# The SYK model and black holes

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PHYSICS
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1. The complex SYK model

    A. *Large N theory*

    B. *Fluctuations and the Schwarzian*

2. Charged black holes

    A. *Einstein-Maxwell-Hawking theory*

    B. *Semiclassical quantum gravity*

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# The complex SYK model

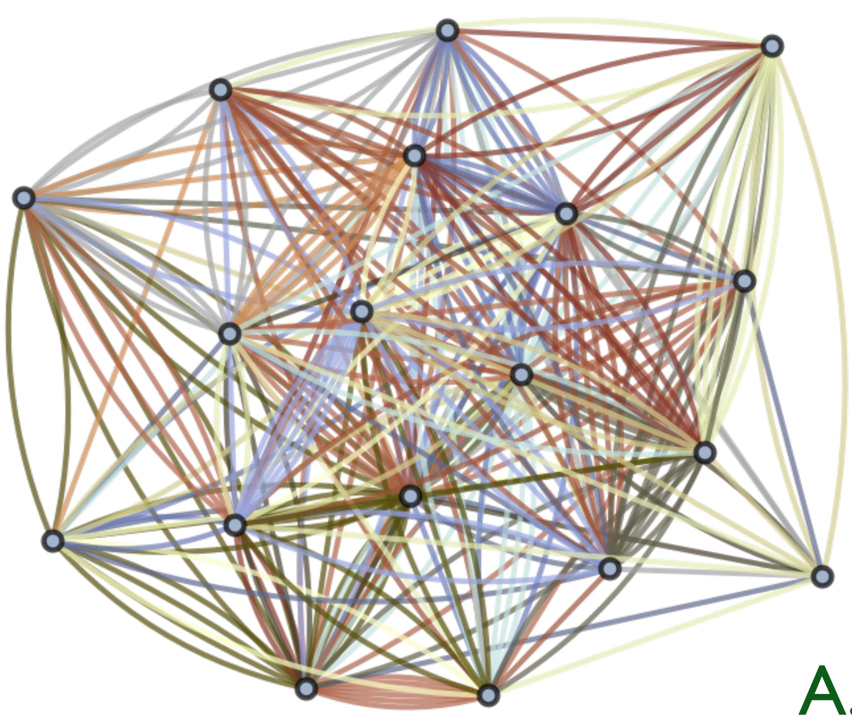
(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large $N$ limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

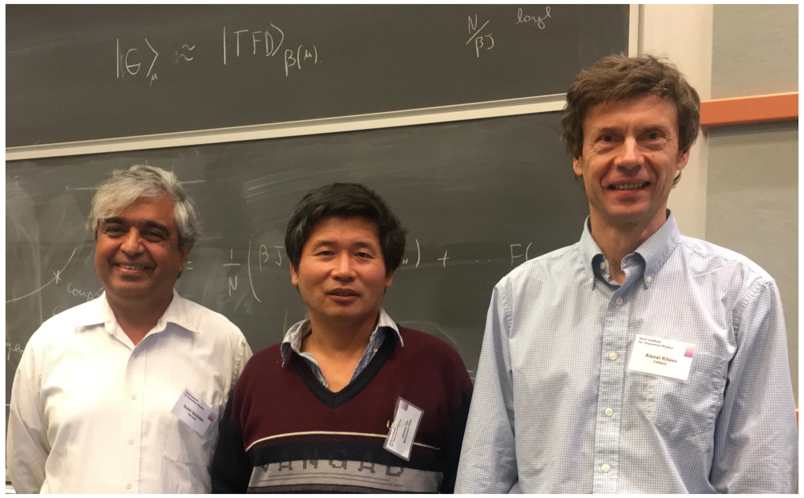
$$H = \frac{1}{(2N)^{3/2}} \sum_{\alpha,\beta,\gamma,\delta=1}^{N} U_{\alpha\beta;\gamma\delta}\, c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta + \epsilon \sum_\alpha c^\dagger_\alpha c_\alpha$$

$$c_\alpha c_\beta + c_\beta c_\alpha = 0 \quad , \quad c_\alpha c^\dagger_\beta + c^\dagger_\beta c_\alpha = \delta_{\alpha\beta}$$

$$\mathcal{Q} = \frac{1}{N} \sum_\alpha c^\dagger_\alpha c_\alpha$$

$U_{\alpha\beta;\gamma\delta}$ are independent random variables with $\overline{U_{\alpha\beta;\gamma\delta}} = 0$ and $\overline{|U_{\alpha\beta;\gamma\delta}|^2} = U^2$
$N \to \infty$ yields critical strange metal.





