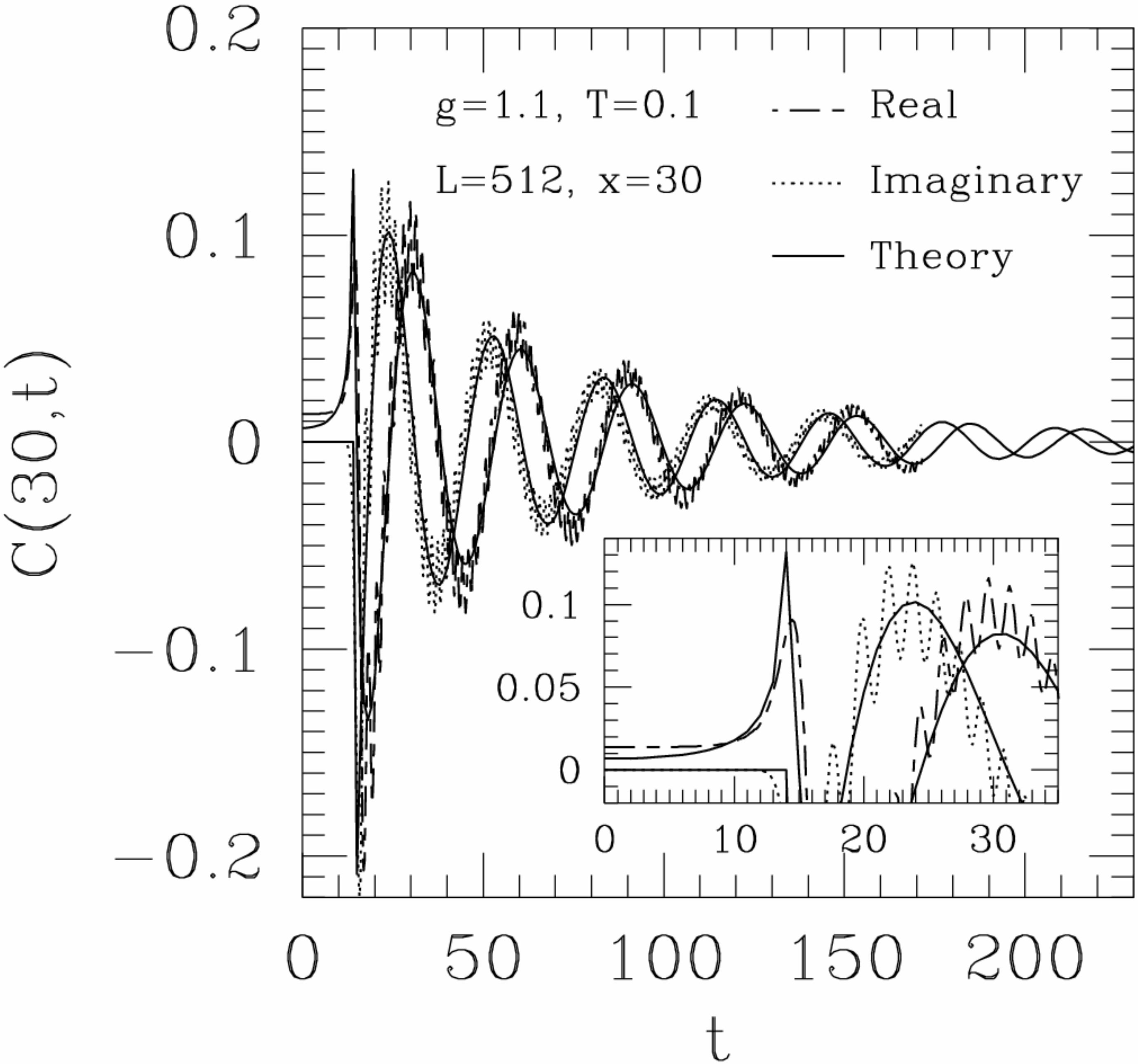


$$C(x,t) = \left\langle (-1)^{\text{number of collisions between } (0,0) \text{ and } (x,t)} \right\rangle C_{\text{single particle}}(x,t)$$


g=1.1, T=0.1
L=512, x=30
Real
Imaginary
Theory
C(30,t)
t
0.2
0.1
0
−0.1
−0.2
0
50
100
150
200
0.1
0.05
0
0
10
20
30

Strongly-coupled qubits $\left(g \ll 1\right)$

Ground states:

$$\left|G\uparrow\right\rangle = \left|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\right\rangle$$

$$-\frac{g}{2}\left|\cdots\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\right\rangle - \cdots$$

Ferromagnetic moment

$$N_0 = \left\langle G\right|\sigma^z\left|G\right\rangle \neq 0$$

Second state $\left|G\downarrow\right\rangle$ obtained by $\uparrow \Leftrightarrow \downarrow$

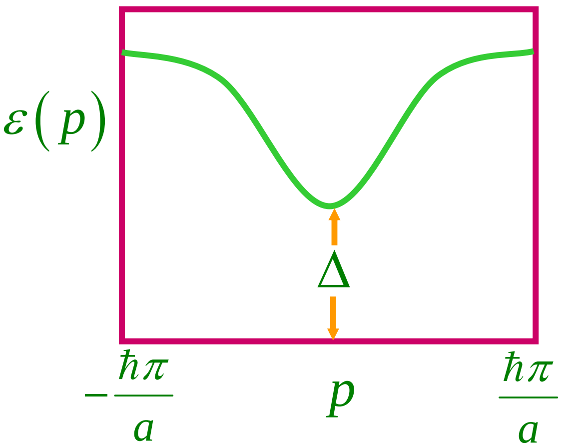
$\left|G\downarrow\right\rangle$ and $\left|G\uparrow\right\rangle$ mix only at order g^N

Lowest excited states: domain walls

$$\left|d_j\right\rangle = \left|\cdots\uparrow\uparrow\uparrow\uparrow\uparrow_j\downarrow\downarrow\downarrow\downarrow\downarrow\cdots\right\rangle + \cdots$$

Coupling between qubits creates new "domain-wall" *quasiparticle* states at momentum p

