Z_2 topological order near the Neel state of the square lattice antiferromagnet

Perimeter Institute, Waterloo
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PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

PHYSICS
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S. Chatterjee, S. Sachdev, and M. S. Scheurer, arXiv:1705.06289

1. Introduction to Z_2 topological order

Z_2 and $\mathrm{U}(1)$ lattice gauge theories

2. The uses of Z_2 topological order

(A) Intertwining topological order and symmetry breaking: CP^1 theory of the square lattice antiferromagnet

(B) The pseudogap: metals with non-Luttinger volume Fermi surfaces

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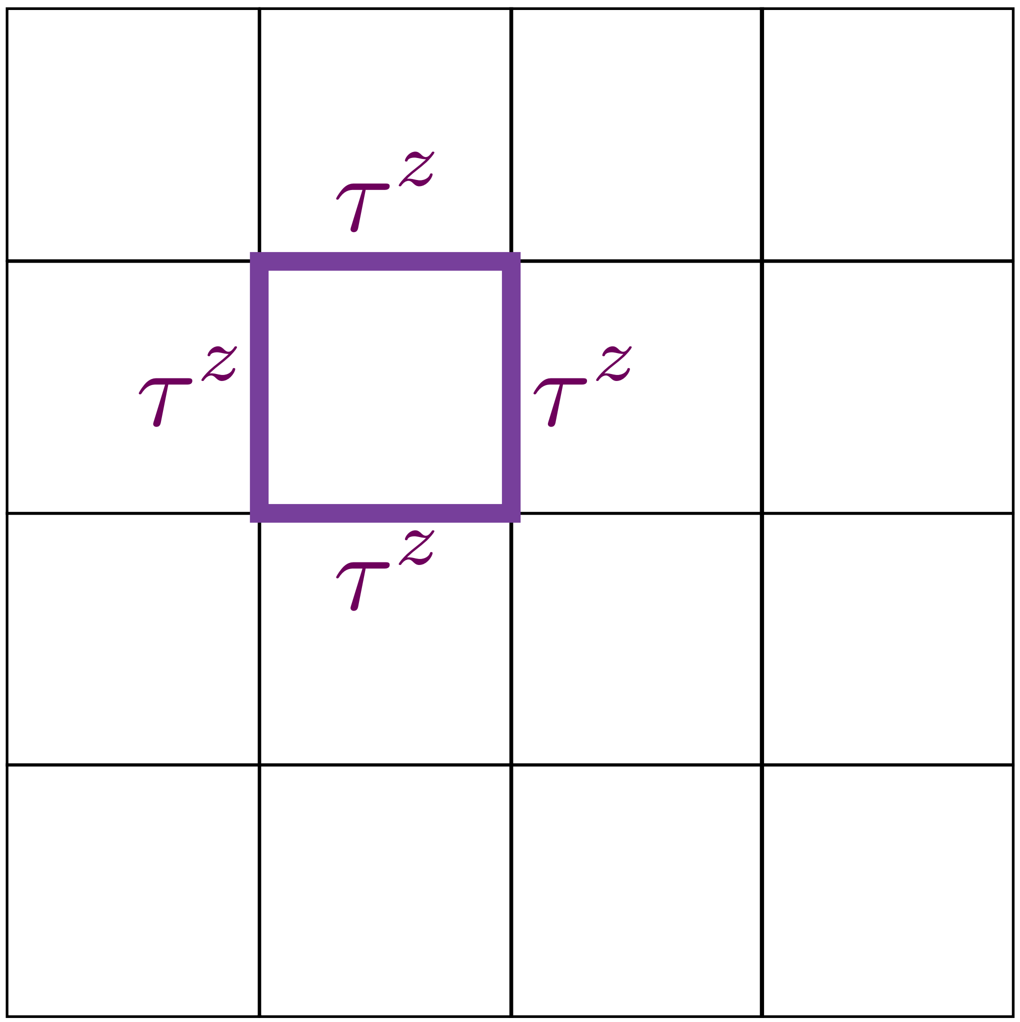
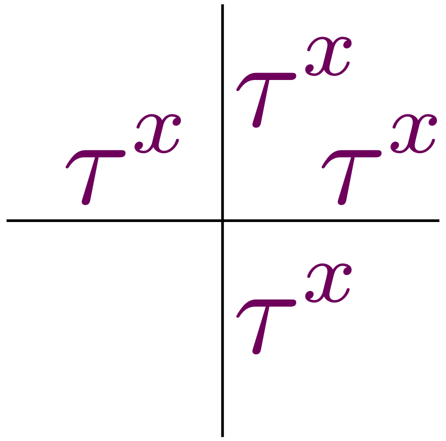
Z_2 lattice gauge theory (Wegner, 1971)



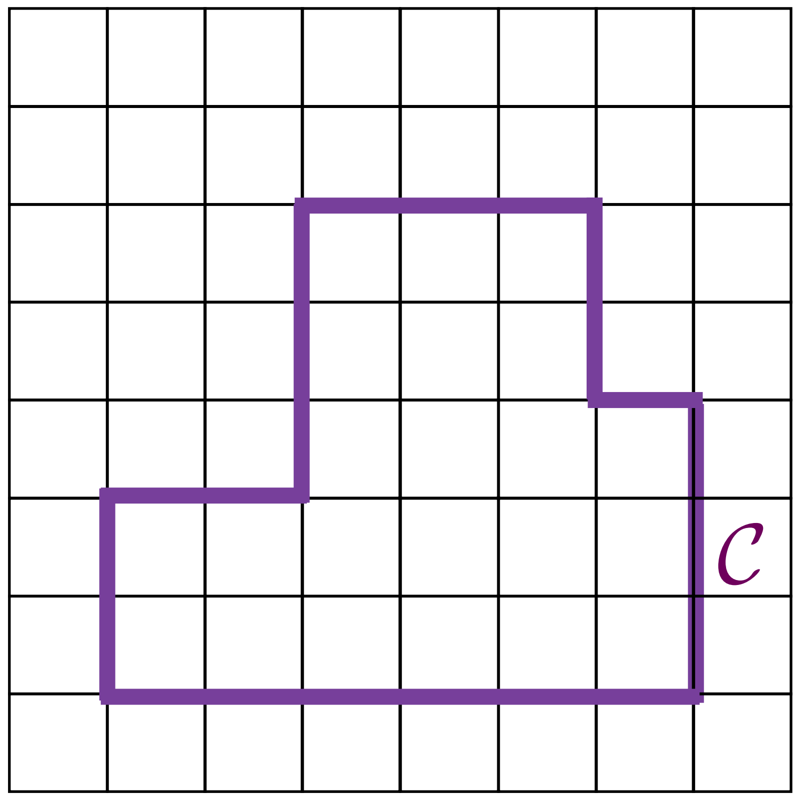
$$H = -\sum_{\Box} \tau^z \tau^z \tau^z \tau^z - g \sum_i \tau^x$$

$G_i =$



Gauss's Law: $[H, G_i] = 0$, $\boxed{G_i = 1}$

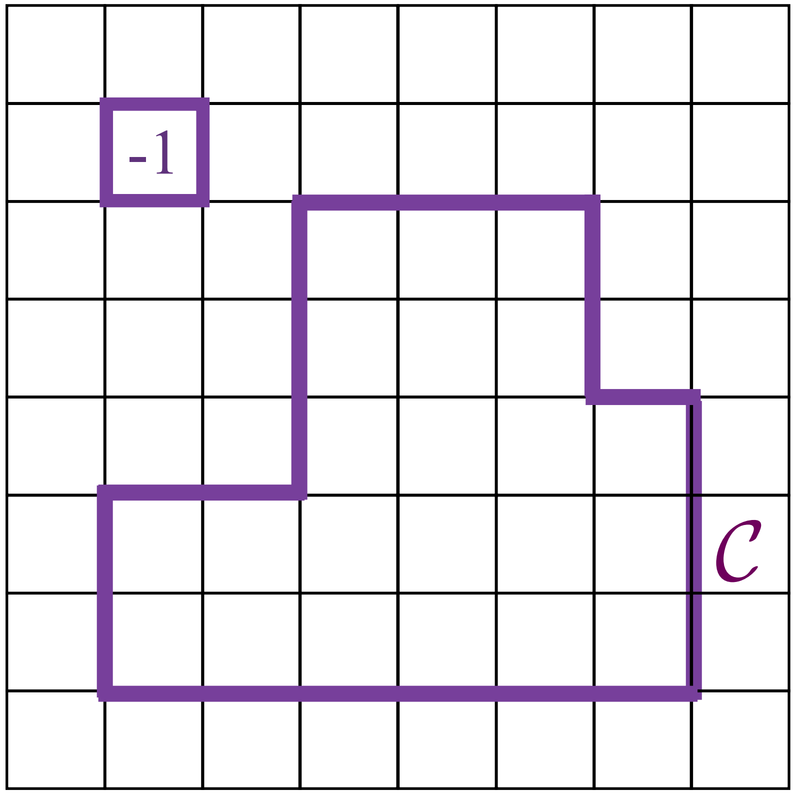
Z_2 lattice gauge theory (Wegner, 1971)



$$W_{\mathcal{C}} = \prod_{\mathcal{C}} \tau^z$$

Z_2 lattice gauge theory

(Wegner, 1971)



$$W_{\mathcal{C}} = \prod_{\mathcal{C}} \tau^z$$

Deconfined phase

$W_{\mathcal{C}} \sim$ Perimeter Law.

Deconfined Z_2 flux

$W = -1$ particles

Confined phase

$W_{\mathcal{C}} \sim$ Area Law

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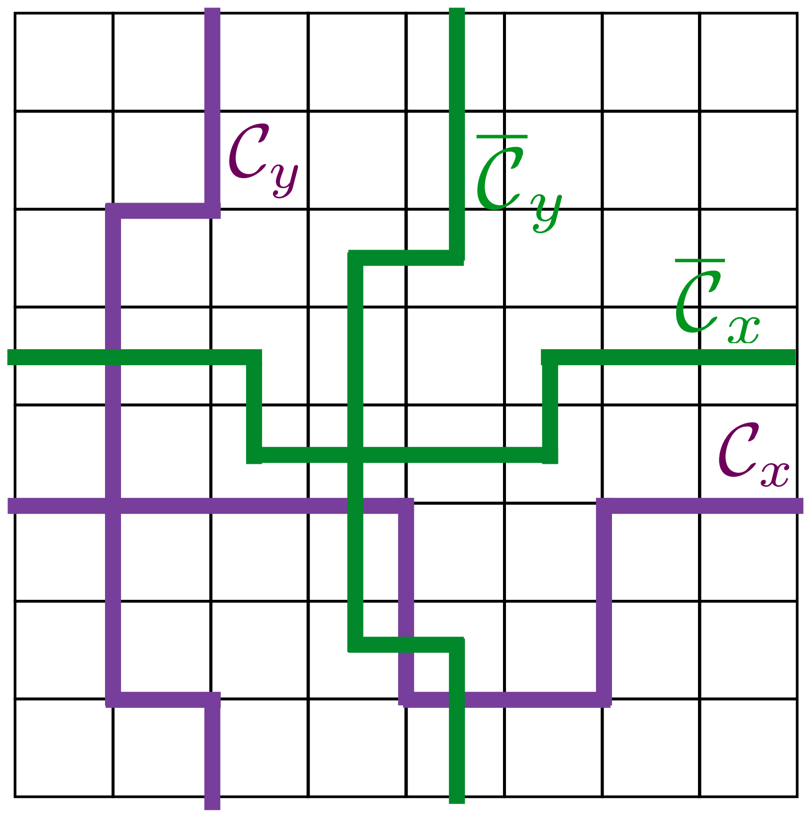
Topological order



Torus

$$V_x = \prod_{\overline{\mathcal{C}}_x} \tau^x \quad , \quad V_y = \prod_{\overline{\mathcal{C}}_y} \tau^x$$

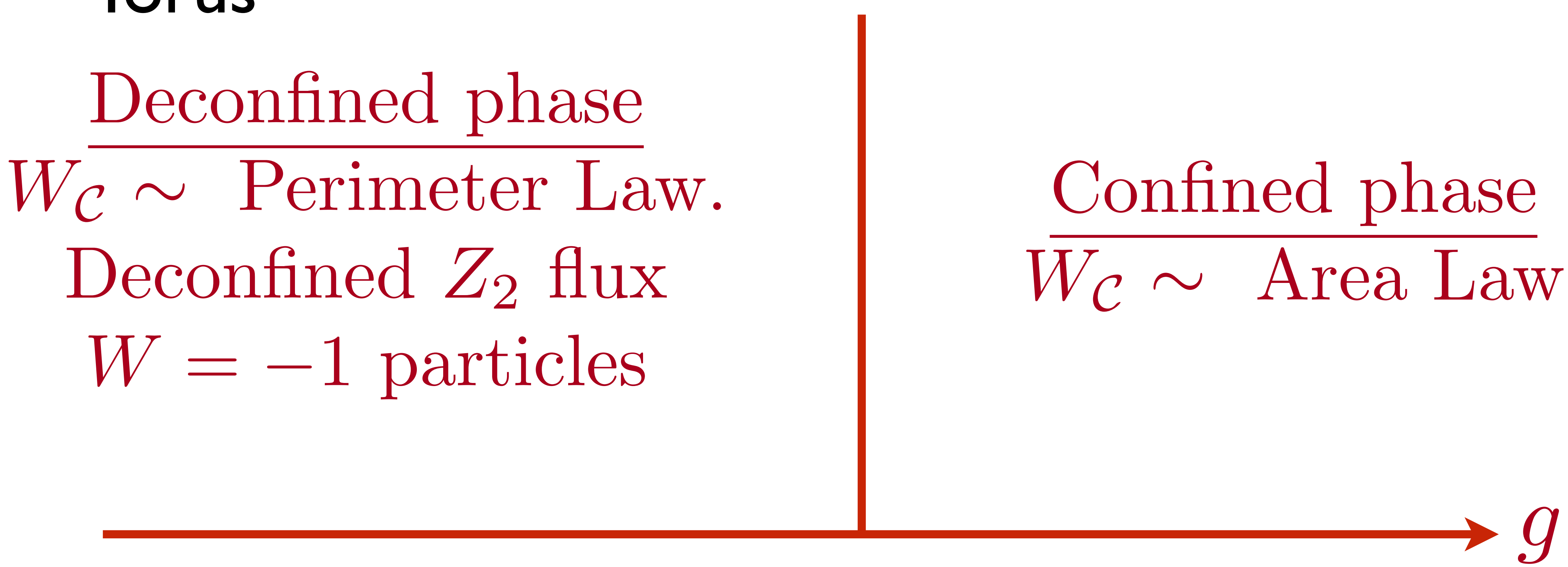
$$W_x = \prod_{\mathcal{C}_x} \tau^z \quad , \quad W_y = \prod_{\mathcal{C}_y} \tau^z$$

$$V_x W_y = -W_y V_x \quad , \quad V_y W_x = -W_x V_y$$

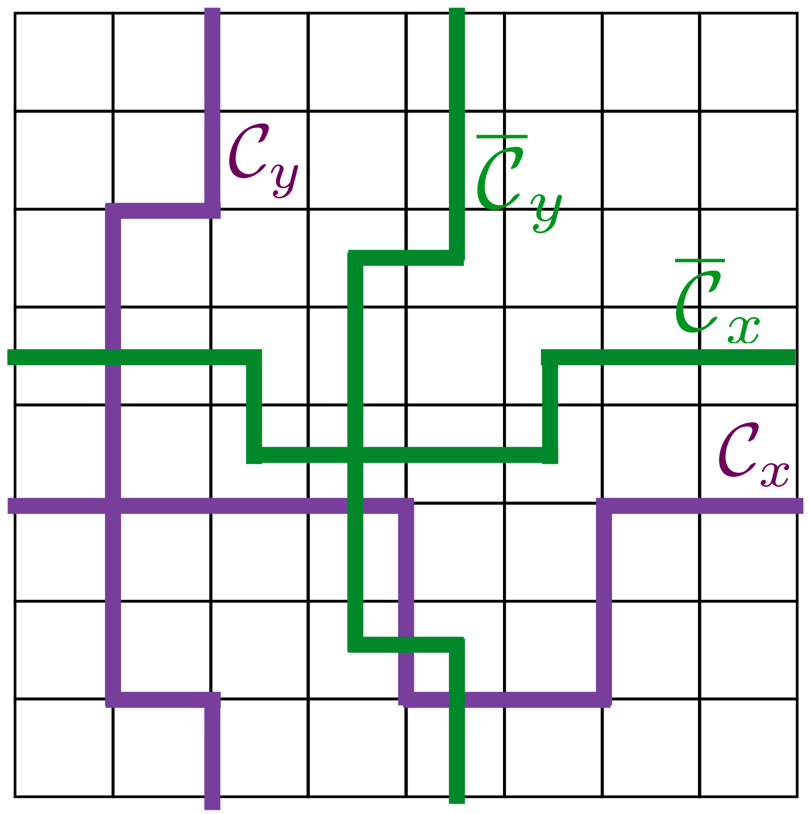
and all other pairs commute.