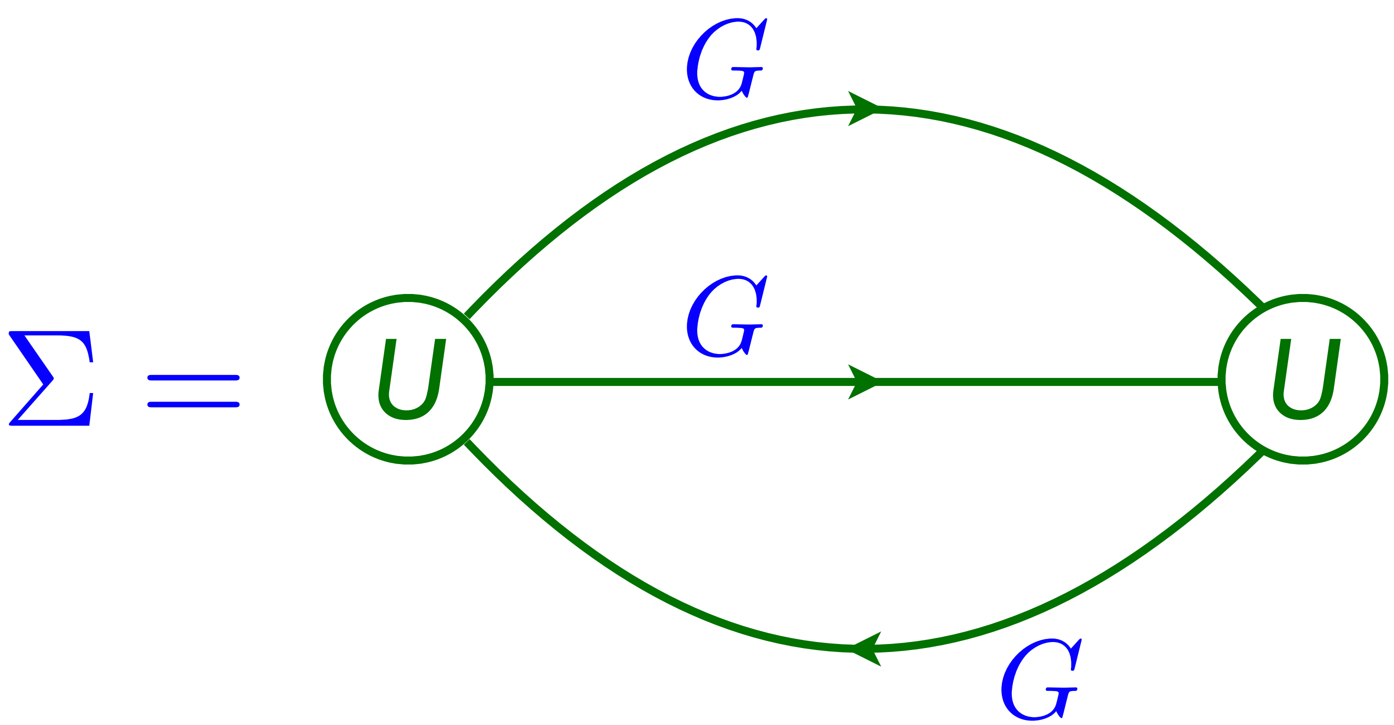
S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

# The complex SYK model

Feynman graph expansion in $U_{\alpha\beta;\gamma\delta}$, and graph-by-graph average, yields exact equations in the large $N$ limit:

$$G(i\omega) = \frac{1}{i\omega - \epsilon - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$




S. Sachdev and J. Ye,
PRL **70**, 3339 (1993)

# The complex SYK model

Solution of these equations, and of the free energy, yields universal results for the SYK model with $q$ fermion terms. These results are *quantitatively* unchanged by adding additional higher $q$ fermion terms:

- At long times, and at $T = 0$, $G(\tau) \sim |\tau|^{-2\Delta}$ with $\Delta = 1/q$ ($\Rightarrow$ indication there are no quasiparticles)