$$\left|p\right\rangle = \sum_j e^{ipx_j/\hbar}\left|d_j\right\rangle$$



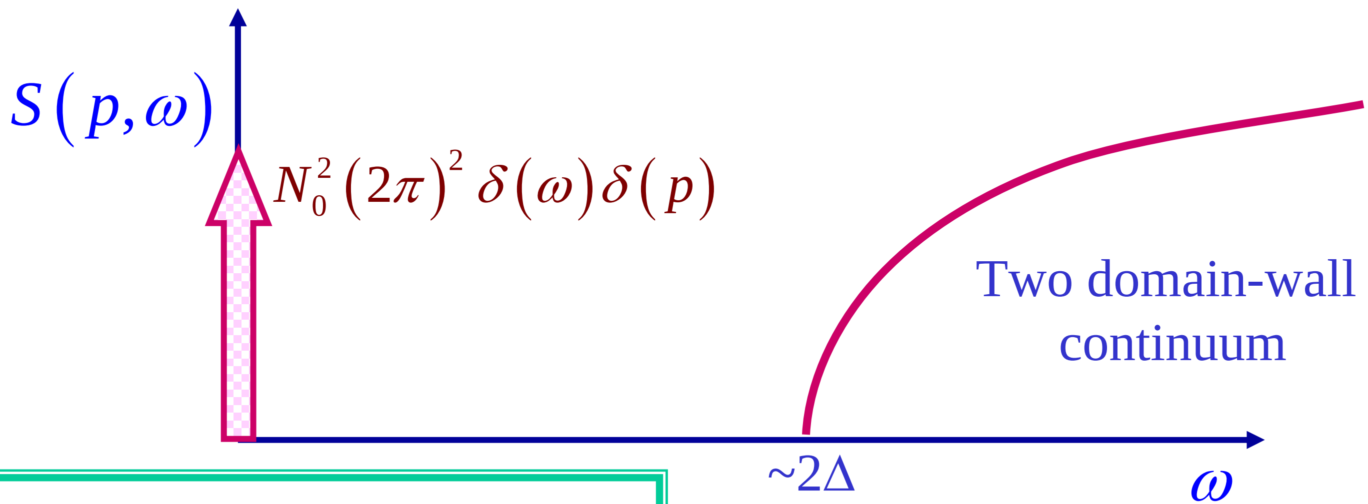
Excitation energy $\varepsilon\left(p\right) = \Delta + 4Jg\sin^2\left(\frac{pa}{2\hbar}\right) + O\left(g^2\right)$

Excitation gap $\Delta = 2J - 2gJ + O\left(g^2\right)$

Dynamic Structure Factor $S(p,\omega)$:

Strongly-coupled qubits $(g \ll 1)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa) while transferring energy $\hbar\omega$ and momentum p



Structure holds to all orders in g

At $T > 0$, motion of domain walls leads to a finite ***phase coherence time*** τ_φ, and broadens coherent peak to a width $1/\tau_\varphi$ where

$$\frac{1}{\tau_\varphi} = \frac{2k_B T}{\pi\hbar} e^{-\Delta/k_B T}$$

S. Sachdev and A.P. Young, *Phys. Rev. Lett.* **78**, 2220 (1997)

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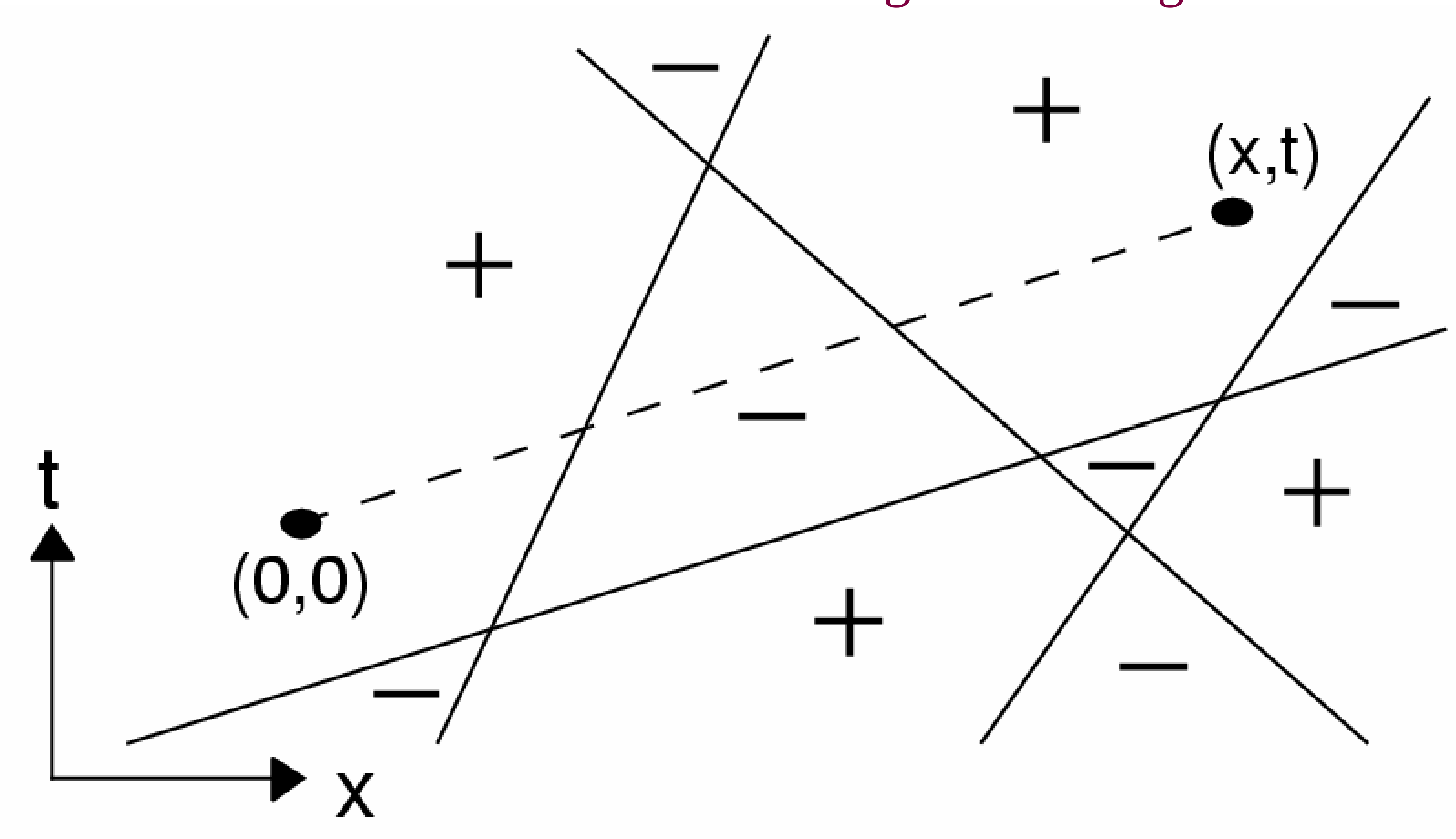