On a torus, there are two additional independent operators, V_x and V_y which commute with the Hamiltonian:

$$[H, V_x] = [H, V_y] = 0$$



Topological order



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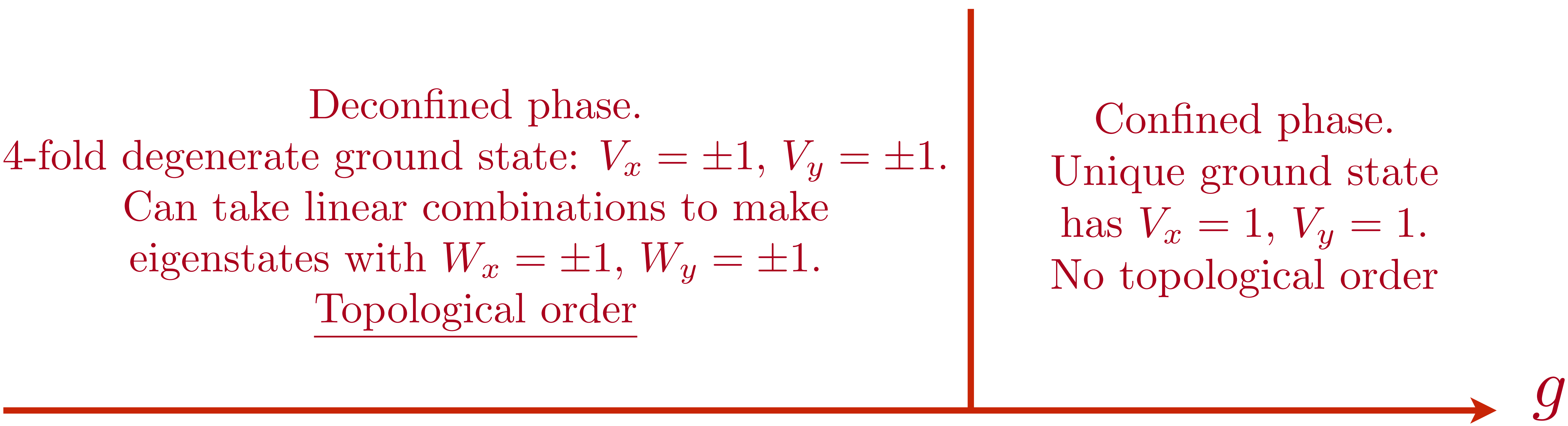
On a torus, there are two additional independent operators, V_x and V_y which commute with the Hamiltonian:

$$[H, V_x] = [H, V_y] = 0$$

Deconfined phase.
4-fold degenerate ground state: $V_x = \pm 1$, $V_y = \pm 1$.
Can take linear combinations to make eigenstates with $W_x = \pm 1$, $W_y = \pm 1$.
Topological order

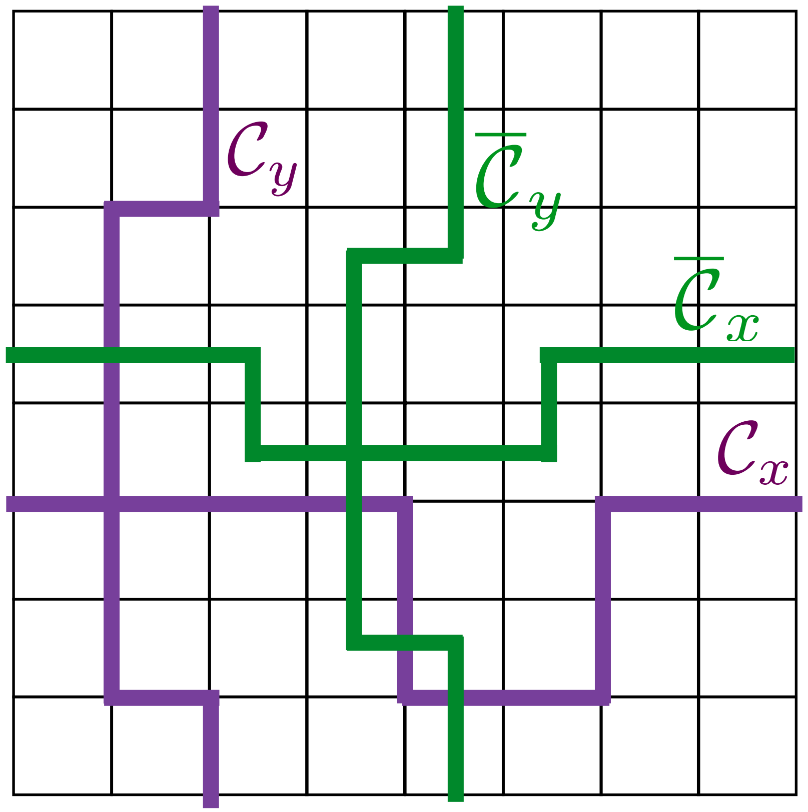
Confined phase.
Unique ground state has $V_x = 1$, $V_y = 1$.
No topological order



Topological order

$$V_x = \prod_{\overline{\mathcal{C}}_x} \tau^x \quad , \quad V_y = \prod_{\overline{\mathcal{C}}_y} \tau^x$$

$$W_x = \prod_{\mathcal{C}_x} \tau^z \quad , \quad W_y = \prod_{\mathcal{C}_y} \tau^z$$



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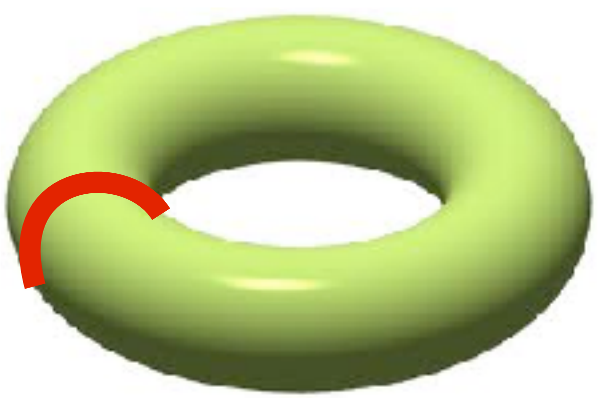


(N. Read and S.S., 1991
Freedman, Nayak, Shtengel, Walker, Wang, 2003)

Topological quantum field theory describes degenerate states with Z_2 flux $W = \pm 1$ through the holes of the torus

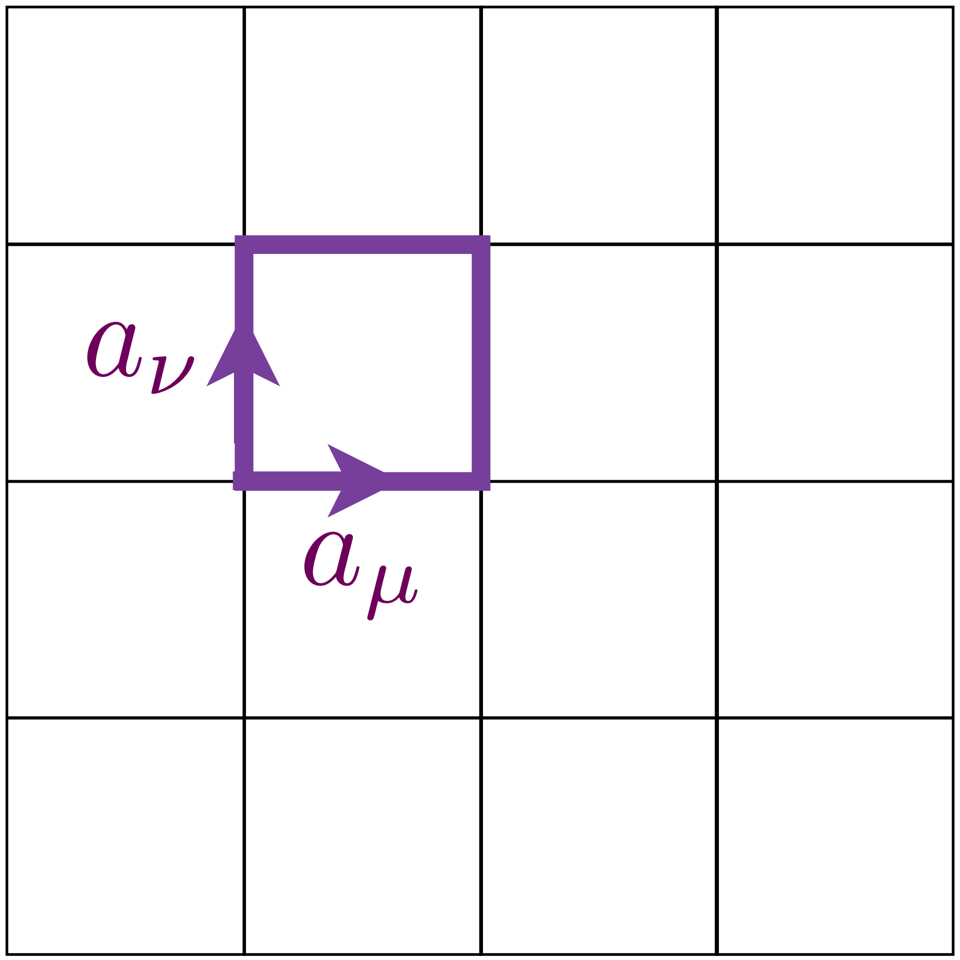
Confined phase.
Unique ground state has $V_x = 1$, $V_y = 1$.
No topological order

g

This criterion can distinguish the phases when dynamical (or even gapless) matter fields are present

Compact U(1) lattice gauge theory

(Fradkin and Shenker, 1979)



$$\mathcal{S} = -\frac{1}{g}\sum_{\square} \cos\left(\Delta_\mu a_\nu - \Delta_\nu a_\mu\right)$$
$$-\sum_{i,\mu} \cos\left(\Delta_\mu \theta - 2a_\mu\right)$$

Same phases as Z_2 gauge theory

Deconfined phase
Monopoles suppressed.
Deconfined π (mod 2π) flux vortices

Confined phase
Monopoles proliferate

g

Compact U(1) lattice gauge theory

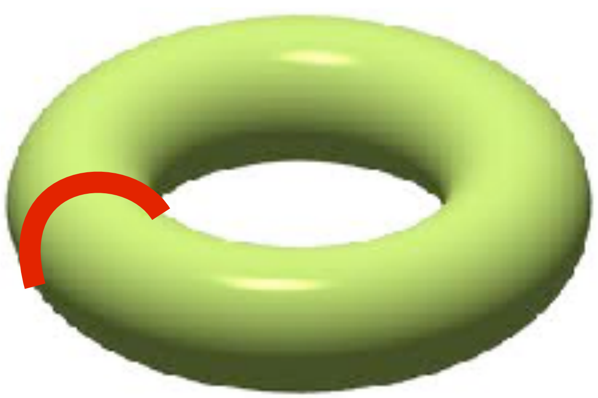
$$\mathcal{S} = -\frac{1}{g}\sum_{\square}\cos\left(\Delta_\mu a_\nu - \Delta_\nu a_\mu\right) - \sum_{i,\mu}\cos\left(\Delta_\mu \theta - 2a_\mu\right)$$

Critical theory with $\Phi \sim e^{i\theta}$.

$$\mathcal{L} = |(\partial_\mu - 2ia_\mu)\Phi|^2 - |\Phi|^4$$

$+$ relevant monopoles.

Ising* criticality



Higgs state with $\langle \Phi \rangle \neq 0$: The phase of Φ winds by 2π around the cycle of the torus, trapping U(1) flux π in the hole of the torus. This leads to 4-fold degeneracy

Confined phase

Monopoles proliferate

No topological order

g

(N. Read and S.S., 1991; X.-G. Wen, 1991; Bais, Van Driel, and de Wild Propitius 1992; T. Senthil and M.P.A. Fisher, 2000; J. Maldacena, G. Moore, N. Seiberg, 2001)

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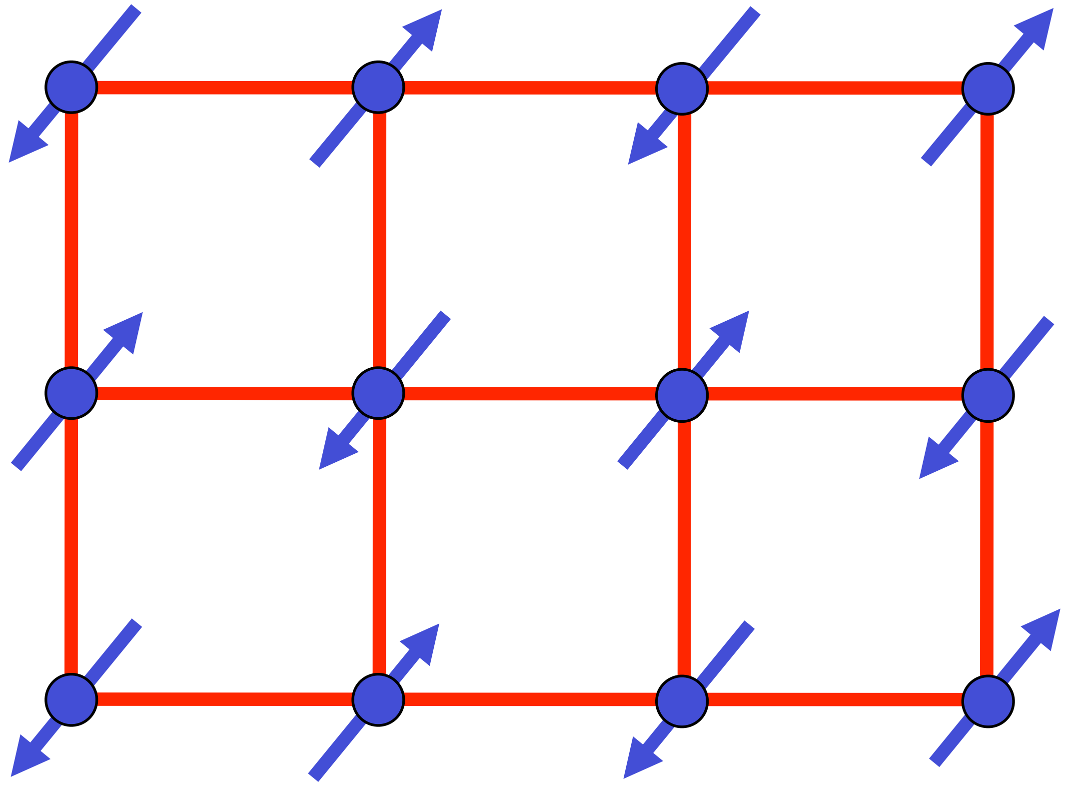


Insulating Antiferromagnet

Néel order parameter $\mathbf{n}(x_i, t) = \eta_i \mathbf{S}_i(t)$, where $\eta_i = \pm 1$ on two sublattices.
O(3) non-linear sigma model:

$$S = \frac{1}{2g} \int d^2x dt \, (\partial_\mu \mathbf{n})^2 \quad , \quad \mathbf{n}^2 = 1.$$

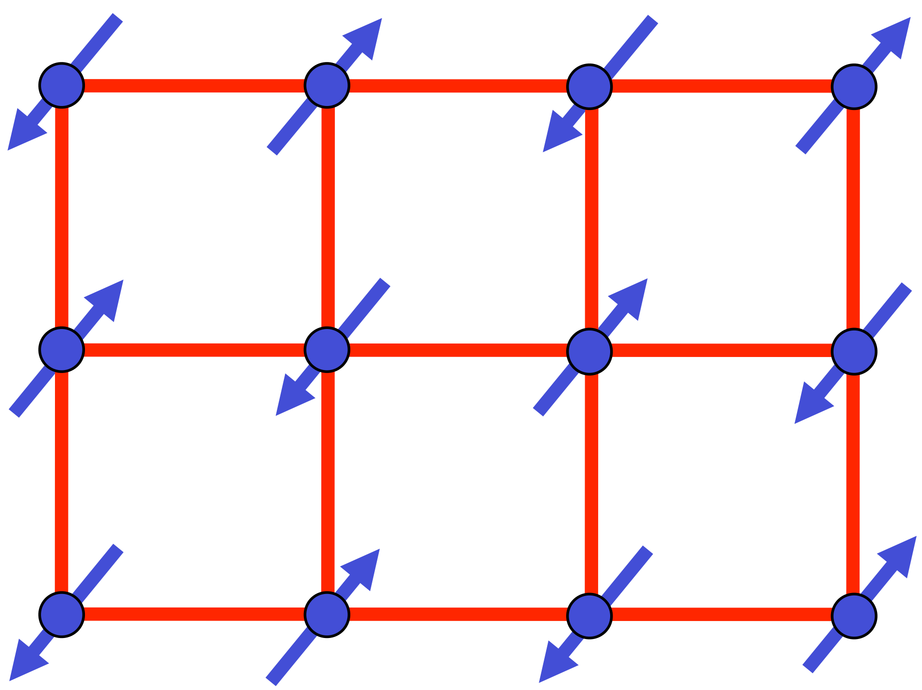
$\mathbb{CP}^1$ model: use $\mathbf{n} = z_\alpha^* \vec{\sigma}_{\alpha\beta} z_\beta$ with $\alpha, \beta = \uparrow, \downarrow$, and then

$$S = \frac{1}{g} \int d^2x dt \, |(\partial_\mu - i a_\mu) z_\alpha|^2 \quad , \quad |z_\alpha|^2 = 1 \, ,$$

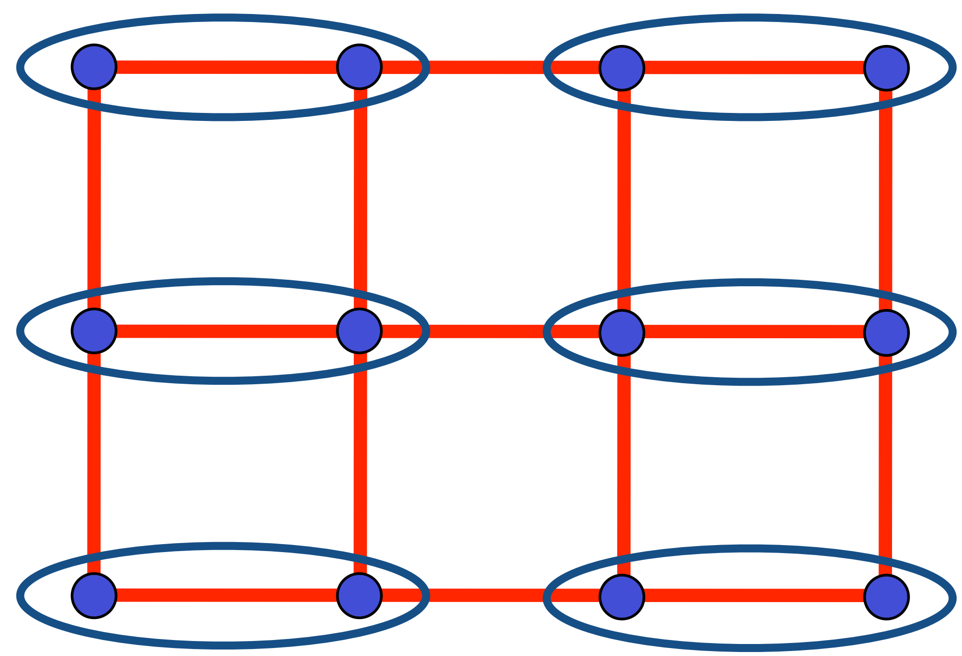
where a_μ is an emergent U(1) gauge field.