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- At general charge $\mathcal{Q}$, there is a spectral symmetry determined by a parameter $\mathcal{E}$:

$$G(\tau) \sim \begin{cases} -\tau^{-2\Delta} & \tau > 0 \\ e^{-2\pi\mathcal{E}}(-\tau)^{-2\Delta} & \tau < 0 \end{cases} \quad , \quad T = 0$$

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- There is a universal 'Luttinger relation' between $-\infty < \mathcal{E} < \infty$ and the total charge $0 < \mathcal{Q} < 1$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)}$$

$$\mathcal{Q} = \frac{1}{2} - \frac{\theta}{\pi} + \left(\Delta - \frac{1}{2}\right)\frac{\sin(2\theta)}{\sin(2\pi\Delta)}$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

# The complex SYK model

Solution of these equations, and of the free energy, yields universal results for the SYK model with $q$ fermion terms. These results are *quantitatively* unchanged by adding additional higher $q$ fermion terms:

- At $T > 0$, we obtain a solution with a conformal structure

$$G(\tau) = -A \frac{e^{-2\pi \mathcal{E} T \tau}}{\sqrt{1 + e^{-4\pi \mathcal{E}}}} \left( \frac{T}{\sin(\pi T \tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \,,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$

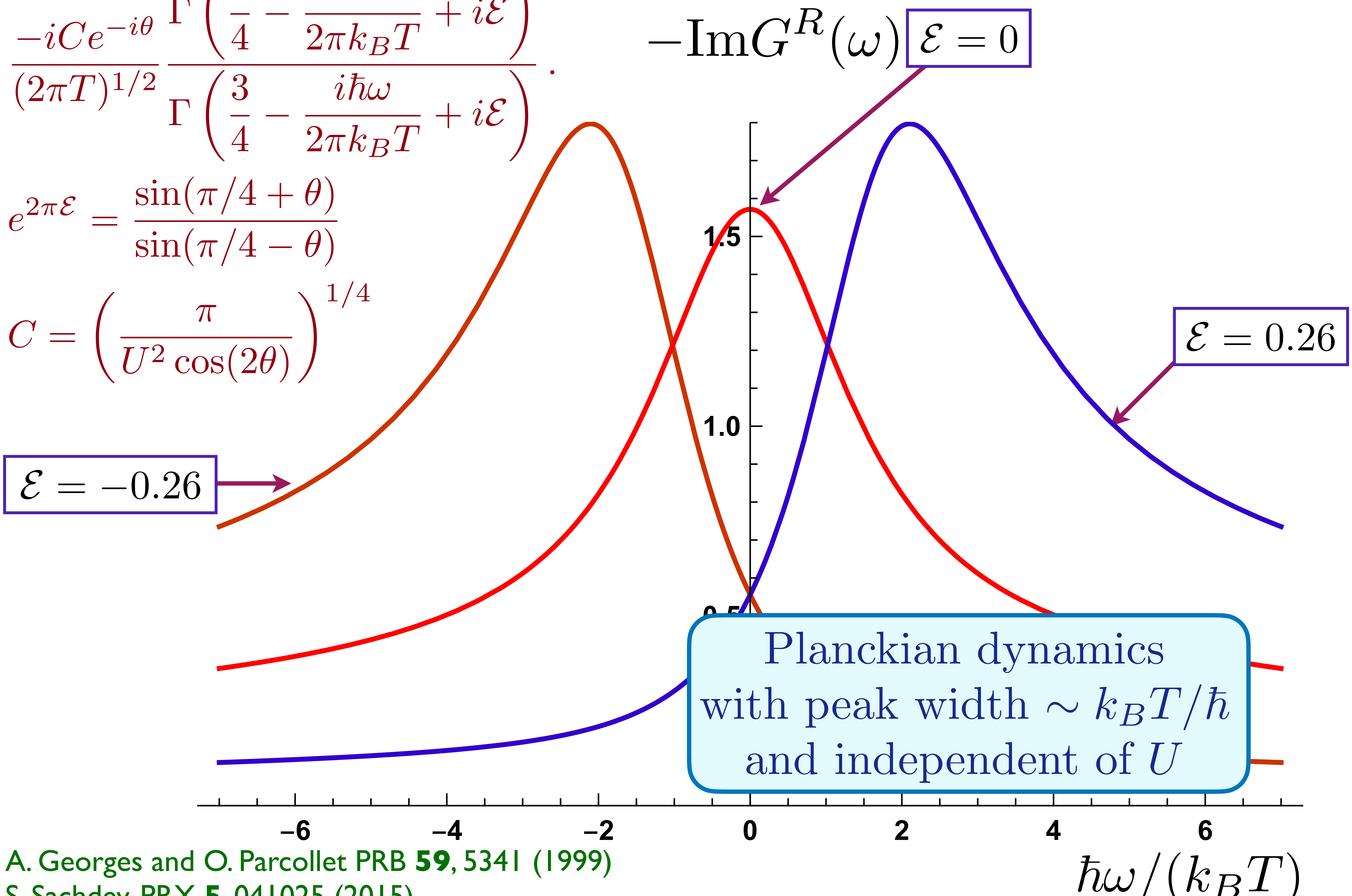
A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