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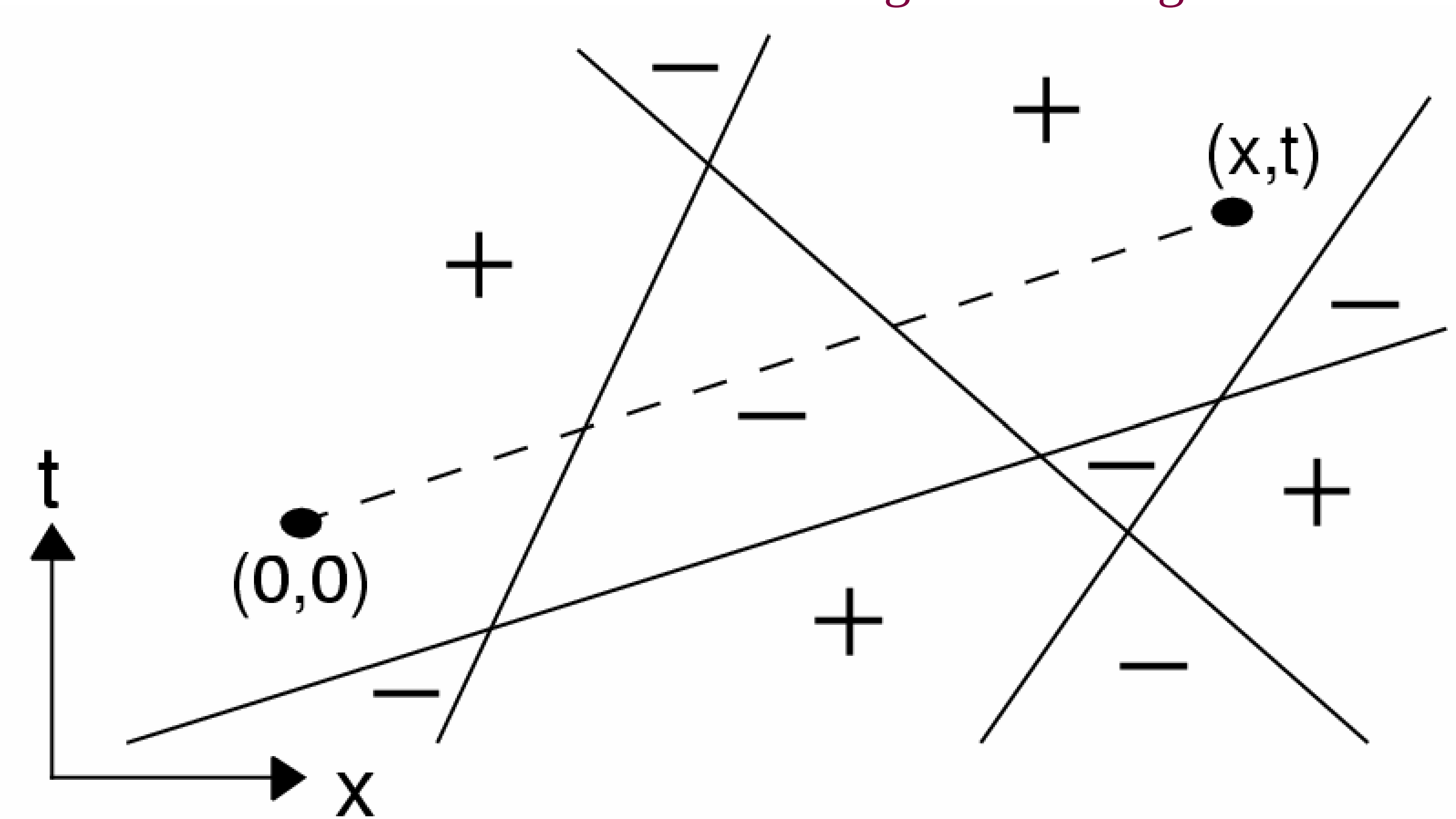
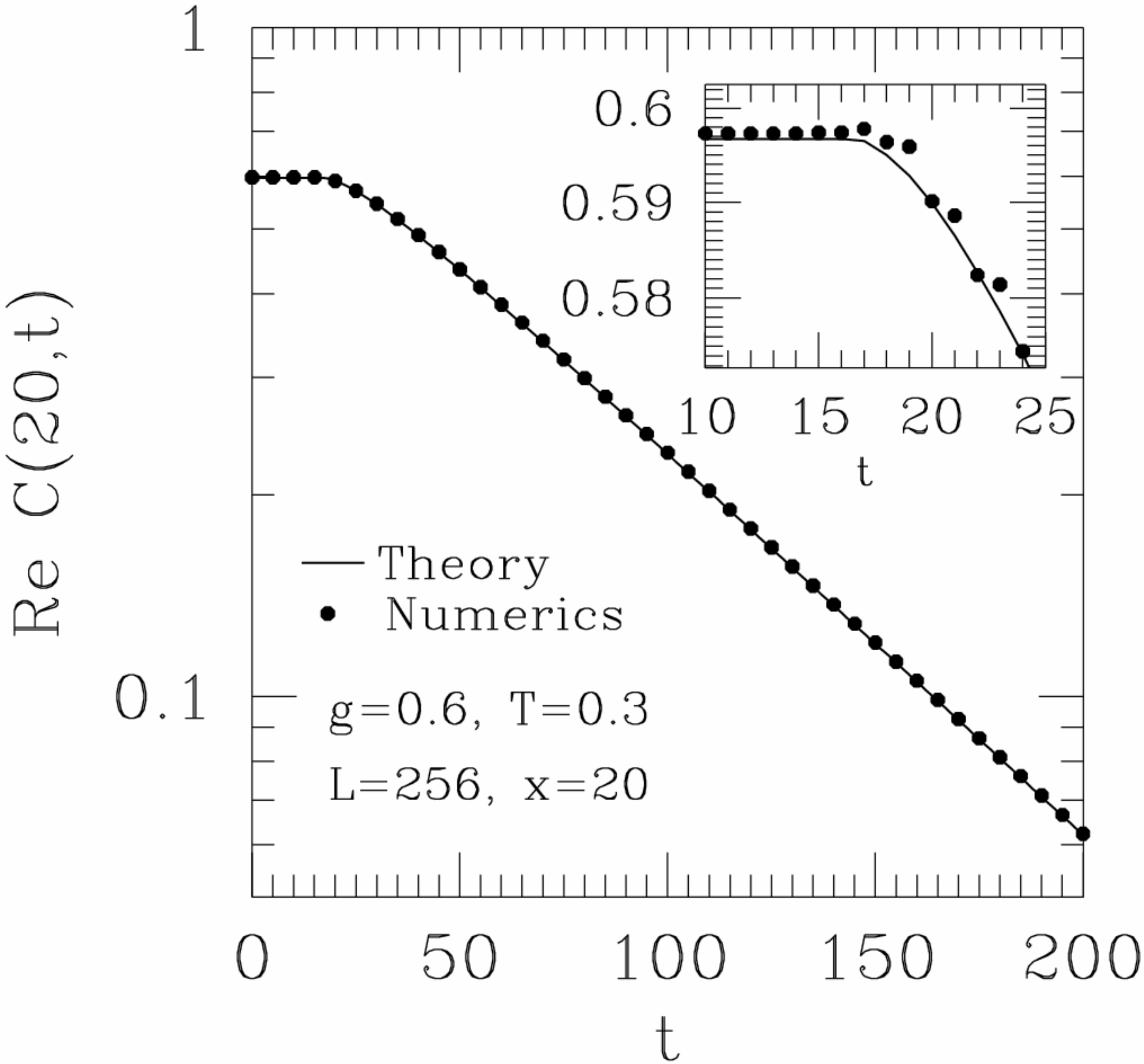


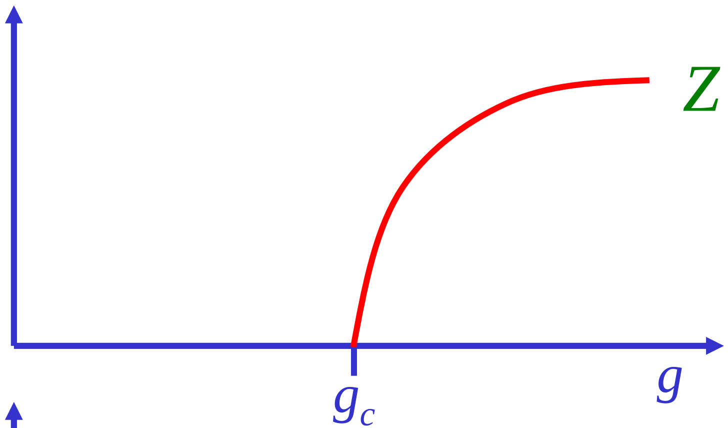
$$C(x,t) = \left\langle (\text{-}1)^{\text{number of domain walls between } (0,0) \text{ and } (x,t)} \right\rangle$$