Theory for $S = 1/2$ antiferromagnet also has spin Berry phase terms

$$S = \frac{1}{g} \int d^2x dt \left| (\partial_\mu - i a_\mu) z_\alpha \right|^2 + \sum_i \int dt \, \eta_i a_{it}$$



Higgs phase with $\langle z_\alpha \rangle \neq 0$
Néel order wih Nambu-Goldstone
(spin-wave) gapless excitations.



Confined phase with $\langle z_\alpha \rangle = 0$
VBS order

g

(N. Read and S.S., 1989; S.S. and R. Jalabert, 1990)

To obtain a Z_2 deconfined phase, we need to condense a Higgs field with U(1) charge 2. The simplest route is to condense spin-singlet pairs of long-wavelength spinons, z_α. There are two candidates for such Higgs fields, corresponding to the operators

$$P \sim \varepsilon_{\alpha\beta} z_\alpha \partial_t z_\beta \quad , \quad Q_a \sim \varepsilon_{\alpha\beta} z_\alpha \partial_a z_\beta \, ,$$

with $a = x, y$. By gauge-invariance and symmetry, we obtain the following effective action with additional tuning parameters s_1 and s_2

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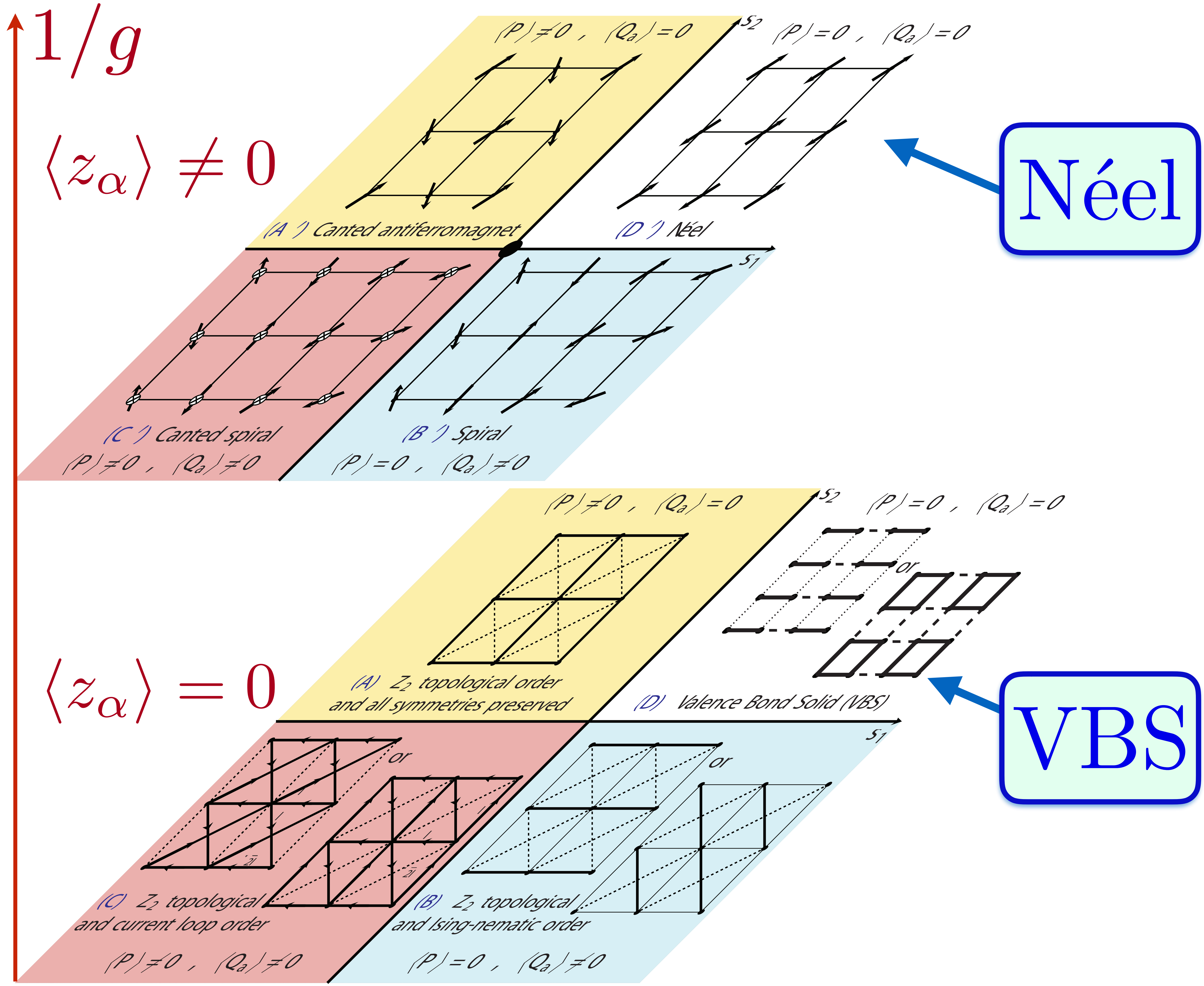
$$\boxed{\begin{aligned} S =& \frac{1}{g} \int d^2x dt \, |(\partial_\mu - i a_\mu) z_\alpha|^2 + \sum_i \int dt \, \eta_i a_{it} \\ & \int d^2x dt \Big[|(\partial_\mu - 2 i a_\mu) P|^2 + |(\partial_\mu - 2 i a_\mu) Q_a|^2 \\ & + \lambda_1 P^* \varepsilon_{\alpha\beta} z_\alpha \partial_t z_\beta + \lambda_2 Q_a^* \cdot \varepsilon_{\alpha\beta} z_\alpha \partial_a z_\beta + \text{H.c.} \\ & - s_1 |P|^2 - s_2 |Q_a|^2 - u_1 |P|^4 - u_2 (|Q_a|^2)^2 + \ldots \Big] \end{aligned}}$$



1/g
⟨z_α⟩ ≠ 0
⟨z_α⟩ = 0
⟨P⟩ ≠ 0 , ⟨Q_a⟩ = 0
s_2
⟨P⟩ = 0 , ⟨Q_a⟩ = 0
(A ') Canted antiferromagnet
(D ') Néel
Néel
s_1
(C ') Canted spiral
⟨P⟩ ≠ 0 , ⟨Q_a⟩ ≠ 0
(B ') Spiral
⟨P⟩ = 0 , ⟨Q_a⟩ ≠ 0
⟨P⟩ ≠ 0 , ⟨Q_a⟩ = 0
s_2
⟨P⟩ = 0 , ⟨Q_a⟩ = 0
or
(A) Z_2 topological order
and all symmetries preserved
(D) Valence Bond Solid (VBS)
VBS
s_1
or
or
(C) Z_2 topological
and current loop order
(B) Z_2 topological
and Ising-nematic order
⟨P⟩ ≠ 0 , ⟨Q_a⟩ ≠ 0
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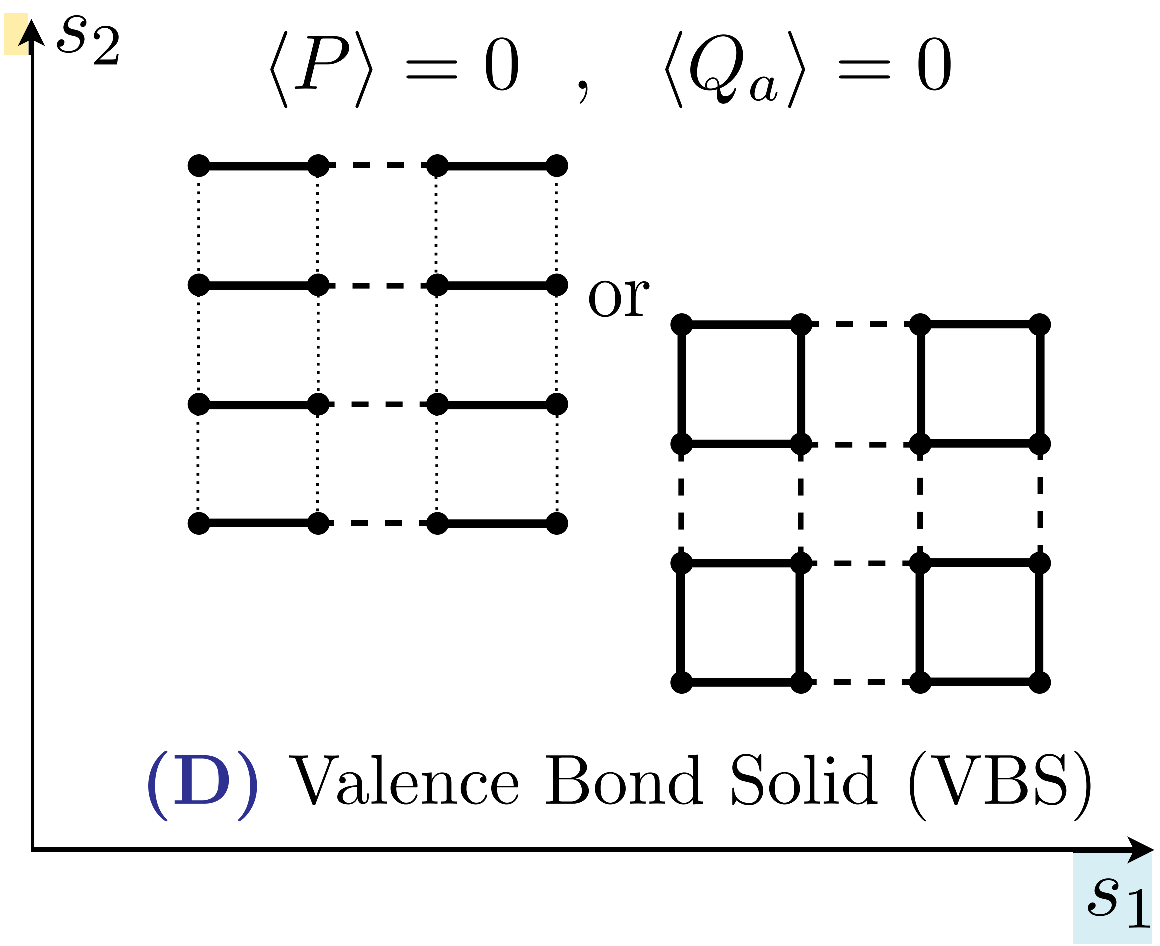
Phase diagram at large g with $\langle z_\alpha \rangle = 0$



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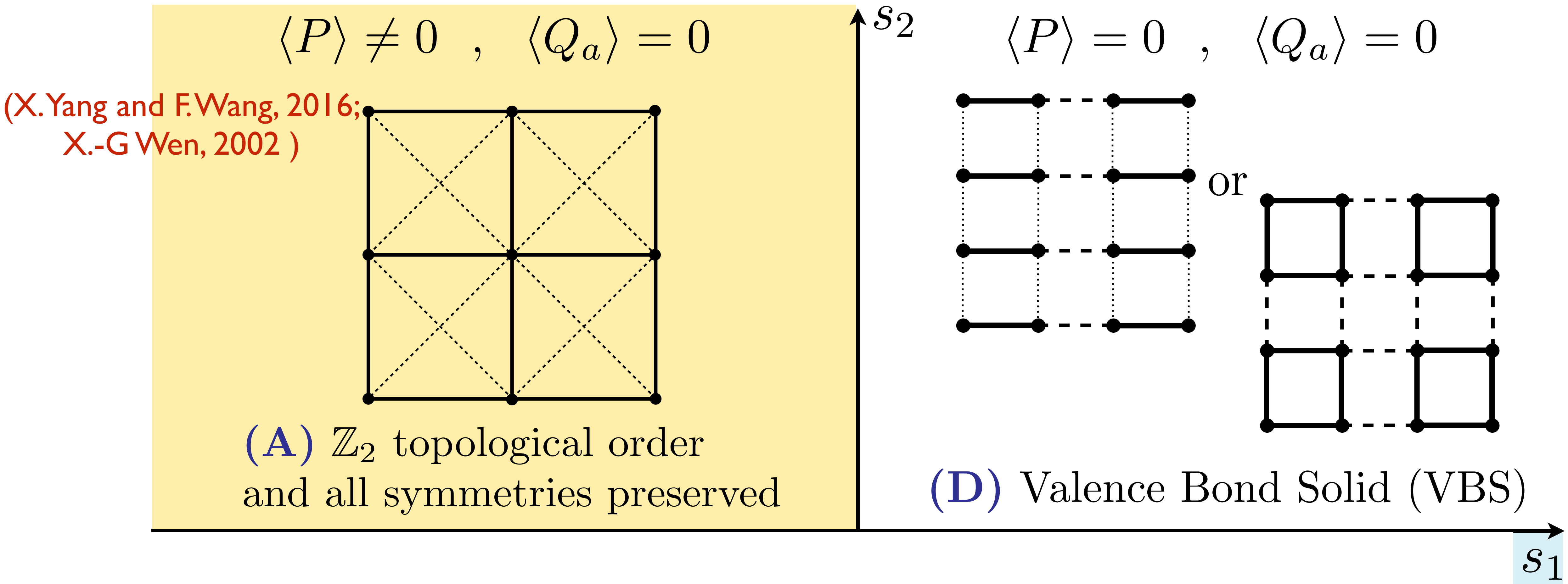