# The complex SYK model

$$G^R_{\text{SYK}}(\hbar\omega/(k_B T)) = \frac{-iCe^{-i\theta}}{(2\pi T)^{1/2}} \frac{\Gamma\left(\frac{1}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}{\Gamma\left(\frac{3}{4} - \frac{i\hbar\omega}{2\pi k_B T} + i\mathcal{E}\right)}.$$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}$$

$$C = \left(\frac{\pi}{U^2 \cos(2\theta)}\right)^{1/4}$$




A. Georges and O. Parcollet PRB **59**, 5341 (1999)
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# The complex SYK model

We now examine the behavior of the chemical potential, $\mu$, as $T \to 0$ at fixed $\mathcal{Q}$. For this we relate the long-time 'conformal' Greens function, (valid for $\tau \gg 1/U$) to its short-time behavior. In particular at $|\omega_n| \gg U$ we have

$$G(i\omega_n) = \frac{1}{i\omega_n} - \frac{\mu}{(i\omega_n)^2} + \ldots$$

which implies for the spectral density of the Green's function, $\rho(\Omega)$

$$\mu = -\int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \, \Omega \rho(\Omega),$$

which makes it evident that $\mu$ depends only upon the particle-hole asymmetric part of the spectral density. Next, we can relate the $\Omega$ integrals to the derivative of the imaginary time correlator

$$\mu = -\partial_\tau G(\tau = 0^+) - \partial_\tau G(\tau = (1/T)^-).$$

# The complex SYK model

We pull out an explicitly particle-hole asymmetric part of $G(\tau)$ by defining

$$G(\tau) \equiv e^{-2\pi \mathcal{E} T \tau} G_c(\tau) \quad , \quad 0 < \sigma < \frac{1}{T}.$$

where $G_c$ will be given by a particle-hole symmetric conformal form at low $T$ and low $\omega$. Then we obtain

$$\begin{aligned} \mu &= 2\pi \mathcal{E} T \left[ G(\tau = 0^+) + G(\tau = (1/T)^-) \right] \\ &\qquad\qquad + \text{terms dependent on } G_c \\ &= -2\pi \mathcal{E} T + \text{terms dependent on } G_c \end{aligned}$$

It can be shown that all the terms dependent upon $G_c$ have a $T$ dependence that is weaker than linear in $T$ provided $\mathcal{Q}$ is held fixed. Hence we have

$$\mu = \mu_0 - 2\pi \mathcal{E} T + \text{terms vanishing as } T^p \text{ with } p > 1$$

with $\mu_0$ a non-universal constant. From this relation we obtain

# The complex SYK model

with $\mu_0$ a non-universal constant. From this relation we obtain

$$\left(\frac{\partial \mu}{\partial T}\right)_{\mathcal{Q}} = -2\pi\mathcal{E} \quad , \quad T \to 0,$$

Using a Maxwell relation we then have

$$\frac{1}{N}\left(\frac{\partial S}{\partial \mathcal{Q}}\right)_{T} = 2\pi\mathcal{E} \neq 0 \ \text{ as } T \to 0.$$

# The complex SYK model

Solution of these equations and corresponding evaluation of the free energy yields the following universal results (*i.e.* all results are *quantitatively* unchanged by adding additional higher $q$ fermion terms):