Re C(20,t)
t
Theory
Numerics
g=0.6, T=0.3
L=256, x=20
1
0.1
0
50
100
150
200
0.6
0.59
0.58
10
15
20
25
t

Entangled states at g of order unity

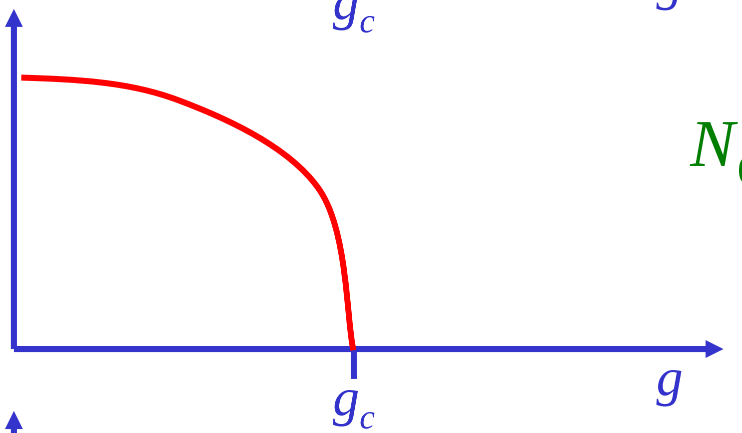
"Flipped-spin" Quasiparticle weight Z



$$Z \sim (g - g_c)^{1/4}$$

A.V. Chubukov, S. Sachdev, and J.Ye, *Phys. Rev.* B **49**, 11919 (1994)

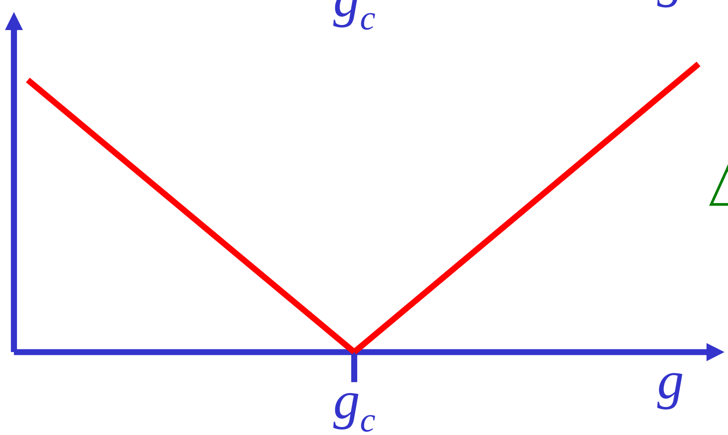
Ferromagnetic moment N_0



$$N_0 \sim (g_c - g)^{1/8}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

Excitation energy gap Δ



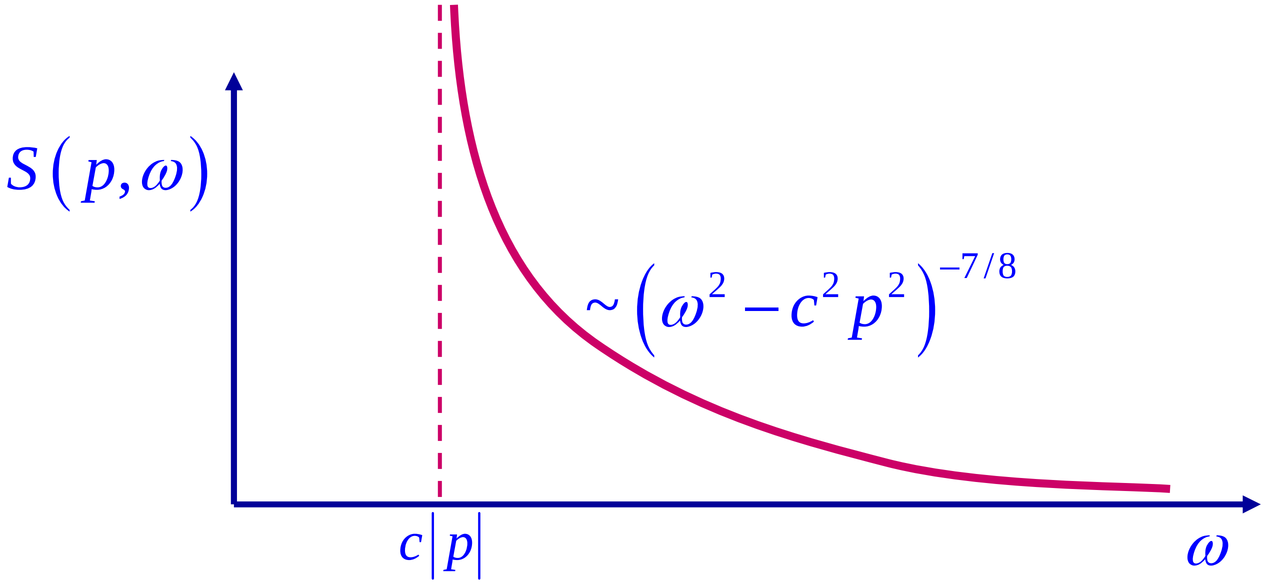
$$\Delta \sim |g - g_c|$$

Dynamic Structure Factor $S(p,\omega)$:

Critical coupling $(g = g_c)$

Cross-section to flip a $|\rightarrow\rangle$ to a $|\leftarrow\rangle$ (or vice versa)

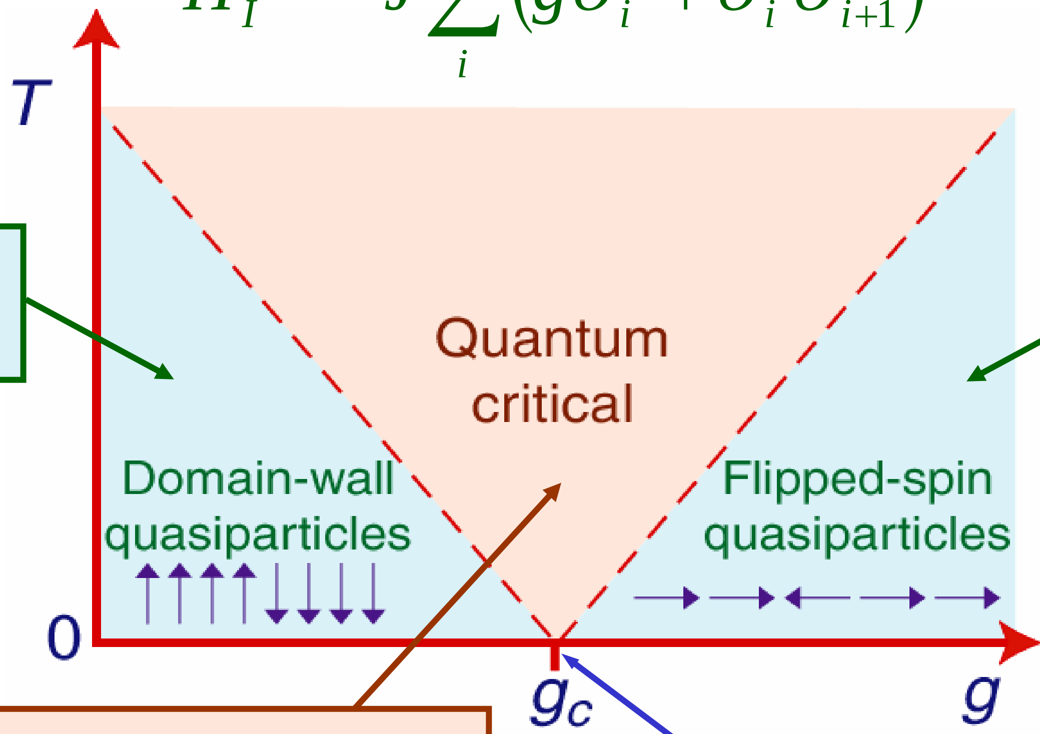
while transferring energy $\hbar\omega$ and momentum p