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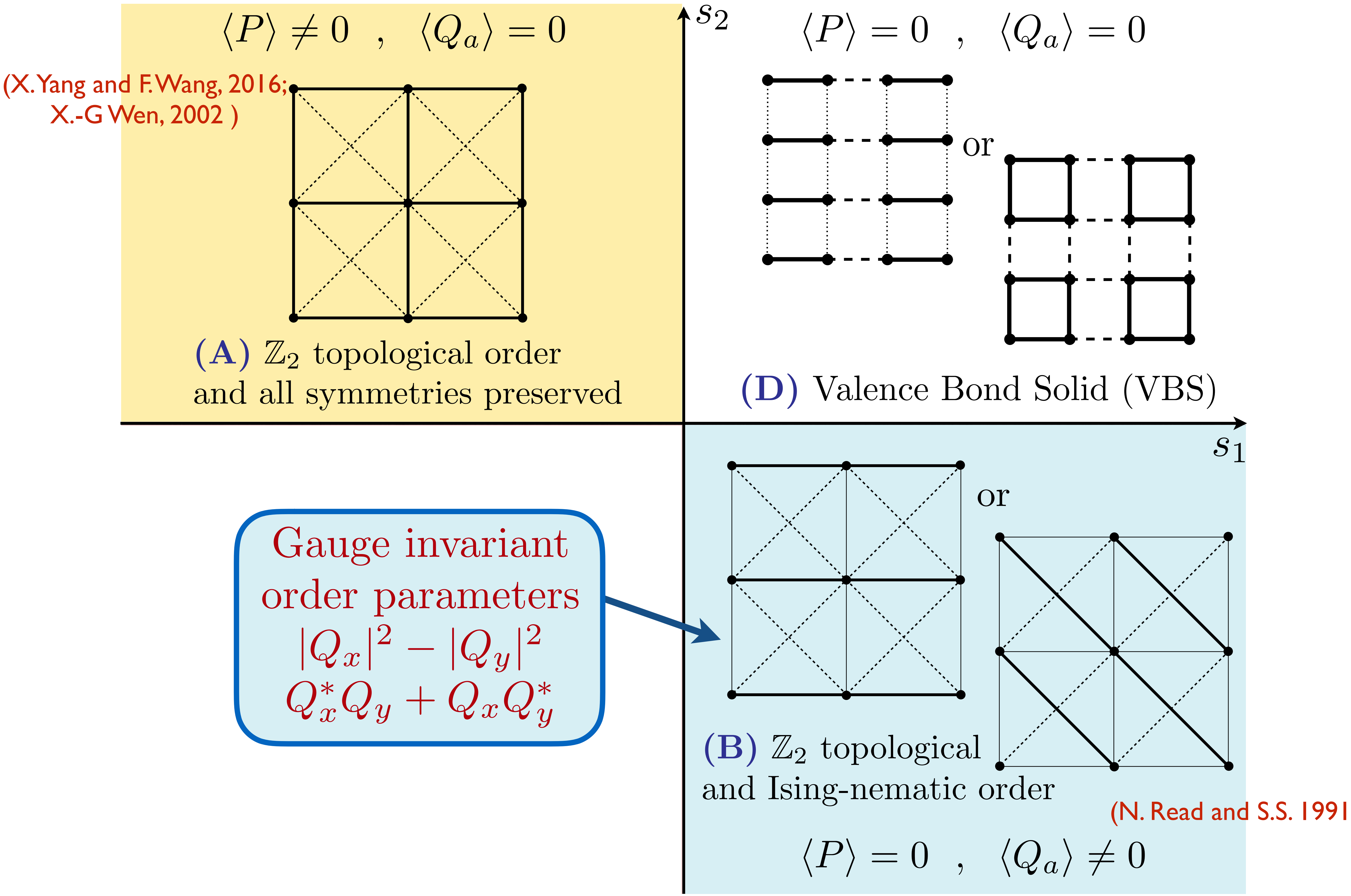
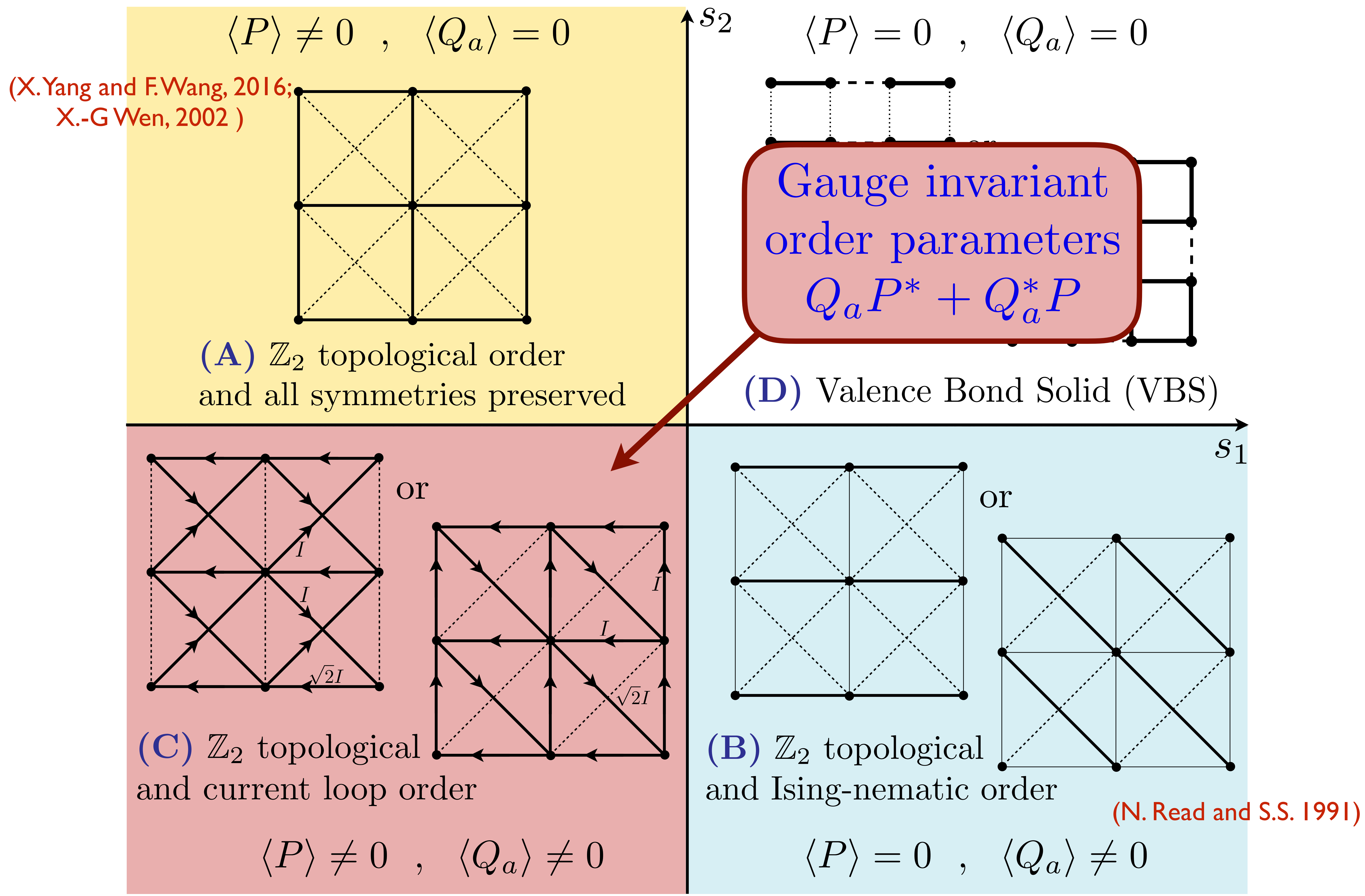


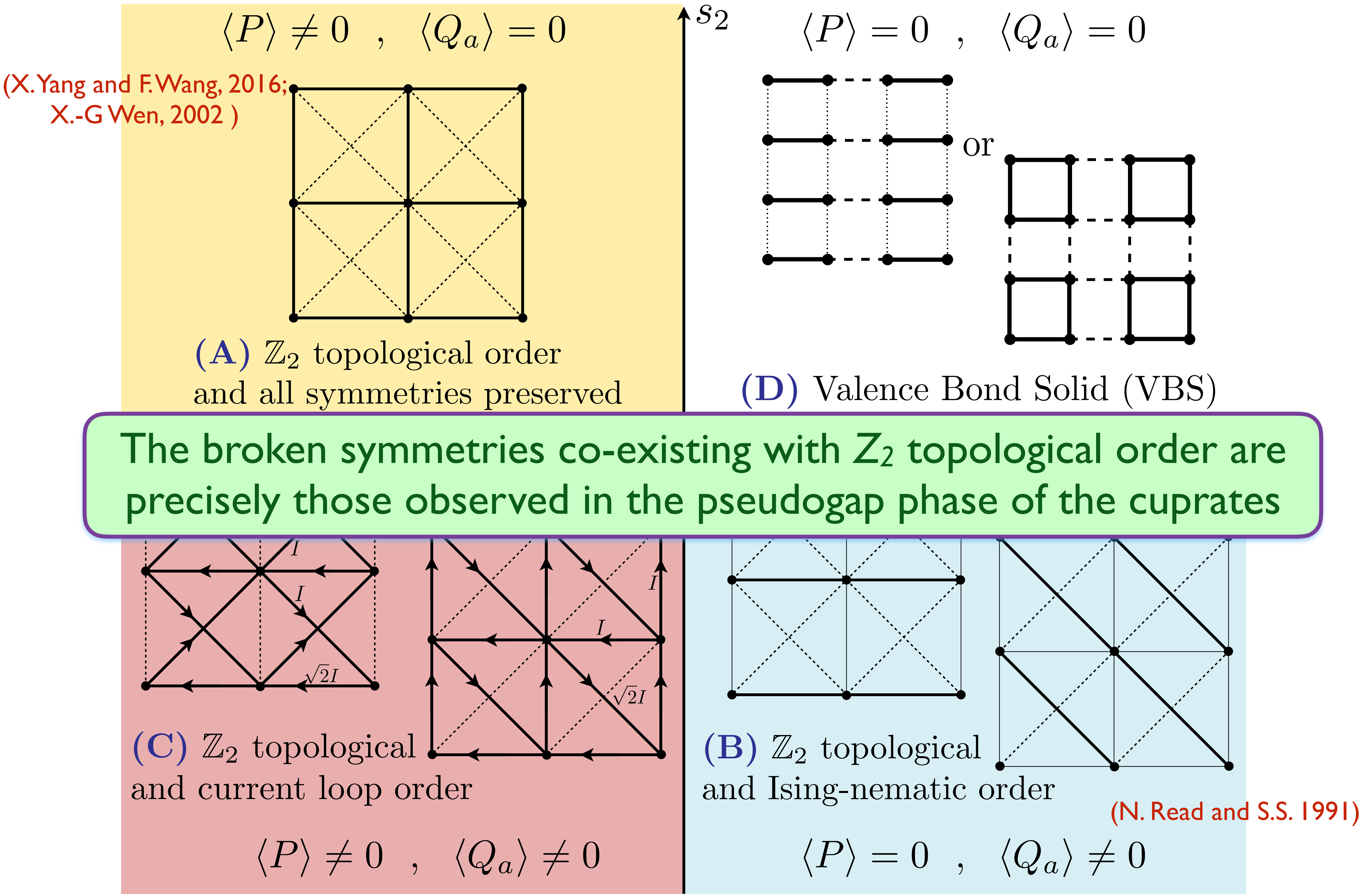
Phase diagram at large g with $\langle z_\alpha \rangle = 0$

Three phases with Z_2 topological order



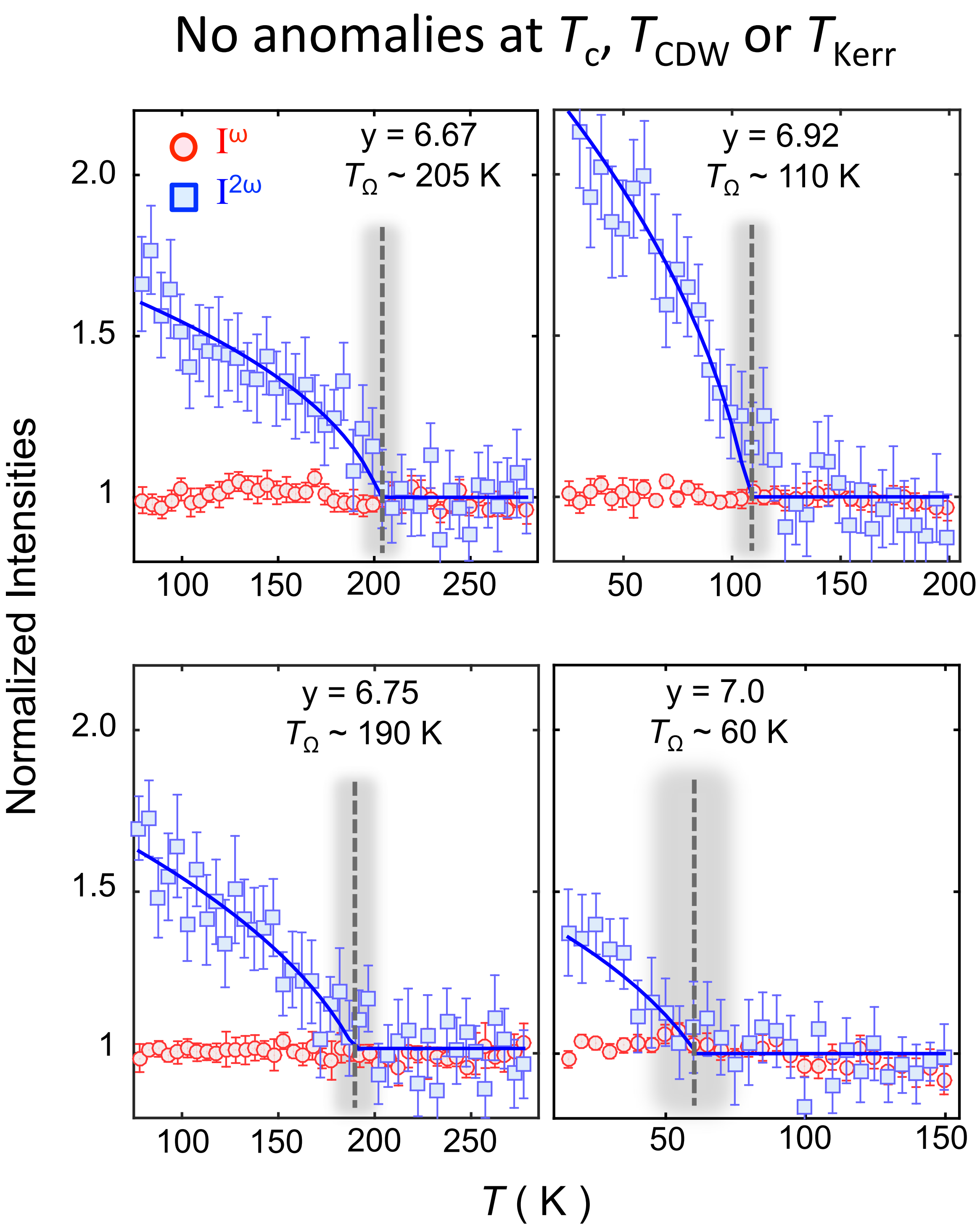
Phase diagram at large g with $\langle z_\alpha \rangle = 0$

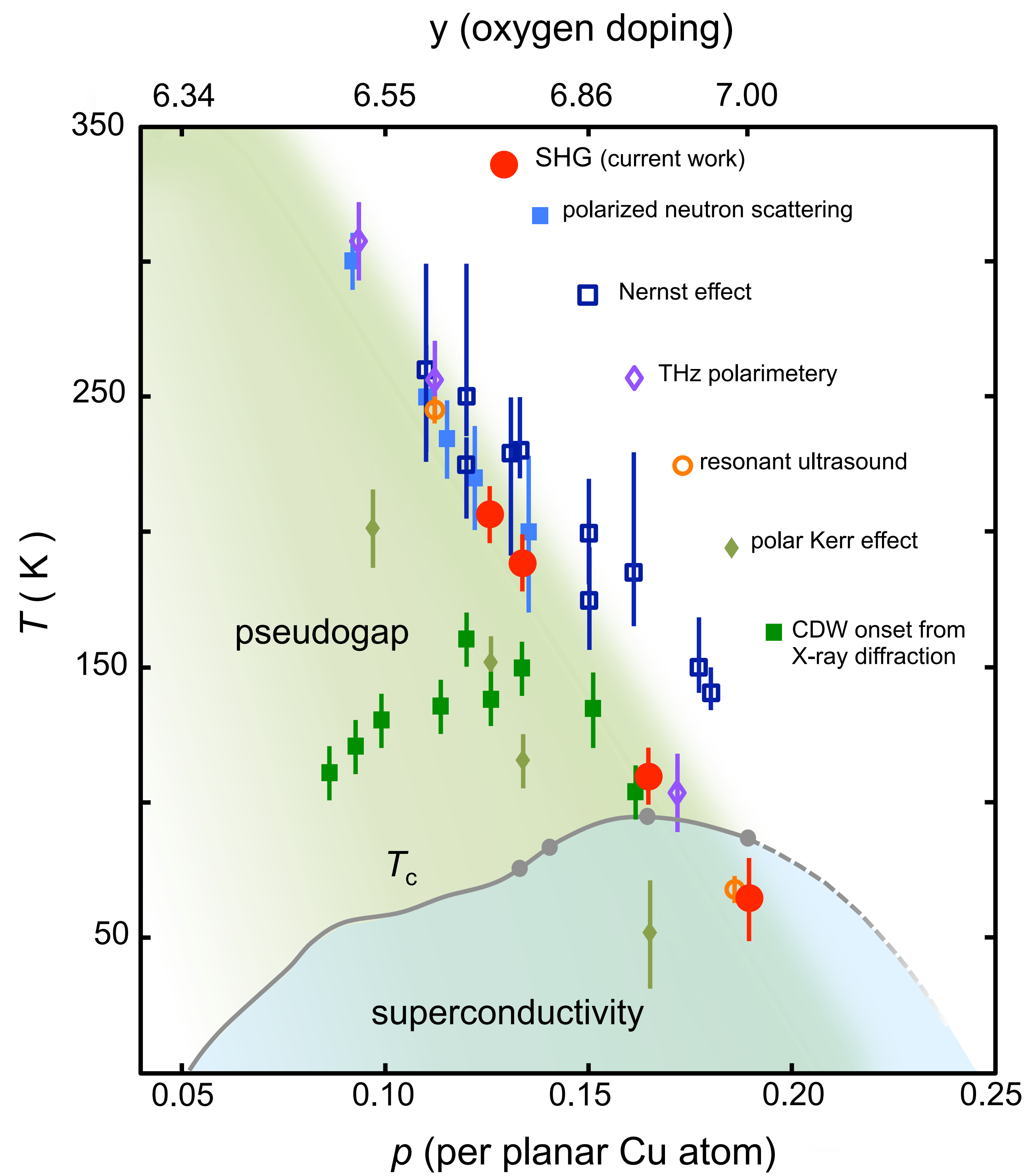
Three phases with Z_2 topological order



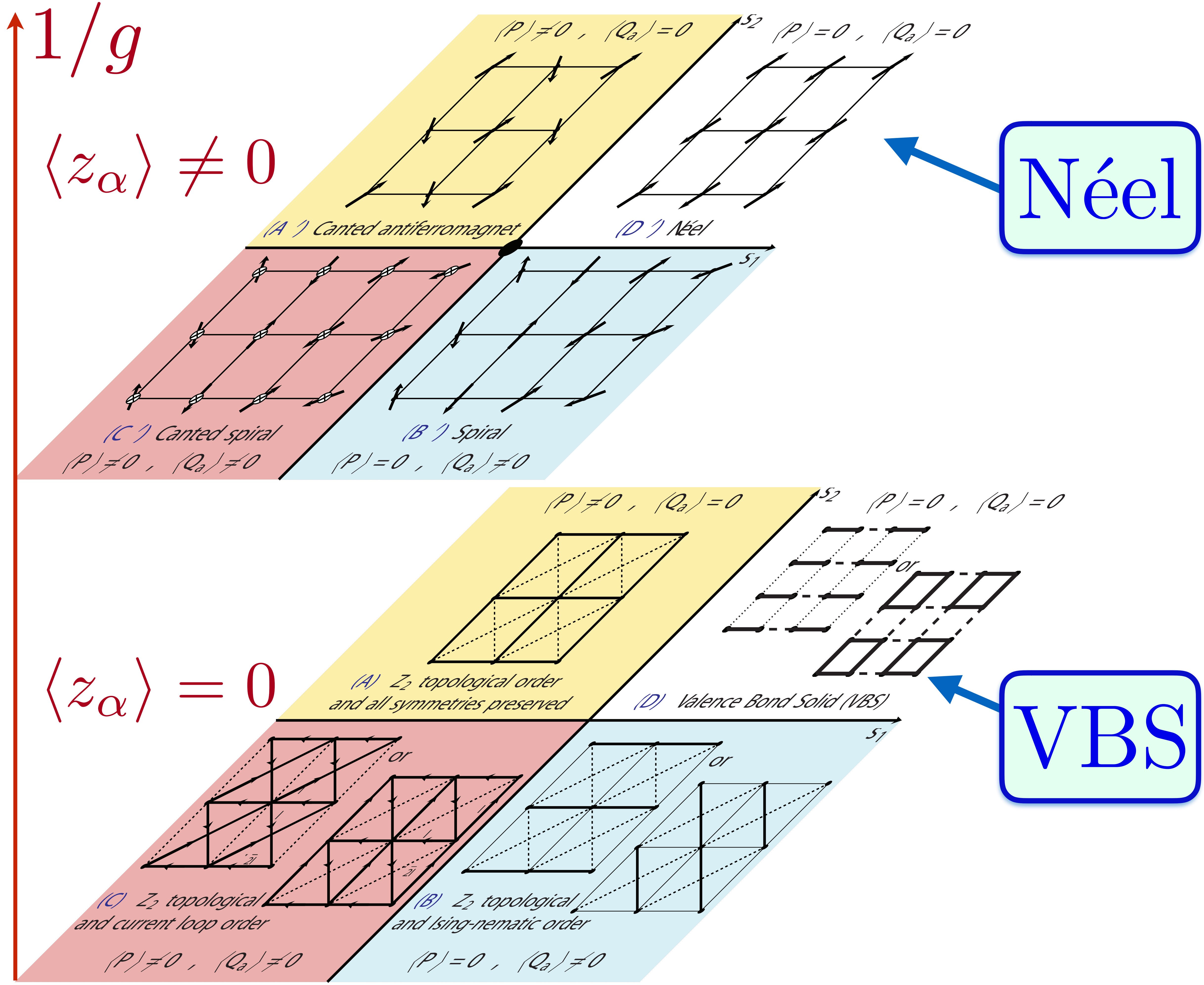
The broken symmetries co-existing with Z_2 topological order are precisely those observed in the pseudogap phase of the cuprates