- There is a non-vanishing entropy in the zero temperature limit

$$S(T \to 0) = N S_0 + \ldots$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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- There is a non-vanishing entropy in the zero temperature limit

$$S(T \to 0) = N S_0 + \ldots$$

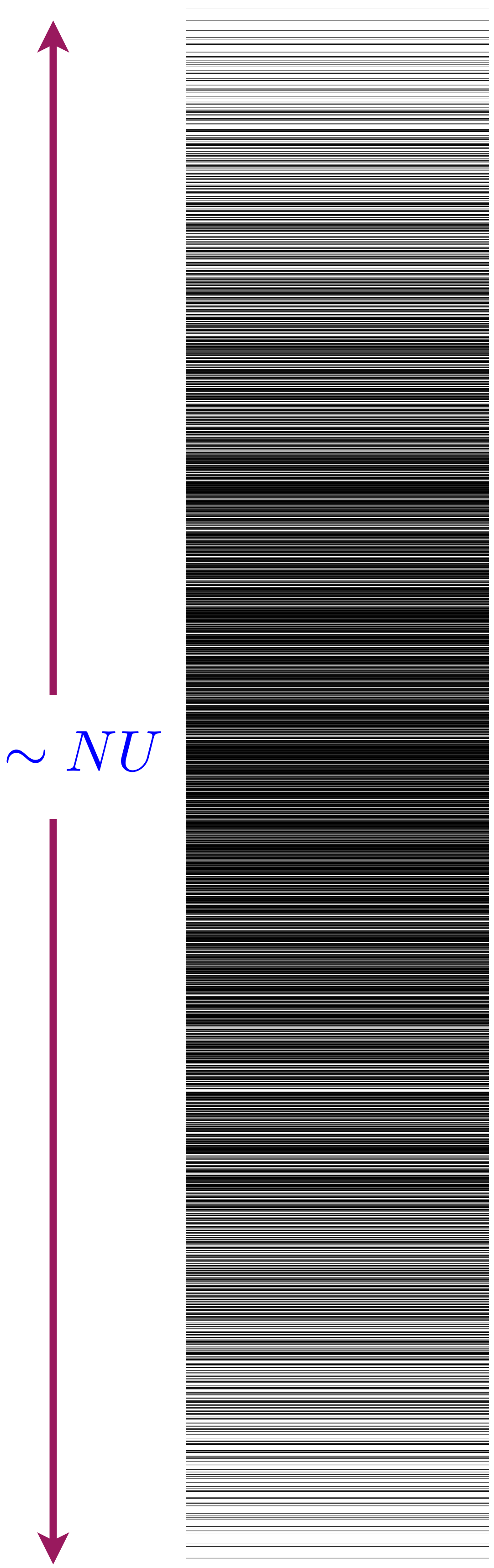
- The saddle point equations imply the relation

$$\boxed{\frac{dS_0}{d\mathcal{Q}} = 2\pi\mathcal{E}}$$

Integrating this relation from $S_0 = 0$, $\mathcal{Q} = 0$, allows us to compute $S_0$ as a function of $\mathcal{Q}$.

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

# The complex SYK model




Many-body level spacing $\sim 2^{-N} = e^{-N \ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-NS_0}$

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

There are $2^N$ many body levels with energy $E$. Shown are all values of $E$ for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = NS_0$, where $S_0 < \ln 2$ is determined by integrating

$$\frac{dS_0}{d\mathcal{Q}} = 2\pi\mathcal{E}\,.$$

At $\mathcal{Q} = 1/2$,

$$S_0 \quad = \quad \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where $G$ is Catalan's constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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# Symmetries of the saddle point

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

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$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

At frequencies $\ll U$, the $i\omega + \mu$ can be dropped, and without it equations are invariant under the reparametrization and gauge transformations. The singular part of the self-energy and the Green's function obey

$$\int_0^\beta d\tau_2 \, \Sigma_{\text{sing}}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma_{\text{sing}}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

A. Kitaev, 2015

S. Sachdev, PRX **5,** 041025 (2015)

# Symmetries of the saddle point

$$\int_0^\beta d\tau_2 \, \Sigma(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

These equations are invariant under

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
By using $f(\sigma) = \tan(\pi T \sigma)/(\pi T)$ and
$g(\sigma) = e^{-2\pi \mathcal{E} T \sigma}$, we can now obtain
the $T > 0$ solution from the $T = 0$ solution.