No quasiparticles --- dissipative critical continuum

$$H_I = -J\sum_i \left(g\sigma_i^x + \sigma_i^z \sigma_{i+1}^z \right)$$

Quasiclassical dynamics



Quasiclassical dynamics

$$\chi(\omega) = \frac{i}{\hbar}\sum_k \int_0^\infty dt \left\langle \left[\sigma_j^z(t), \sigma_k^z(0) \right] \right\rangle e^{i\omega t}$$

$$= \frac{A}{T^{7/4}\left(1 - i\omega/\Gamma_R + \ldots\right)}$$

$$\Gamma_R = \left(2\tan\frac{\pi}{16} \right) \frac{k_B T}{\hbar}$$

$$\left\langle \sigma_j^z \sigma_k^z \right\rangle \sim \frac{1}{|j-k|^{1/4}}$$

P. Pfeuty *Annals of Physics*, **57**, 79 (1970)

S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992).
S. Sachdev and A.P. Young, Phys. Rev. Lett. **78**, 2220 (1997).

II. Quantum O(3) non-linear σ model

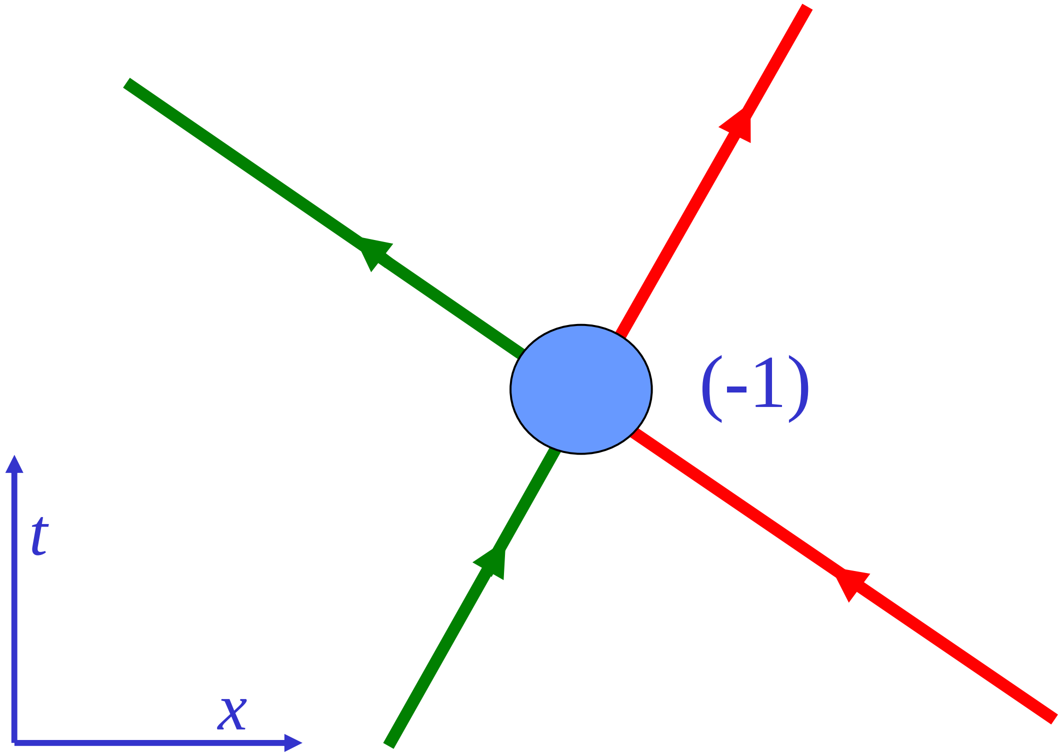
Quantum field theory of a unit vector field $\mathbf{n}(x,t)$, with the constraint $\mathbf{n}^2 = 1$. Imaginary time Feynman path integral

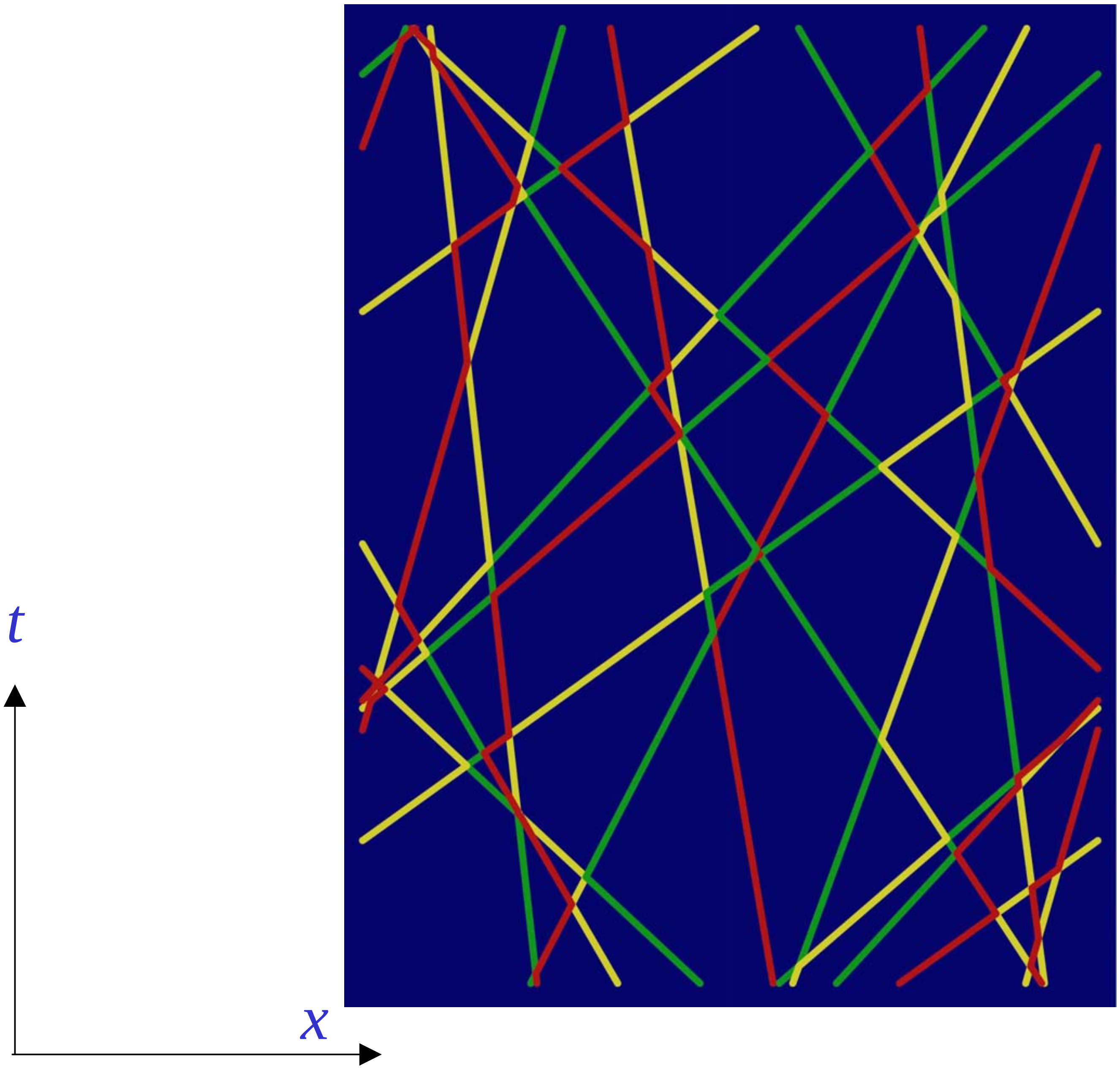
$$\mathcal{Z} = \int \mathcal{D}\mathbf{n}(x,\tau)\delta(\mathbf{n}^2 - 1) \times \exp\left(-\frac{1}{2g}\int dx \int d\tau \left[c^2 (\partial_x \mathbf{n})^2 + (\partial_\tau \mathbf{n})^2\right]\right)$$