Broken inversion symmetry below T^* in $YBa_2Cu_3O_y$





L. Zhao, C. A. Belvin, R. Liang, D. A. Bonn, W. N. Hardy, N. P. Armitage, and D. Hsieh, Nature Physics **13**, 250 (2017)



$1/g$
$\langle z_\alpha \rangle \neq 0$
$\langle z_\alpha \rangle = 0$
$\langle P \rangle \neq 0$, $\langle Q_a \rangle = 0$
s_2
$\langle P \rangle = 0$, $\langle Q_a \rangle = 0$
(A ') Canted antiferromagnet
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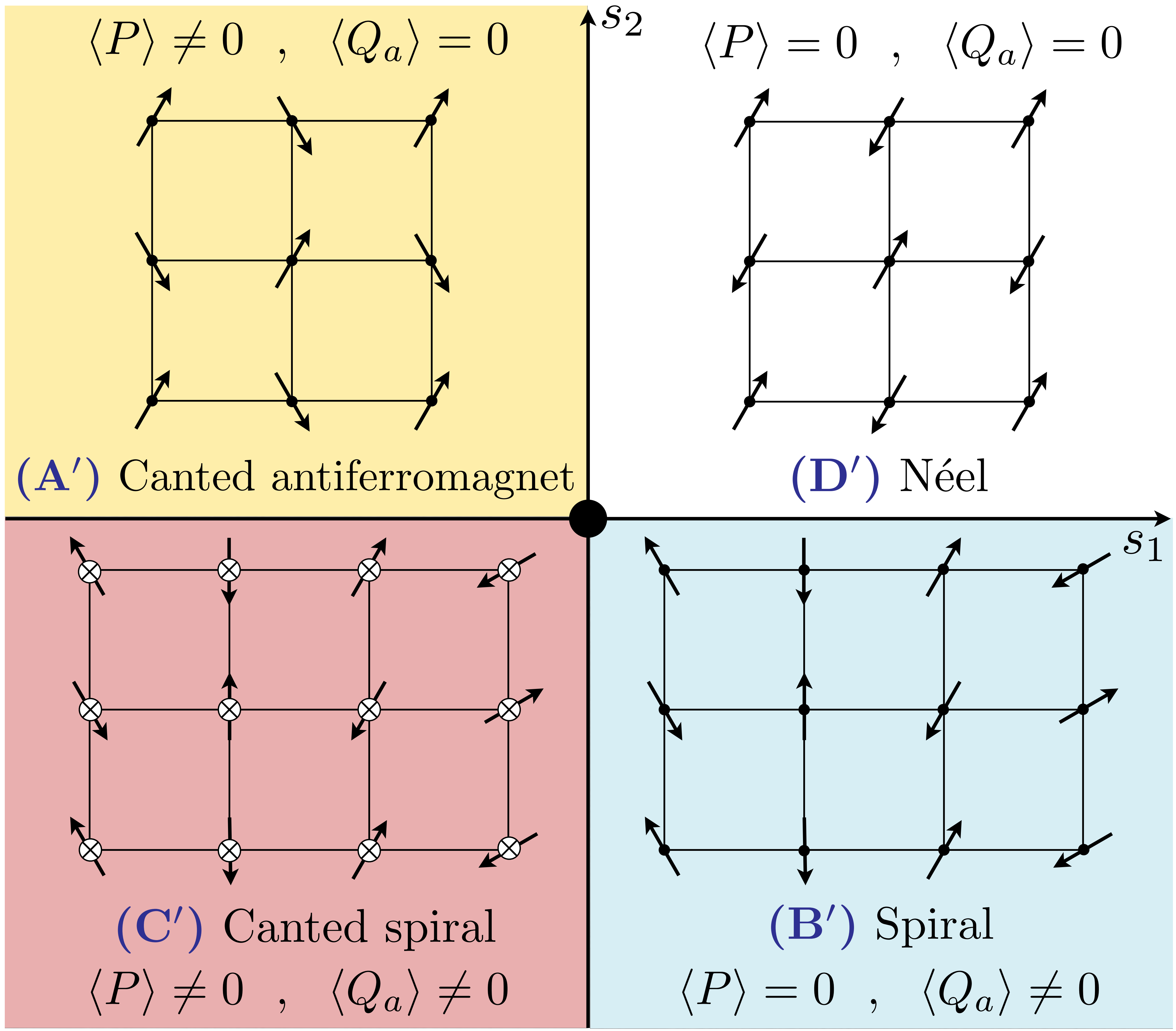
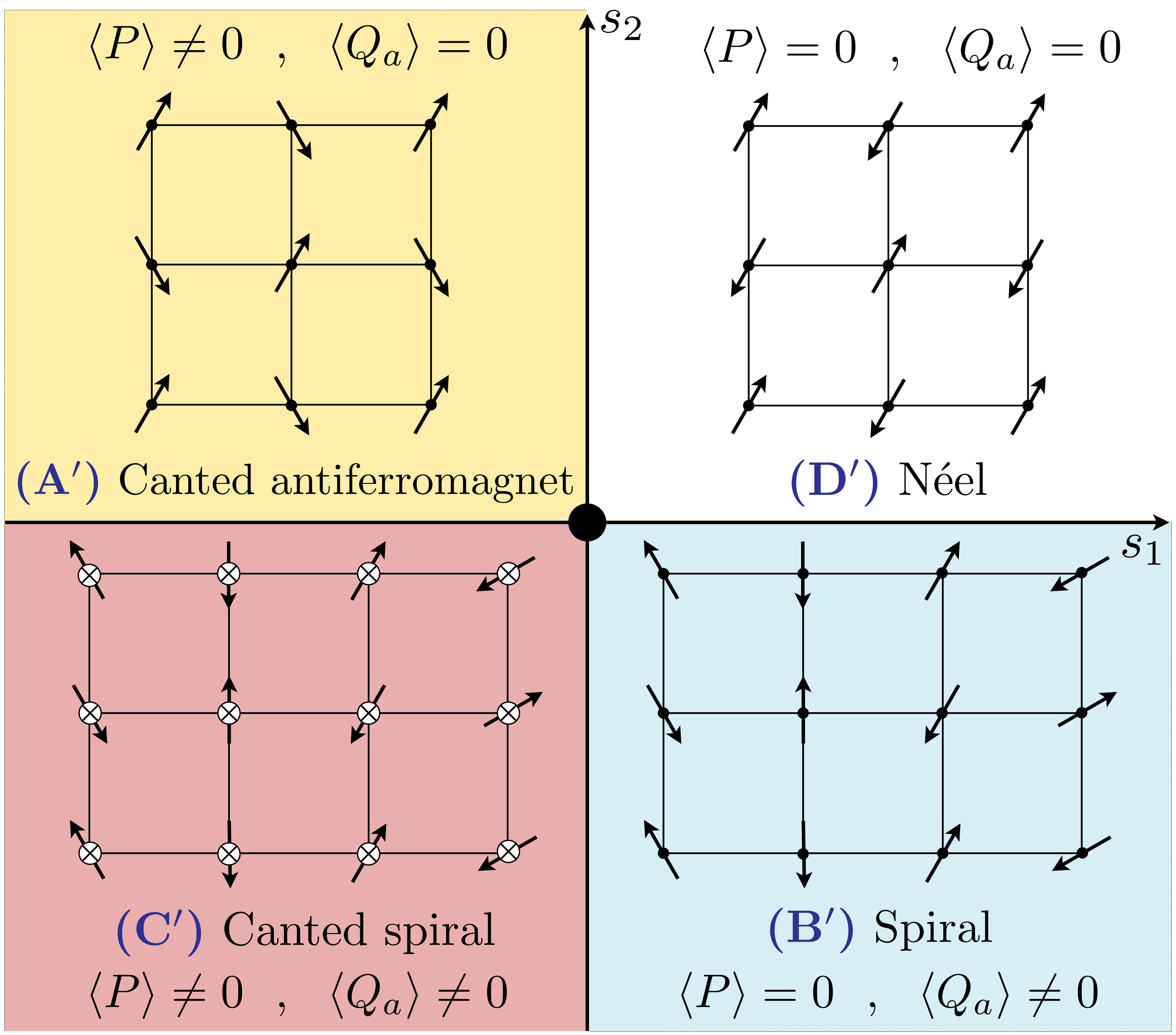
or
(A) Z_2 topological order and all symmetries preserved
(D) Valence Bond Solid (VBS)
VBS
s_1
or
or
(C) Z_2 topological and current loop order
(B) Z_2 topological and Ising-nematic order
$\langle P \rangle \neq 0$, $\langle Q_a \rangle \neq 0$
$\langle P \rangle = 0$, $\langle Q_a \rangle \neq 0$

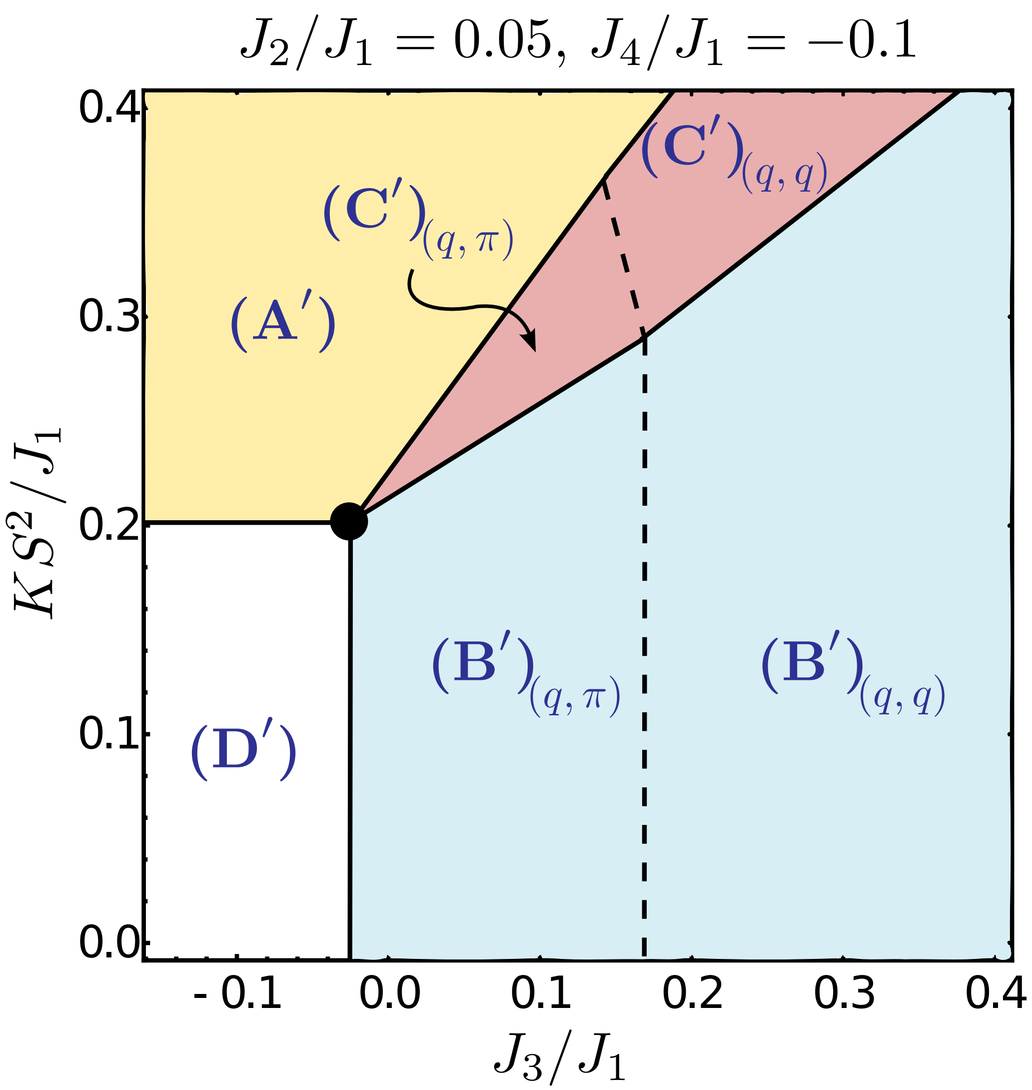
Phase diagram at small g with $\langle z_\alpha \rangle \neq 0$



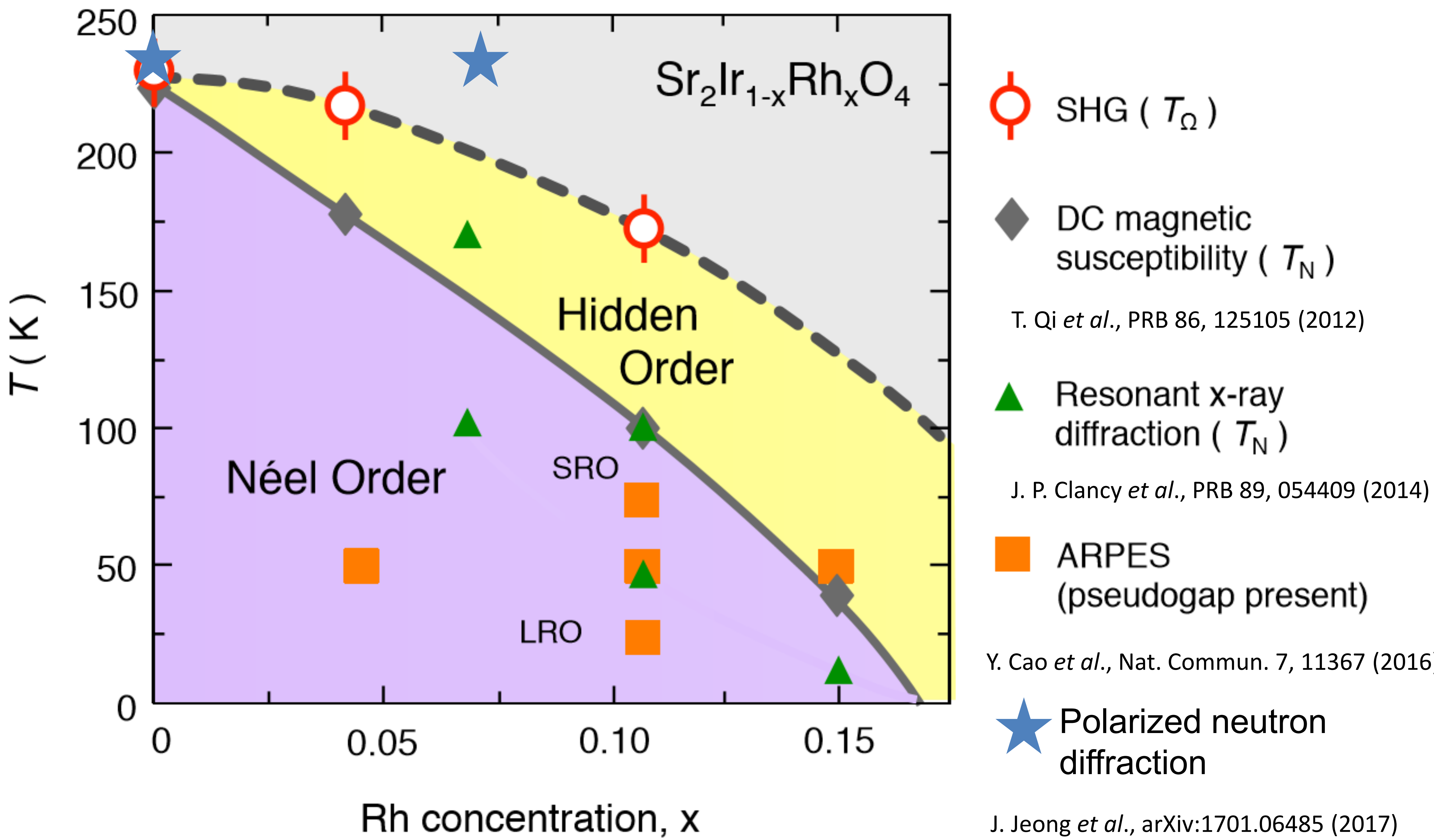
Phase diagram of $\mathbb{CP}^1$ model at small g, coupled to Higgs fields P and Q_a $(a = x, y)$. All phases have $\langle z_\alpha \rangle \neq 0$