A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

# Symmetries of the saddle point

Let us write the large $N$ saddle point solutions of $S$ as

$$
\begin{aligned}
G_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-1/2} \\
\Sigma_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-3/2}.
\end{aligned}
$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for $\Sigma$). It turns out this is true only for the SL(2, R) transformations under which

$$
f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.
$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

A. Kitaev

# Symmetries of the saddle point

- The saddle-point

$$G(\tau_1 - \tau_2) = -A \frac{e^{-2\pi\mathcal{E}T(\tau_1-\tau_2)}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left( \frac{T}{\sin(\pi T(\tau_1 - \tau_2))} \right)^{2\Delta}$$

is invariant only under PSL(2, R) transformations which map the thermal circle onto itself, and an associated gauge transformation

$$\frac{\tan(\pi T f(\tau))}{\pi T} = \frac{a \dfrac{\tan(\pi T \tau)}{\pi T} + b}{c \dfrac{\tan(\pi T \tau)}{\pi T} + d} \quad , \quad ad - bc = 1,$$

$$-i\phi(\tau) = -i\phi_0 + 2\pi\mathcal{E}T(\tau - f(\tau))$$

A. Kitaev, 2015

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

# $G$-$\Sigma$ path integral

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp\left[ -\sum_{ia} \int_0^\beta d\tau\, c_{ia}^\dagger \left( \frac{\partial}{\partial\tau} - \mu \right) c_{ia} \right.$$
$$\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2) \exp\left[ -N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1,\tau_2) \Big( G(\tau_2,\tau_1) \right.$$
$$\left. + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \Big) \right] .$$

# $G$-$\Sigma$ path integral

Then the partition function can be written as a path integral with an action $S$ analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det \left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2)G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = \left[f'(\sigma_1)f'(\sigma_2)\right]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = \left[f'(\sigma_1)f'(\sigma_2)\right]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

# $G$-$\Sigma$ path integral

## Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, $G_s$, $\Sigma_s$, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1)-i\phi(\tau_2)}$$

(and similarly for $\Sigma$). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-E_0/T+Ns_0-NS_{\text{eff}}[f,\phi]} ,$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

# $G$-$\Sigma$ path integral

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\rm eff}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau \, \{\tan(\pi T f(\tau)), \tau\},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, the couplings $K$, $\gamma$, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g, \tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

# Main result I

Low temperature thermodynamics: for $k_B T \ll U$

$$\begin{aligned} \mathcal{Z} &= \text{Tr} \exp\left(-\frac{\mathcal{H}}{k_B T}\right) \\ &= \exp\left(N \frac{S_0}{k_B}\right) \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) \exp\left(-\frac{1}{\hbar} S_{\text{eff}}\left[f(\tau), \phi(\tau)\right]\right) \end{aligned}$$

- Feynman path integral over $f(\tau)$, the reparameterization of the time of the SYK model, and $\phi(\tau)$ a phase conjugate to the total charge $\mathcal{Q}$.


S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)
A. Kitaev (2015)
J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)