There is always an energy gap above the ground state $\Delta \sim \Lambda e^{-2\pi/g}$.

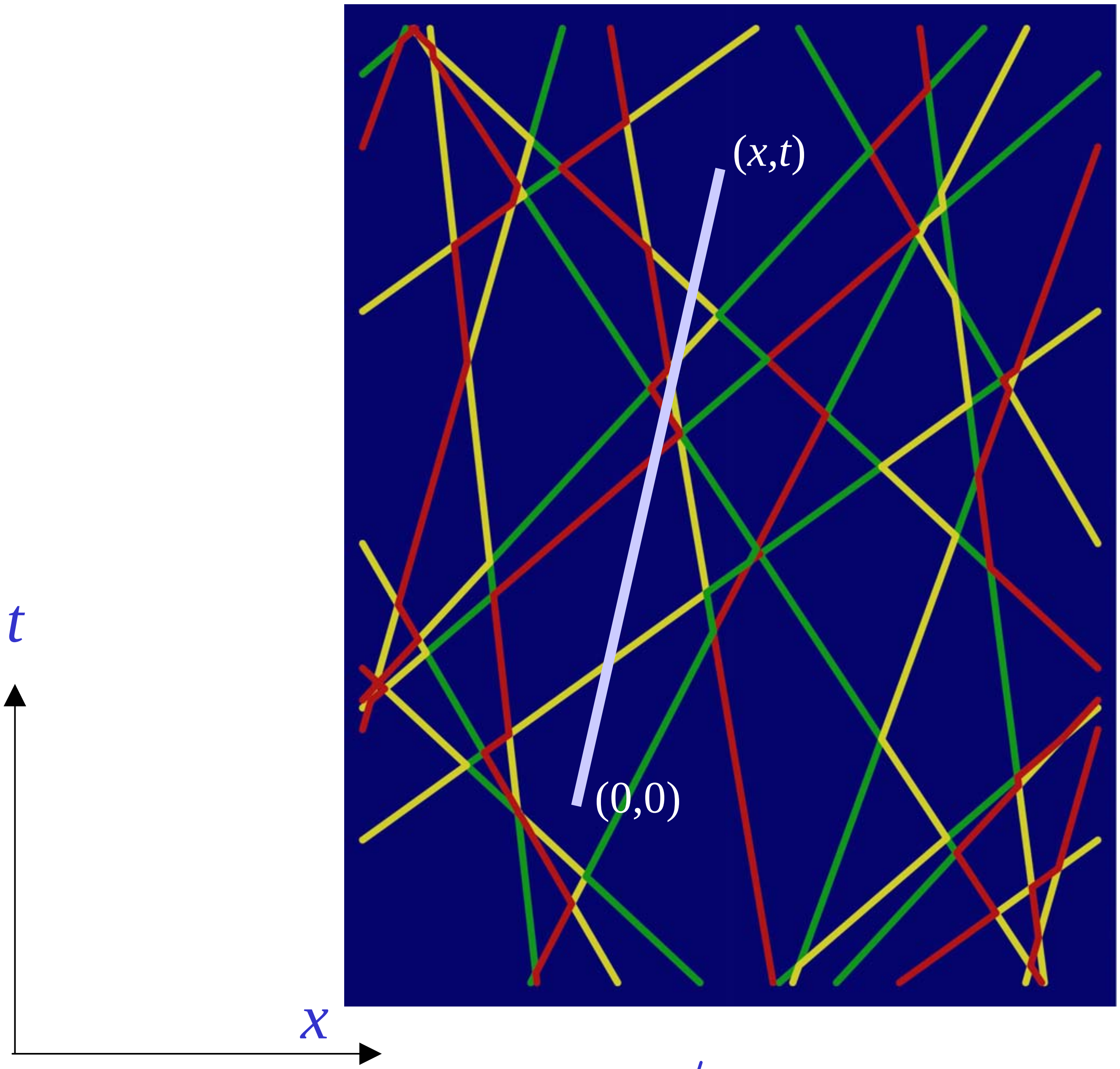
Excitations are quasi-particles with spin $S = 1$ (three-fold degenerate).

Quantum mechanical S-matrix
has a universal form at low momenta
(in one dimension)





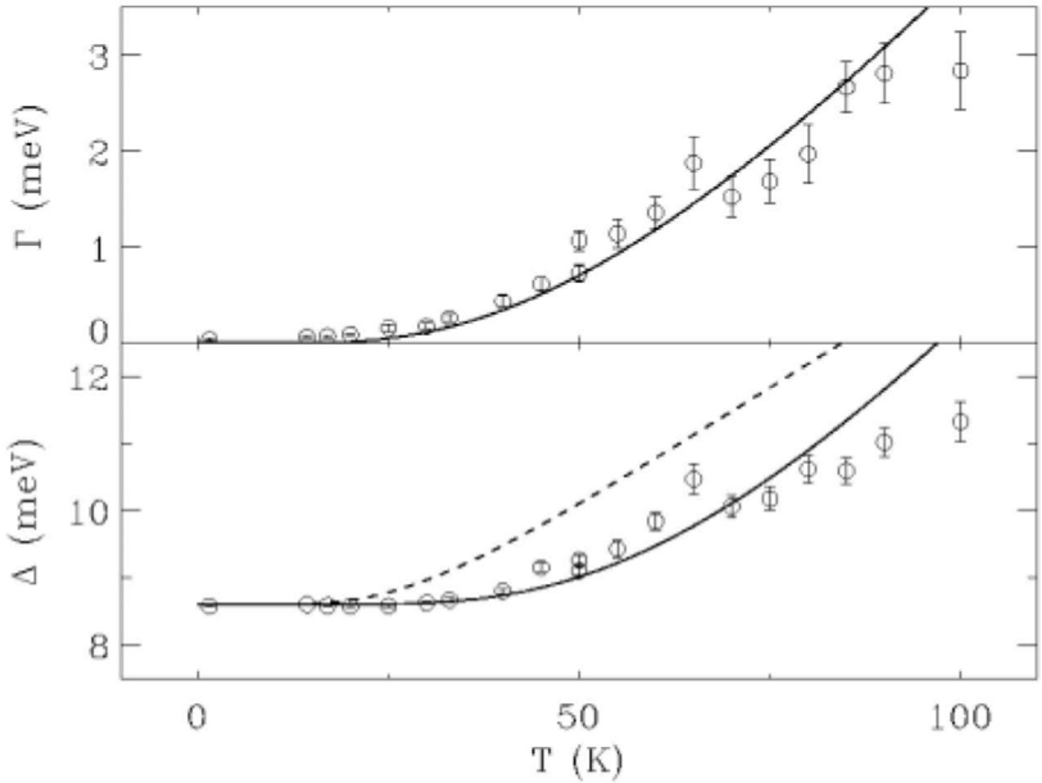
t
x



$$C(x,t) = C(x,t)\big|_{T=0}\left\langle (-1)^{\text{number of collisions with particles of same color}} \right\rangle$$