Classical phase diagram of square lattice antiferromagnet with near-neighbor exchanges J_1, J_2, J_3, J_4 and ring-exchange K



Phase diagram of $Sr_2Ir_{1-x}Rh_xO_4$



L. Zhao, D. H. Torchinsky, H. Chu, V. Ivanov, R. Lifshitz, R. Flint, T. Qi, G. Cao, and D. Hsieh, Nature Physics **12**, 32 (2016)

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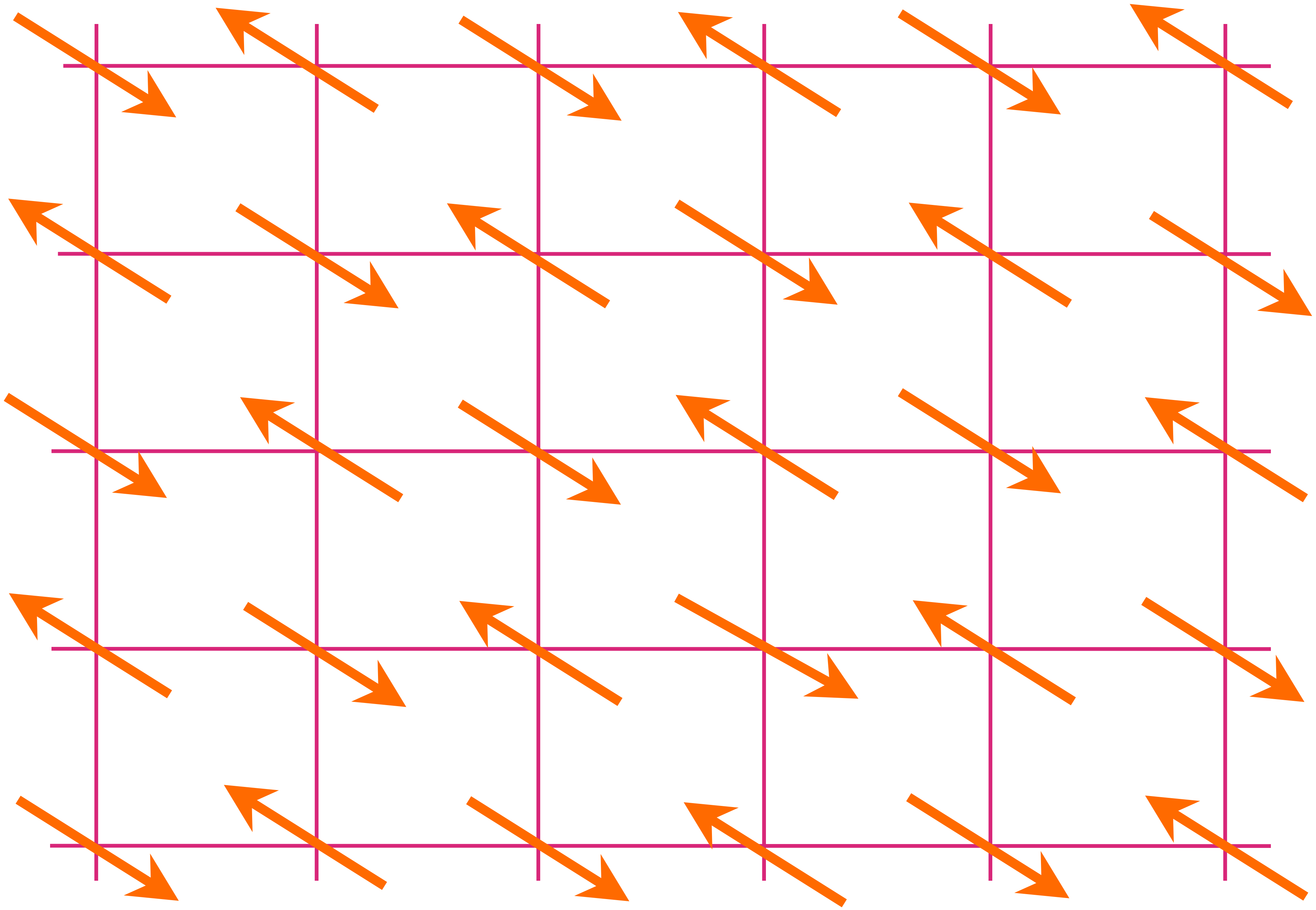
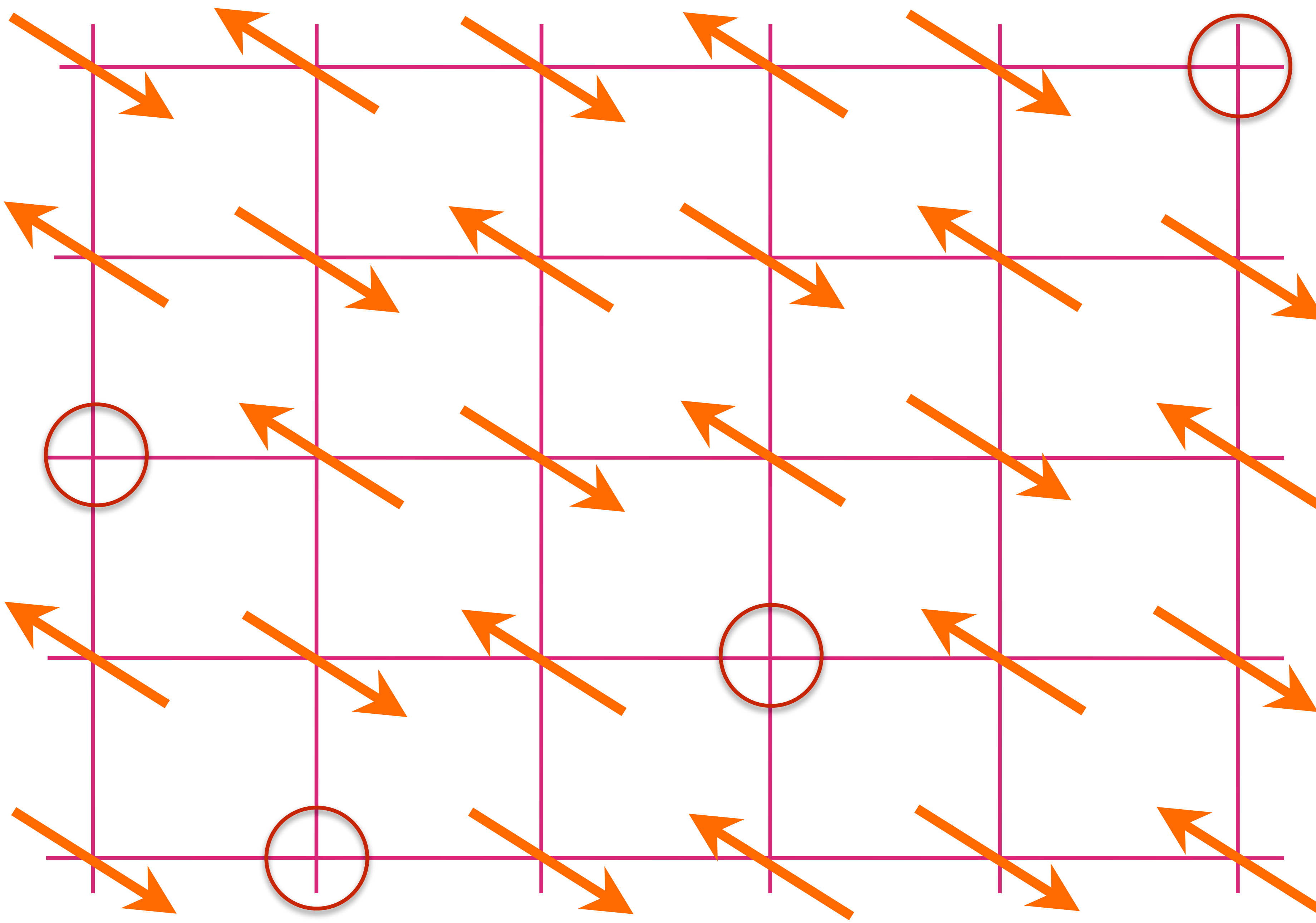
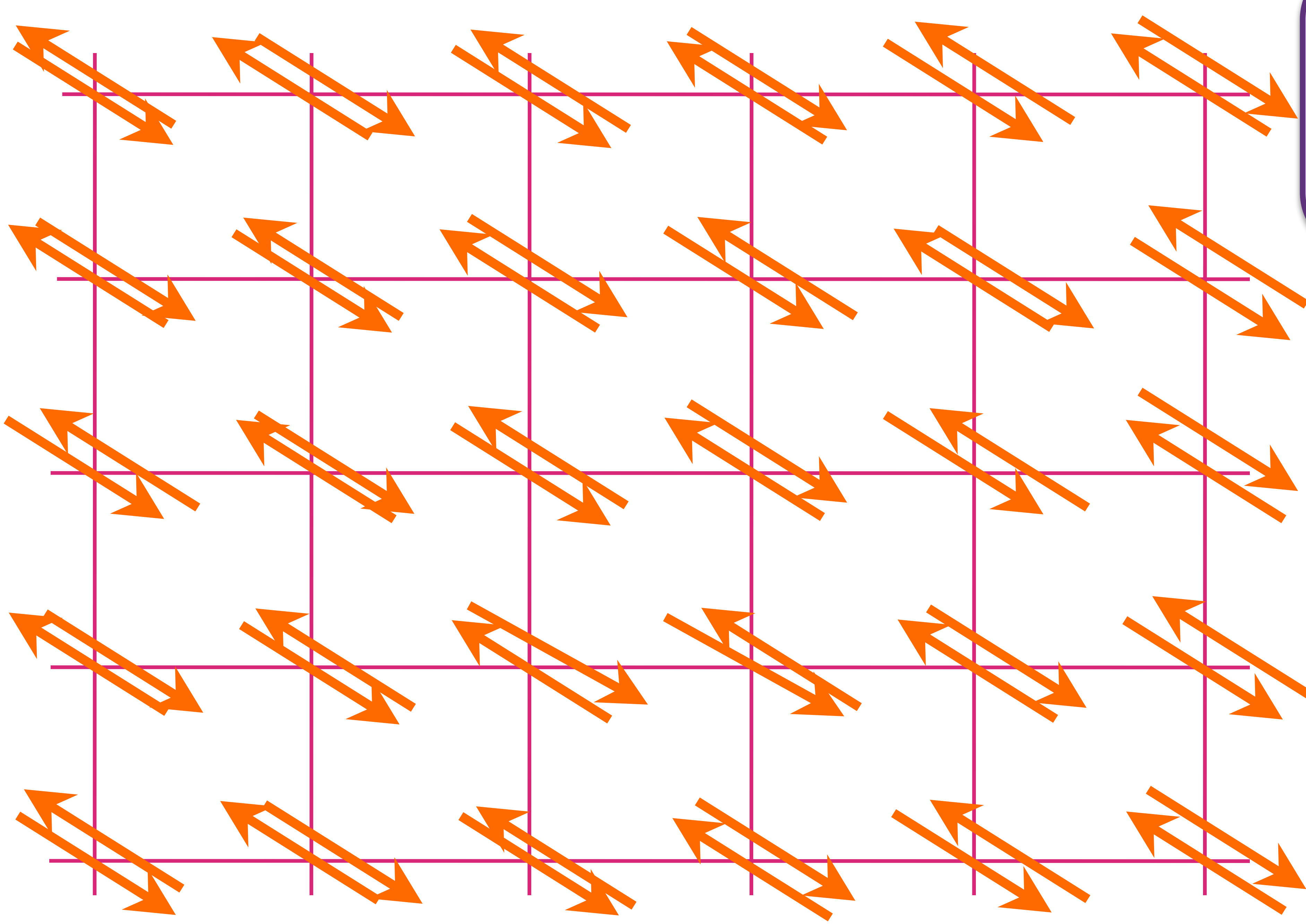


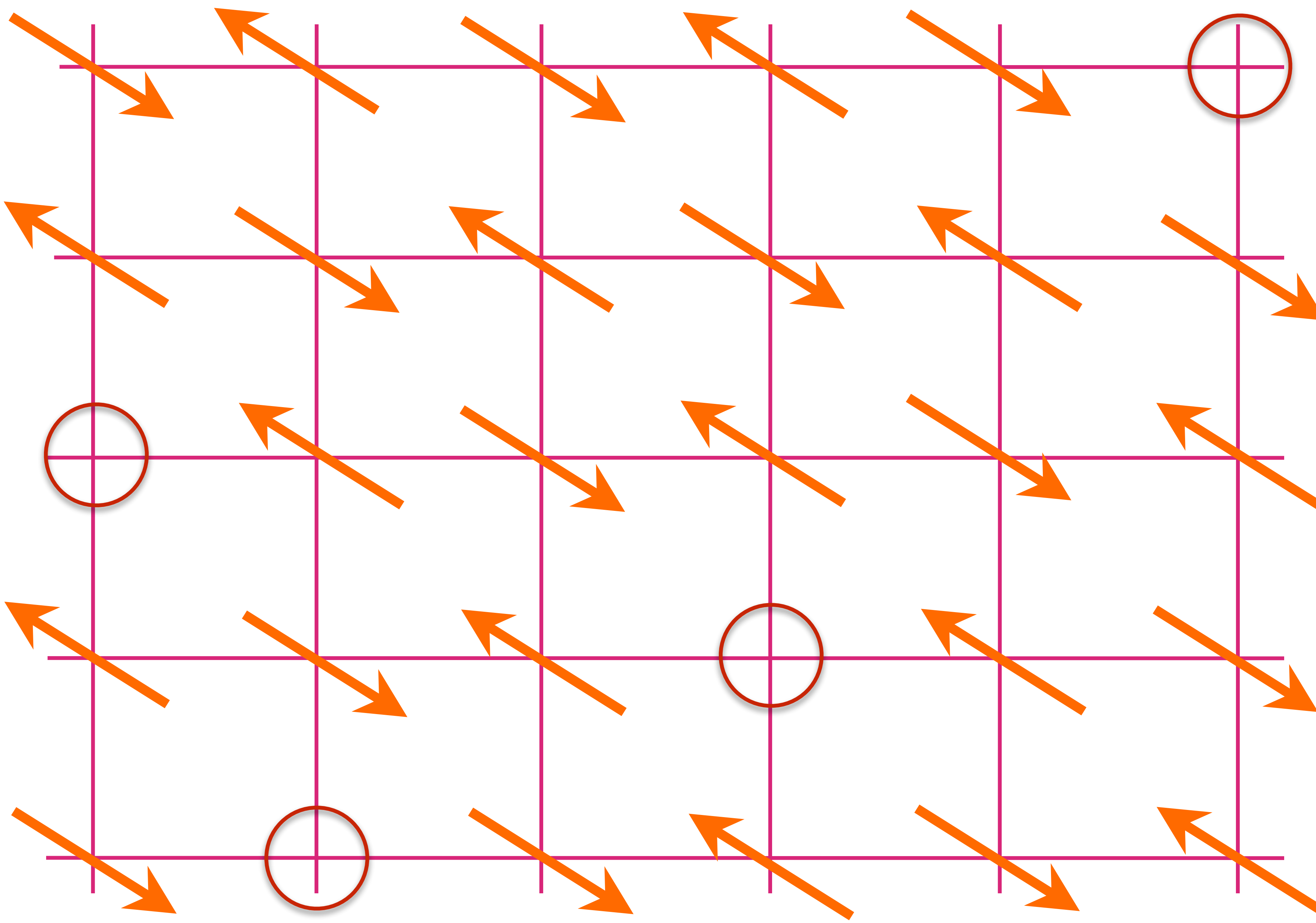
“Undoped” insulating anti-ferromagnet



Anti-ferromagnet with p mobile holes per square

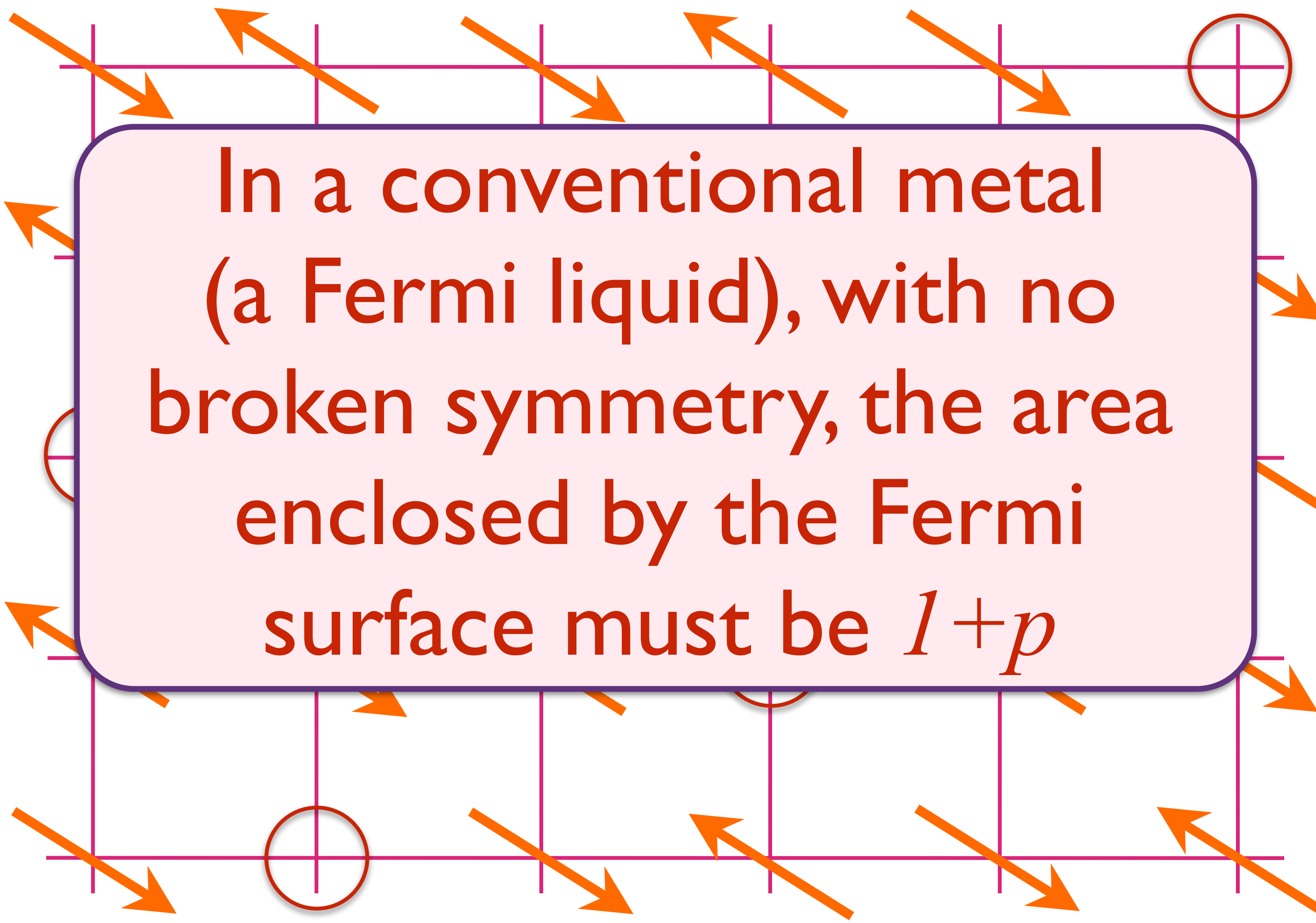


Filled
Band



Anti-ferromagnet with p mobile holes per square

But relative to the band insulator, there are $1+p$ holes per square



In a conventional metal (a Fermi liquid), with no broken symmetry, the area enclosed by the Fermi surface must be $1+p$

Anti-ferromagnet with p mobile holes per square

But relative to the band insulator, there are $1+p$ holes per square

Begin with the "spin-fermion" model. **Electrons** $c_{i\alpha}$ on the square lattice with dispersion

$$\mathcal{H}_c = -\sum_{i,\rho} t_\rho \left(c_{i,\alpha}^\dagger c_{i+\boldsymbol{v}_\rho,\alpha} + c_{i+\boldsymbol{v}_\rho,\alpha}^\dagger c_{i,\alpha} \right) - \mu \sum_i c_{i,\alpha}^\dagger c_{i,\alpha} + \mathcal{H}_{\rm int}$$

are coupled to an **antiferromagnetic order parameter** $\Phi^\ell(i)$**,** $\ell = x, y, z$

$$\mathcal{H}_{\rm int} = -\lambda \sum_i \eta_i \Phi^\ell(i) c_{i,\alpha}^\dagger \sigma_{\alpha\beta}^\ell c_{i,\beta} + V_\Phi$$

where $\eta_i = \pm 1$ on the two sublattices.

When $\Phi^\ell(i)$ =constant independent of i, we have long-range AFM, and a gap in the fermion spectrum at the anti-nodes.

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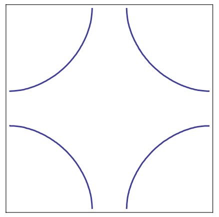

Fermi surface size $1+p$

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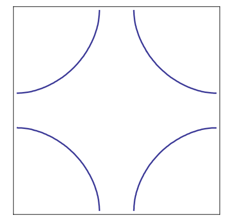
When $\Phi^\ell(i)$ =constant independent of i, we have long-range AFM, and a gap in the fermion spectrum at the anti-nodes.

Begin with the "spin-fermion" model. **Electrons** $c_{i\alpha}$ on the square lattice with dispersion

$$\mathcal{H}_c = -\sum_{i,\rho} t_\rho \left(c^\dagger_{i,\alpha} c_{i+\boldsymbol{v}_\rho,\alpha} + c^\dagger_{i+\boldsymbol{v}_\rho,\alpha} c_{i,\alpha} \right) - \mu \sum_i c^\dagger_{i,\alpha} c_{i,\alpha} + \mathcal{H}_{\rm int}$$

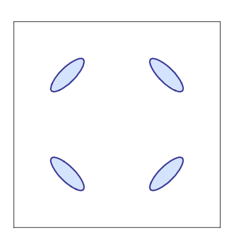


are coupled to an **antiferromagnetic order parameter** $\Phi^\ell(i)$**,** $\ell = x, y, z$

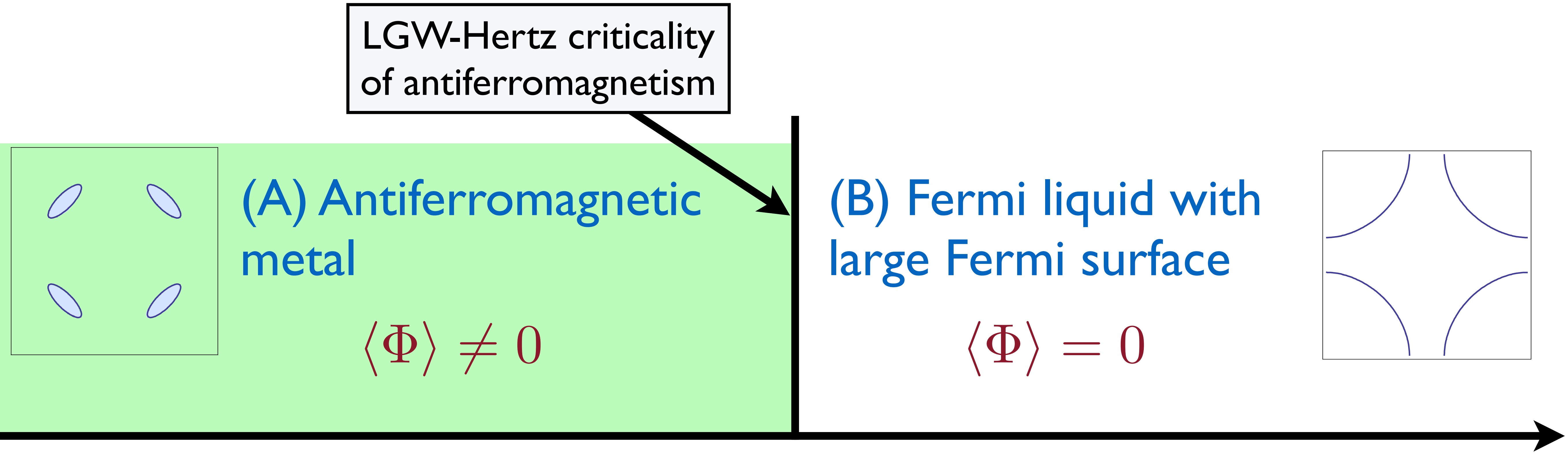
$$\mathcal{H}_{\rm int} = -\lambda \sum_i \eta_i \Phi^\ell(i) c^\dagger_{i,\alpha} \sigma^\ell_{\alpha\beta} c_{i,\beta} + V_\Phi$$

where $\eta_i = \pm 1$ on the two sublattices.

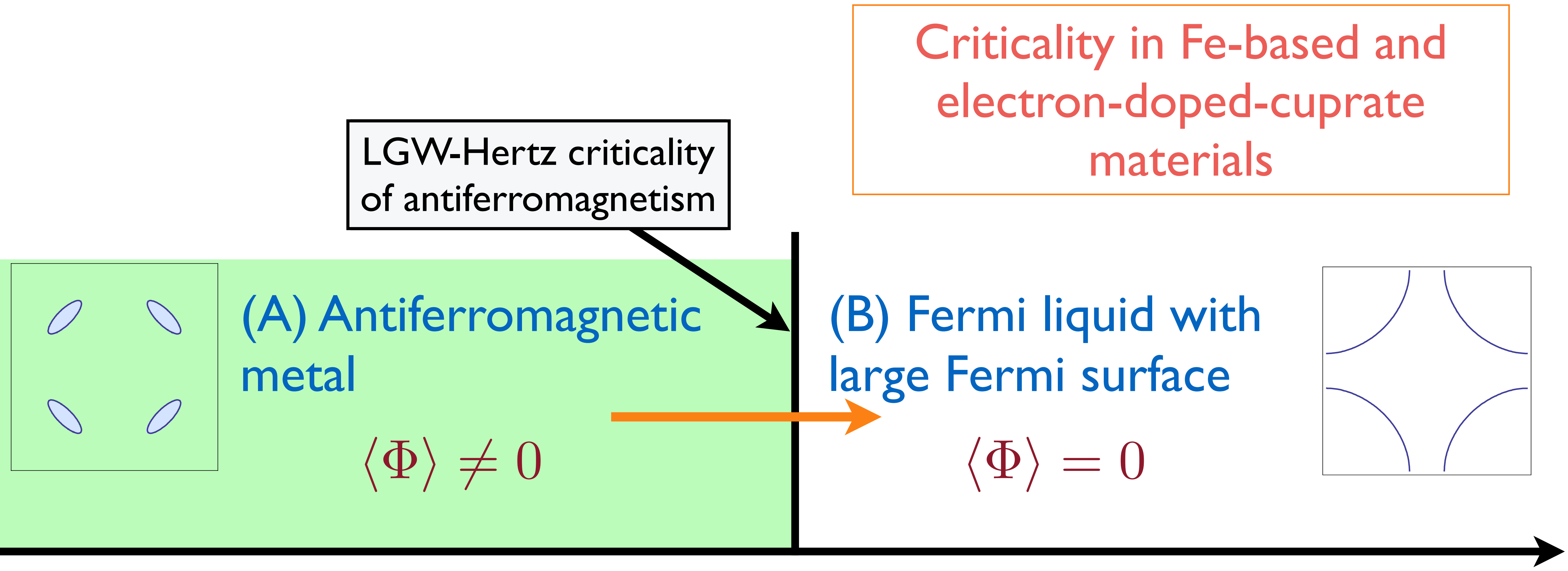
When $\Phi^\ell(i)$ =constant independent of i, we have long-range AFM, and a gap in the fermion spectrum at the anti-nodes.



Fermi surface size p



LGW-Hertz criticality
of antiferromagnetism
(A) Antiferromagnetic
metal
$\langle \Phi \rangle \neq 0$
(B) Fermi liquid with
large Fermi surface
$\langle \Phi \rangle = 0$



Criticality in Fe-based and electron-doped-cuprate materials
LGW-Hertz criticality of antiferromagnetism
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Can we get a stable zero temperature state with "fluctuating antiferromagnetism" and a Fermi surface of size p *i.e.* a pseudogap metal ?

Yes, provided the metal has topological order

T. Senthil, M. Vojta and S. Sachdev, PRB **69**, 035111 (2004)

For fluctuating antiferromagnetism, we transform to a rotating reference frame using the SU(2) rotation R_i

$$\left(\begin{array}{c} c_{i\uparrow} \\ c_{i\downarrow} \end{array} \right) = R_i \left(\begin{array}{c} \psi_{i,+} \\ \psi_{i,-} \end{array} \right),$$

in terms of fermionic "chargons" ψ_s and a **Higgs field** $H^a(i)$

$$\sigma^\ell \Phi^\ell(i) = R_i \, \sigma^a H^a(i) \, R_i^\dagger$$

The Higgs field is the AFM order in the rotating reference frame.

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Note that this representation is ambiguous up to a SU(2) gauge transformation, V_i

$$\left(\begin{array}{c} \psi_{i,+} \\ \psi_{i,-} \end{array} \right) \rightarrow V_i \left(\begin{array}{c} \psi_{i,+} \\ \psi_{i,-} \end{array} \right)$$

$$R_i \rightarrow R_i V_i^\dagger$$

$$\sigma^a H^a(i) \rightarrow V_i \, \sigma^b H^b(i) \, V_i^\dagger .$$

Fluctuating antiferromagnetism

The simplest effective Hamiltonian for the fermionic chargons is the same as that for the electrons, with the AFM order replaced by the Higgs field.

$$\mathcal{H}_\psi = -\sum_{i,\rho} t_\rho \left(\psi^\dagger_{i,s} \psi_{i+\boldsymbol{v}_\rho,s} + \psi^\dagger_{i+\boldsymbol{v}_\rho,s} \psi_{i,s} \right) - \mu \sum_i \psi^\dagger_{i,s} \psi_{i,s} + \mathcal{H}_{\rm int}$$

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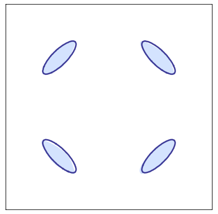
IF we can transform to a rotating reference frame in which $H^a(i) =$ a constant independent of i and time, **THEN** the ψ fermions in the presence of fluctuating AFM will inherit the anti-nodal gap of the electrons in the presence of static AFM.



Fermi surface size p

Fluctuating antiferromagnetism

We cannot always find a single-valued SU(2) rotation R_i to make the Higgs field $H^a(i)$ a constant !

A.V. Chubukov, T. Senthil and S. Sachdev, PRL **72**, 2089 (1994);
S. Sachdev, E. Berg, S. Chatterjee, and Y. Schattner, PRB **94**, 115147 (2016)

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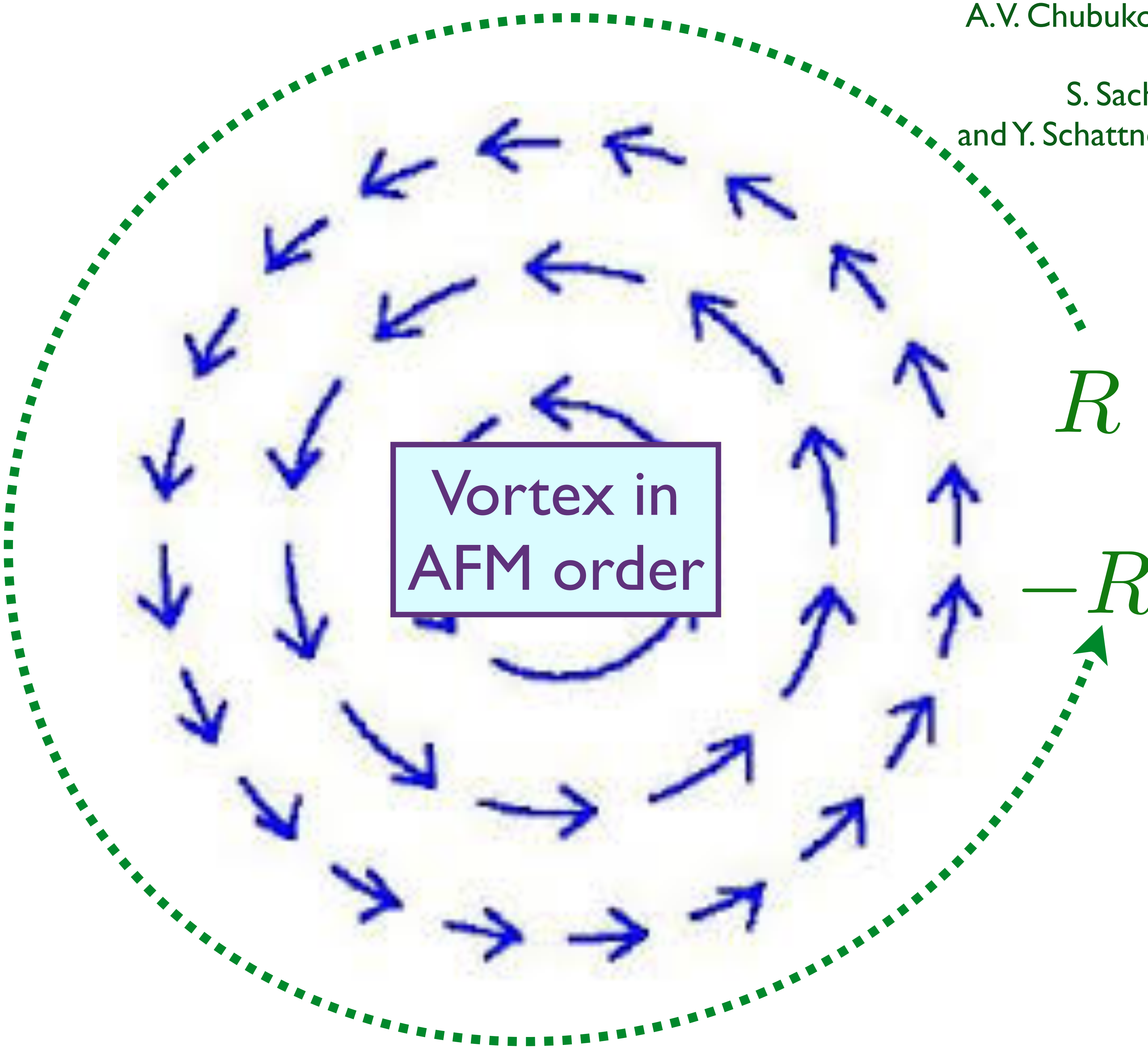