Excitation energy Δ and linewidth Γ for Y_2BaNiO_5

C. Broholm et al. (unpublished)



Solid line for Γ – theory with no free parameters

III. Quantum sine-Gordon model

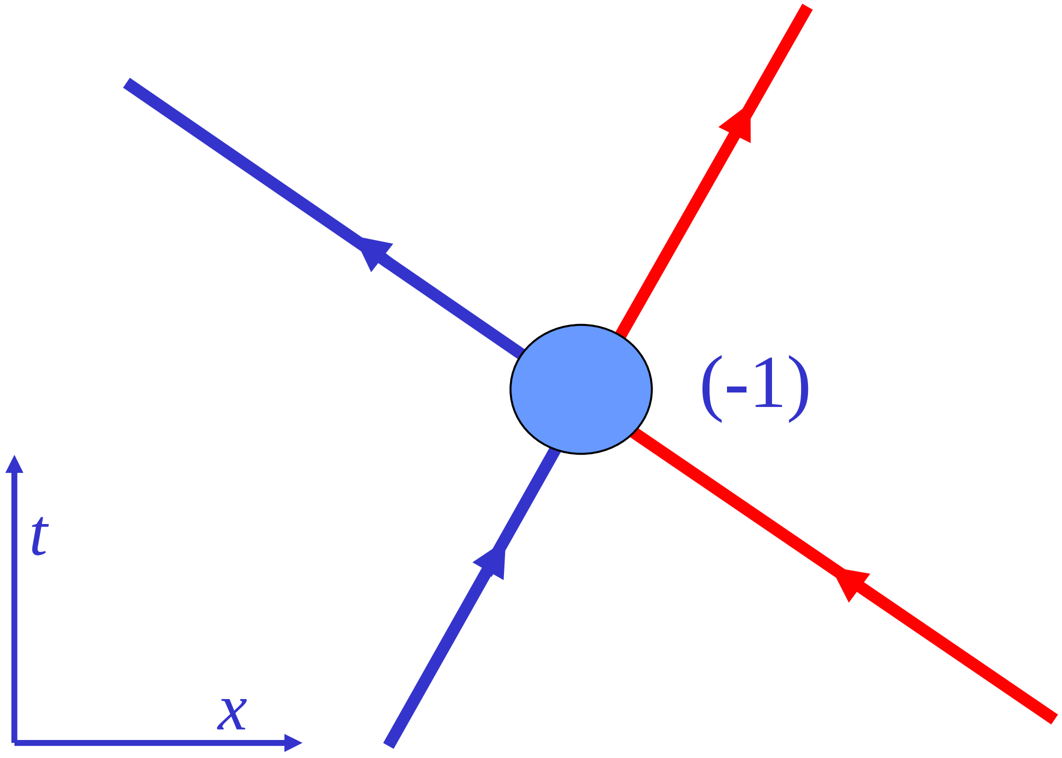
Quantum field theory of a "phase" field $\Phi(x,t)$. Imaginary time Feynman path integral

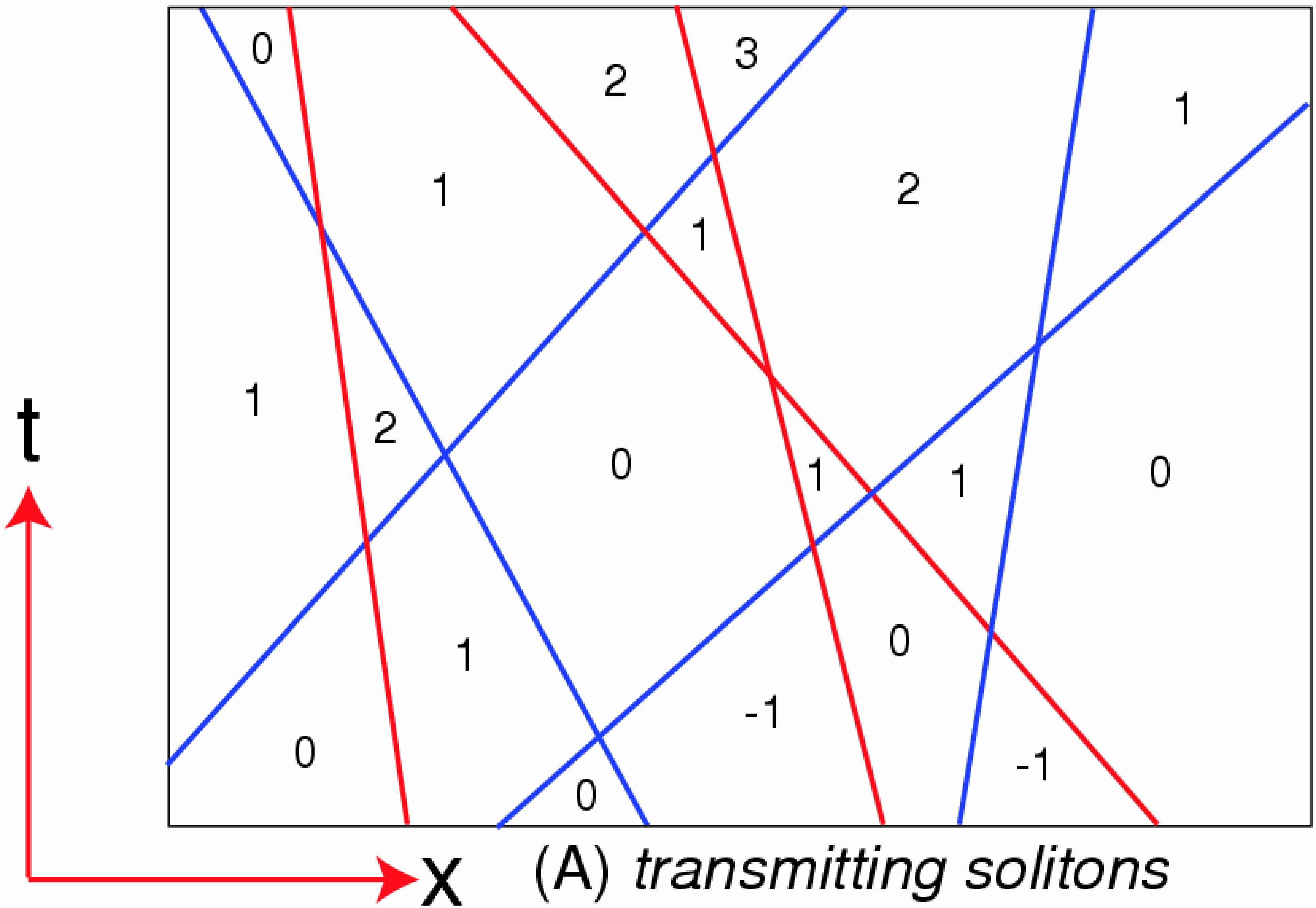
$$\mathcal{Z} = \int \mathcal{D}\Phi(x,\tau)$$
$$\times \quad \exp\left(-\frac{c}{16\pi}\int d\tau dx \left[(\partial_x\Phi)^2 + \frac{1}{c^2}(\partial_\tau\Phi)^2 - g^2\cos(\gamma\Phi)\right]\right)$$

We focus on the range $\gamma < 1$ where there is an energy gap, and the excitations consist of solitons and anti-solitons.