Topological order

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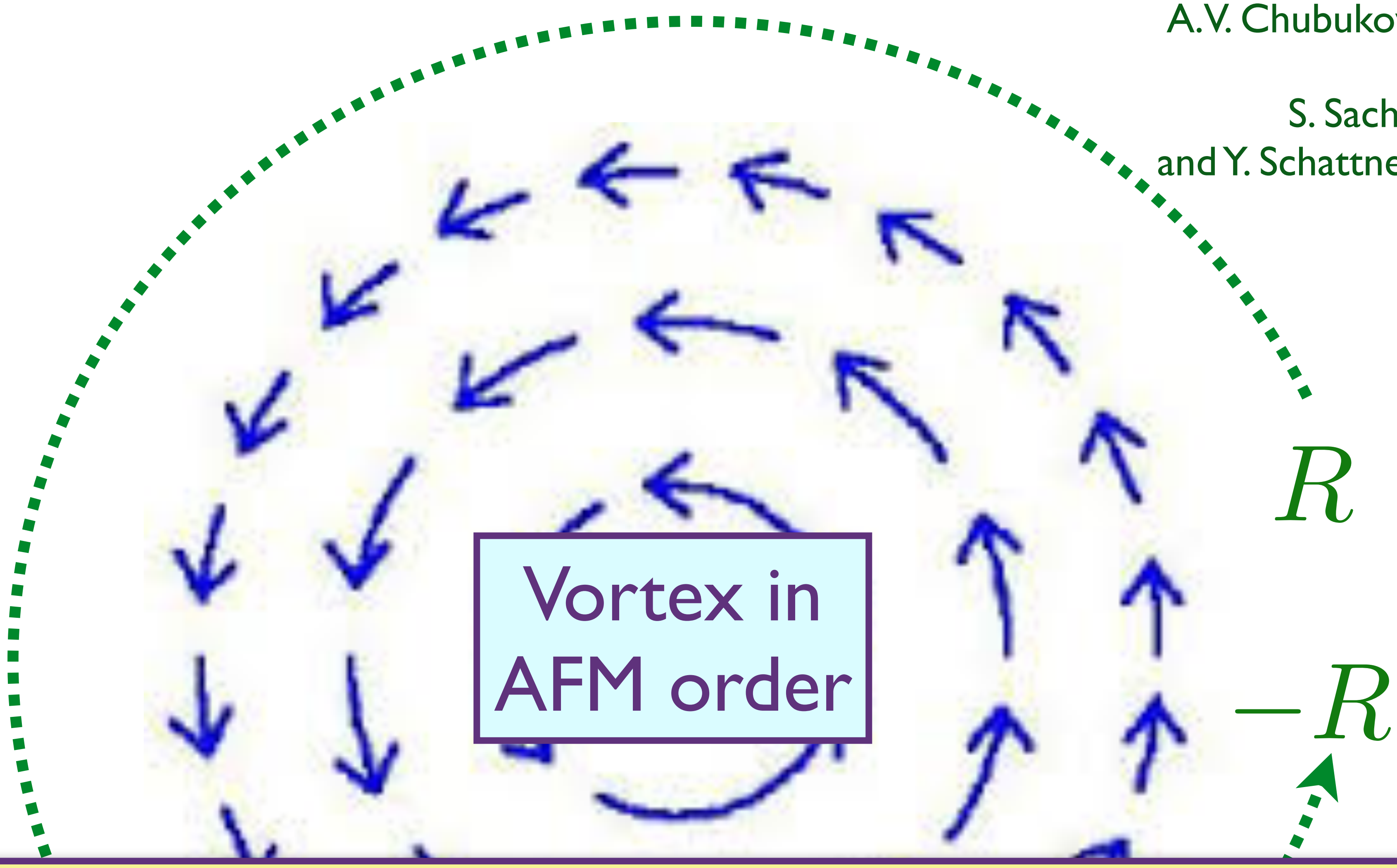
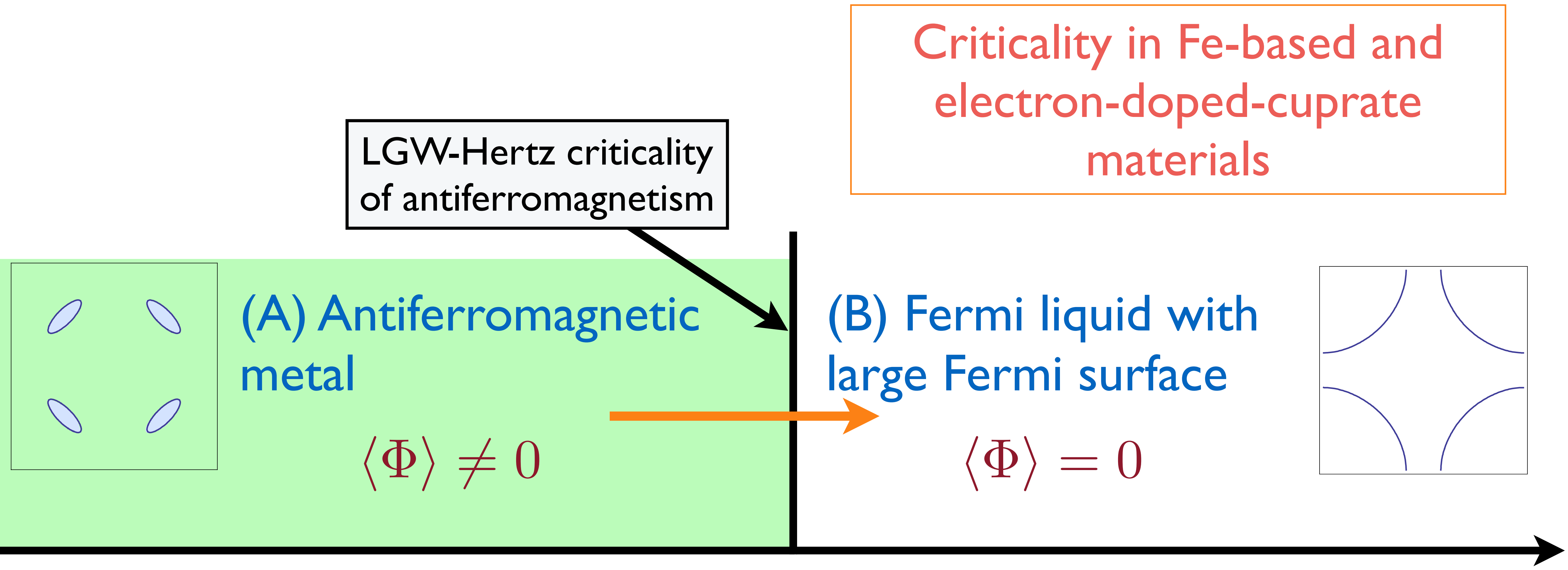


The Higgs field $\in$ SO(3).
Vortices associated with $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$ must be suppressed: such a metal with "fluctuating antiferromagnetism" has $\mathbb{Z}_2$ TOPOLOGICAL ORDER and a pseudogap.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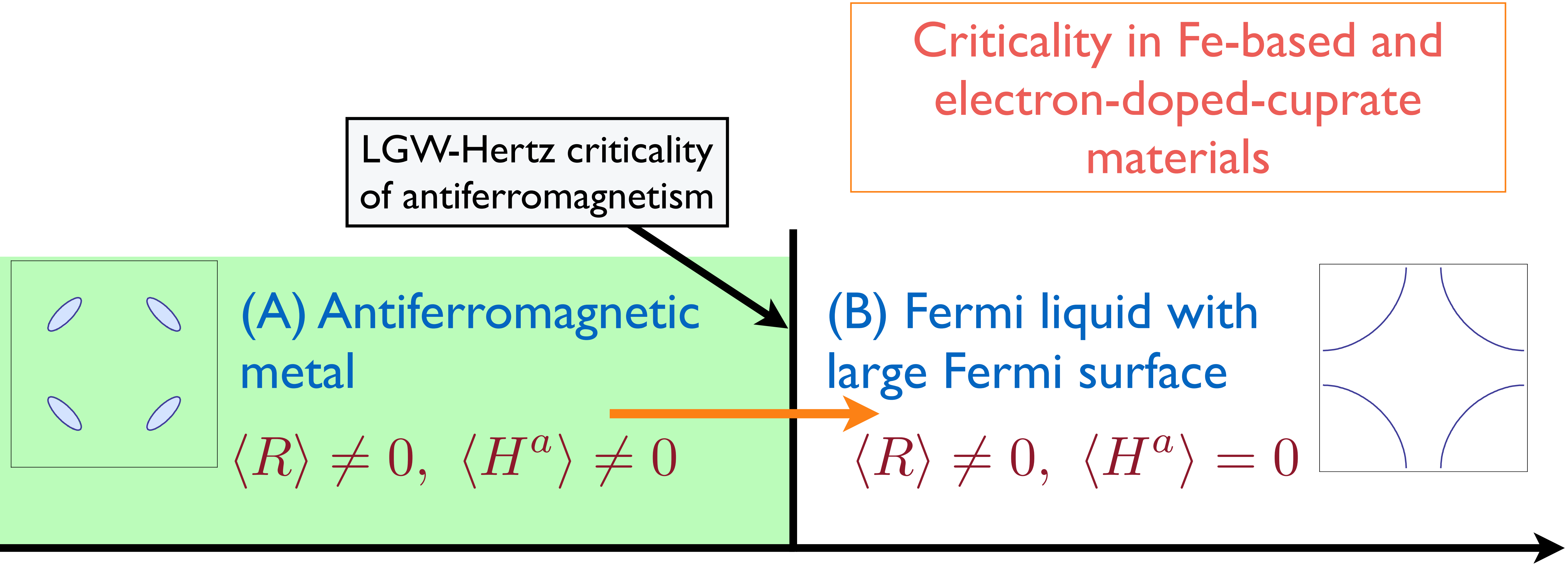
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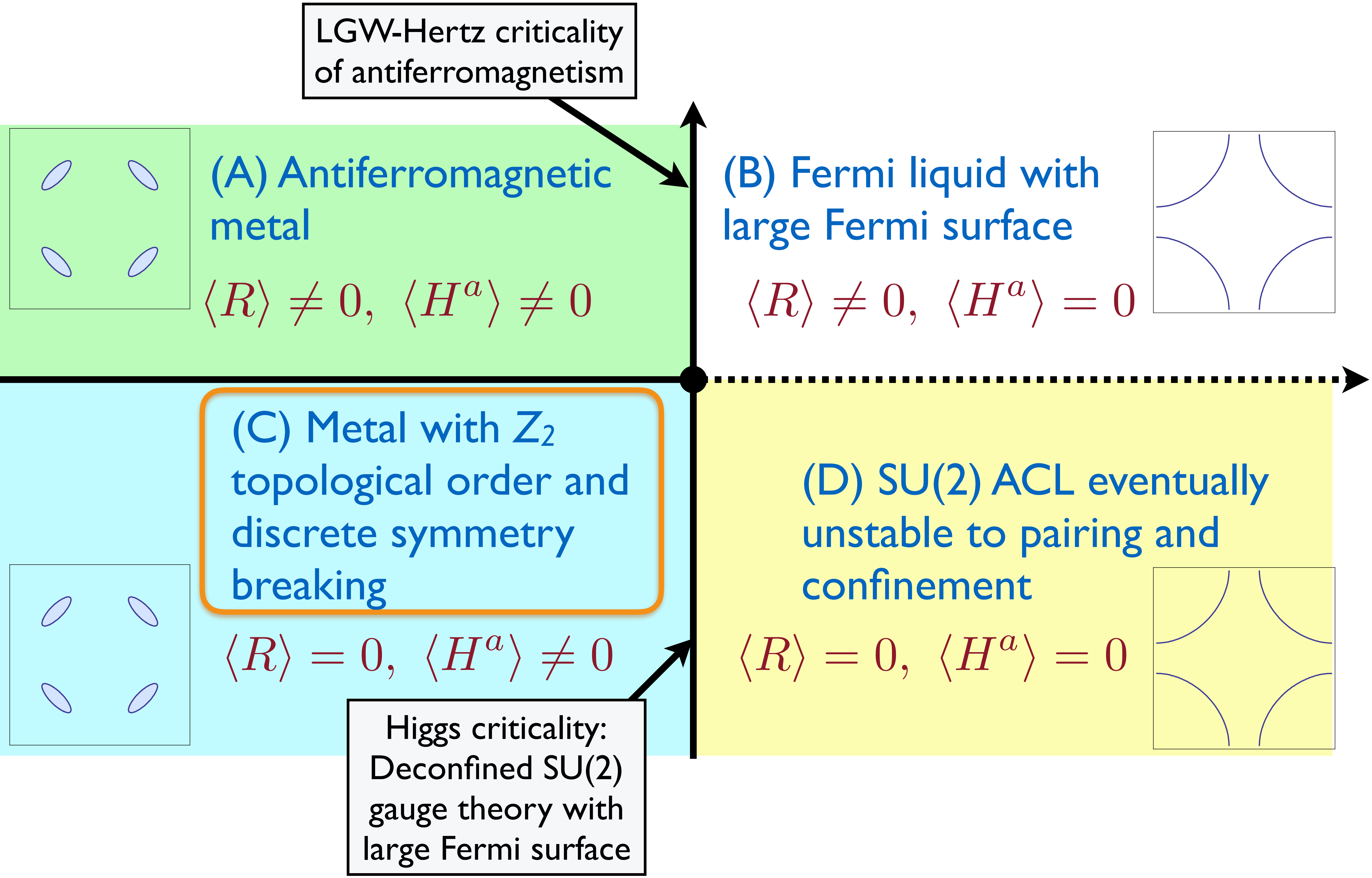
Global phase diagram



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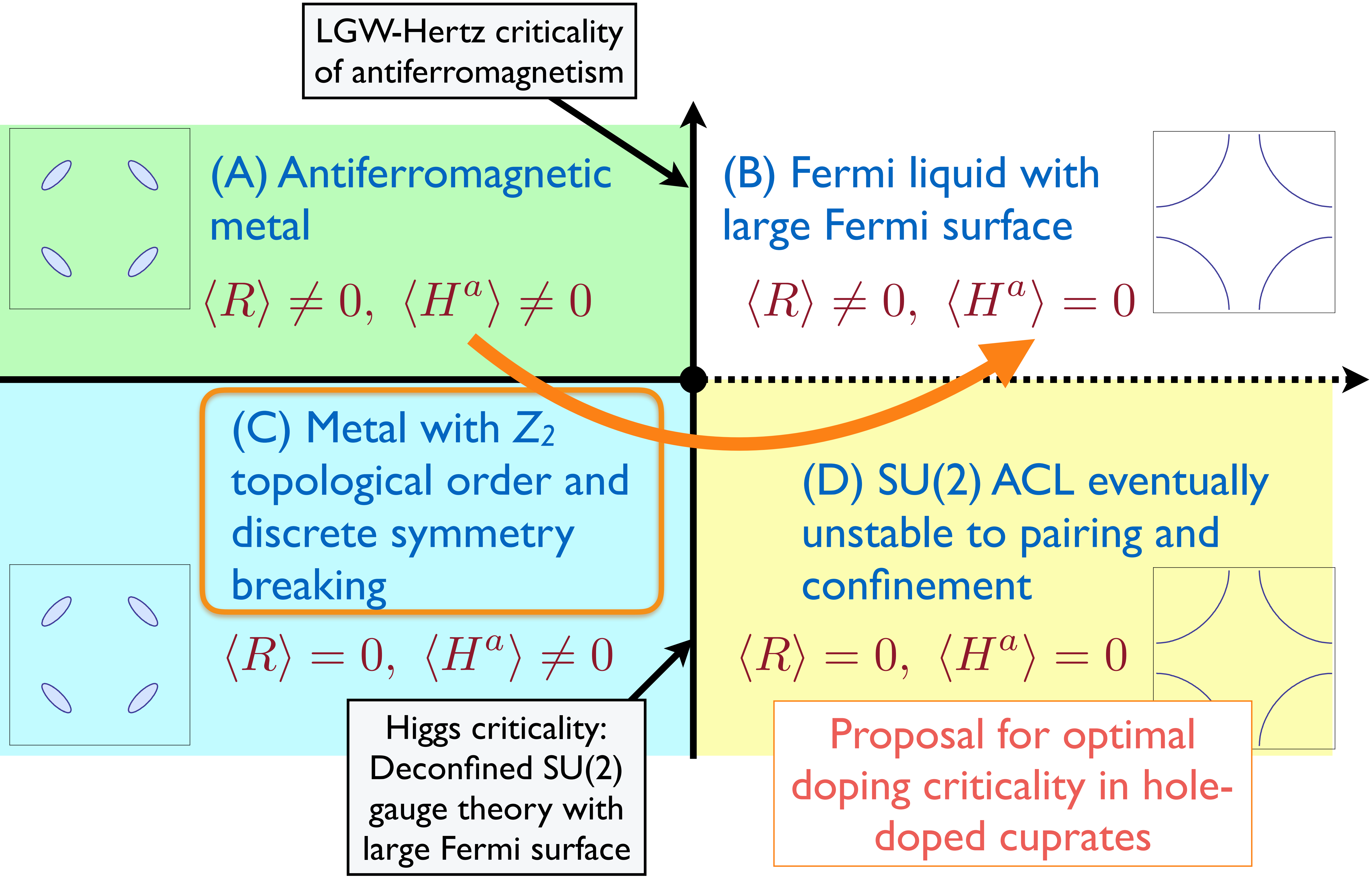

Properties of the hole-doped cuprates:

- The underdoped phase has broken reflection/time-reversal/lattice-rotation symmetry, while preserving translational symmetry.

- The underdoped phase has a pseudogap *i.e.* no gapless fermionic excitations near the anti-nodes.

- With increasing doping, the pseudogap disappears at the same point as the broken symmetries.

These features can be explained by $\mathbb{Z}_2$ topological order found near the square lattice Néel state:

- The topological order can intertwine with the broken symmetry.

- The topological order induce a pseudogap by non-Luttinger volume Fermi surfaces.