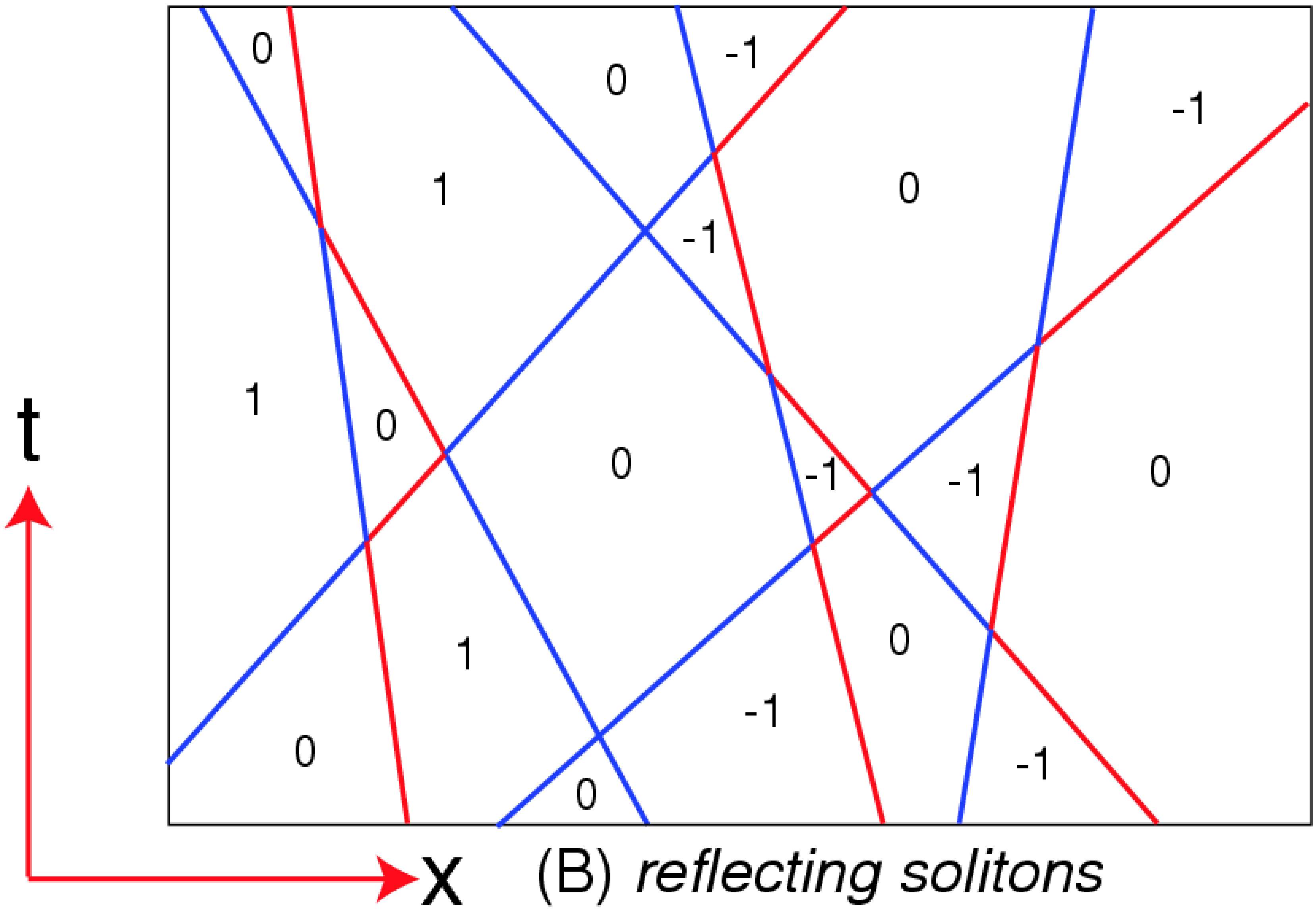
In the semiclassical picture, $\Phi = 2\pi n/\gamma$ (n integer) over most space time. The value of n jumps by ± 1 across solitons/anti-solitons.

Quantum mechanical S-matrix
has a universal form at low momenta
(in one dimension)





(A) *transmitting solitons*



(B) *reflecting solitons*

III. Quantum sine-Gordon model

Given solitons with charges $m_k = \pm 1$ and the "zig-zag" trajectories $x_k(t)$, the sine-Gordon field is

$$\Phi(x,t) = \frac{2\pi}{\gamma} \sum_{k=1} m_k \theta(x - x_k(t)).$$

Averaging over trajectories, we found

$$\begin{aligned} C_\Phi(x,t) &= \langle e^{i\eta\Phi(x,t)} e^{-i\eta\Phi(0,0)} \rangle \\ &= e^{-(\tilde{q}_r + \tilde{q}_l)} \Big[U_0(2i\tilde{q}_r\Theta, 2i\sqrt{\tilde{q}_r\tilde{q}_l}) + \\ & U_0(2i\tilde{q}_l\Theta, 2i\sqrt{\tilde{q}_r\tilde{q}_l}) - iU_1(2i\tilde{q}_r\Theta, 2i\sqrt{\tilde{q}_r\tilde{q}_l}) \\ & -iU_1(2i\tilde{q}_l\Theta, 2i\sqrt{\tilde{q}_r\tilde{q}_l}) - I_0(2\sqrt{\tilde{q}_r\tilde{q}_l}) \Big], \end{aligned}$$

where $\Theta \equiv \cos(2\pi\eta/\gamma)$, $\tilde{q}_r = \rho \int_{-\infty}^{x/t} dv P(v)(x - vt)$, $\tilde{q}_l = \rho \int_{x/t}^{\infty} dv P(v)(x - vt)$, I_0 is the modified Bessel function, $U_{0,1}$ are the *Lommel* functions of two variables, ρ is the density of excitations, and $P(v)$ is their Maxwell-Boltzmann velocity distribution.

At long distances and times, this correlator has a *diffusive* form $\sim t^{-1/2} e^{-x^2/(4Dt)}$ with $D \sim T^{-1/2}$.

Conclusions

1. Large, entangled quantum systems close to equilibrium have an "intrinsic" relaxation rate independent of the strength of the coupling to a heat bath.

2. This relaxation rate has a universal form near interacting quantum critical points

3. In one-dimensional systems

$$\tau_\varphi \sim \begin{cases} \dfrac{\hbar}{k_B T} e^{\Delta / k_B T} & \text{in a gapped state} \\ \dfrac{\hbar}{k_B T} & \text{in a quantum-critical system} \end{cases}$$