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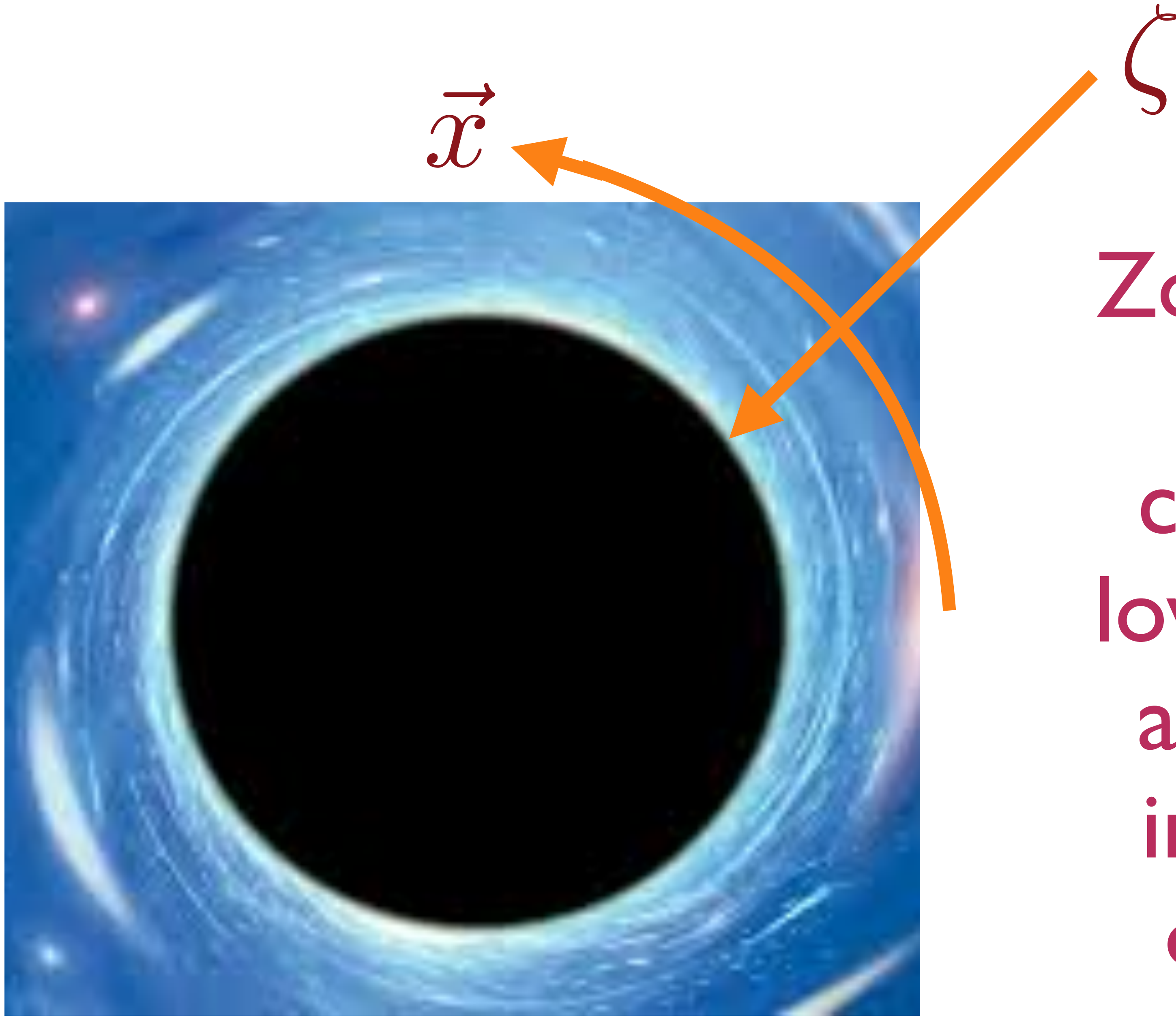
# Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge




Zooming into the near-horizon region of a charged black hole at low temperature, yields a gravitational theory in one space ($\zeta$) and one time dimension


# Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge

$$I_{EM} = \int d^{d+2}x \sqrt{g} \left[ -\frac{1}{2\kappa^2} \left( \mathcal{R}_{d+2} + \frac{d(d+1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right]$$

Metric $g_{\mu\nu}$

Ricci scalar in $d+2$ dimensions, $\mathcal{R}_{d+2}$

Cosmological constant $\Lambda = -d(d+1)/L^2$

U(1) gauge field $A_\mu$

Electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Boundary conditions at spatial infinity:

Metric $\rightarrow$ AdS$_{d+2}$

Electric field $\rightarrow Q/(4\pi r^2)$


# Maxwell's electromagnetism and Einstein's general relativity allow black hole solutions with a net charge

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Quantum gravity is 'defined' by the path integral

$$\mathcal{Z}_{\text{gravity}} = \int \mathcal{D}g\mathcal{D}A \exp(-I_{EM}/\hbar)$$

We will evaluate this integral exactly in a certain low temperature limit for charged black holes.

L.V. Iliesiu, G.J. Turiaci, arXiv:2003.02860; M. Heydeman, L.V. Iliesiu, G.J. Turiaci, Wenli Zhao, arXiv:2011.01953

# Charged black holes

$$I_{EM} = \int d^{d+2}x \sqrt{g} \left[ -\frac{1}{2\kappa^2} \left( \mathcal{R}_{d+2} + \frac{d(d+1)}{L^2} \right) + \frac{1}{4g_F^2} F^2 \right]$$

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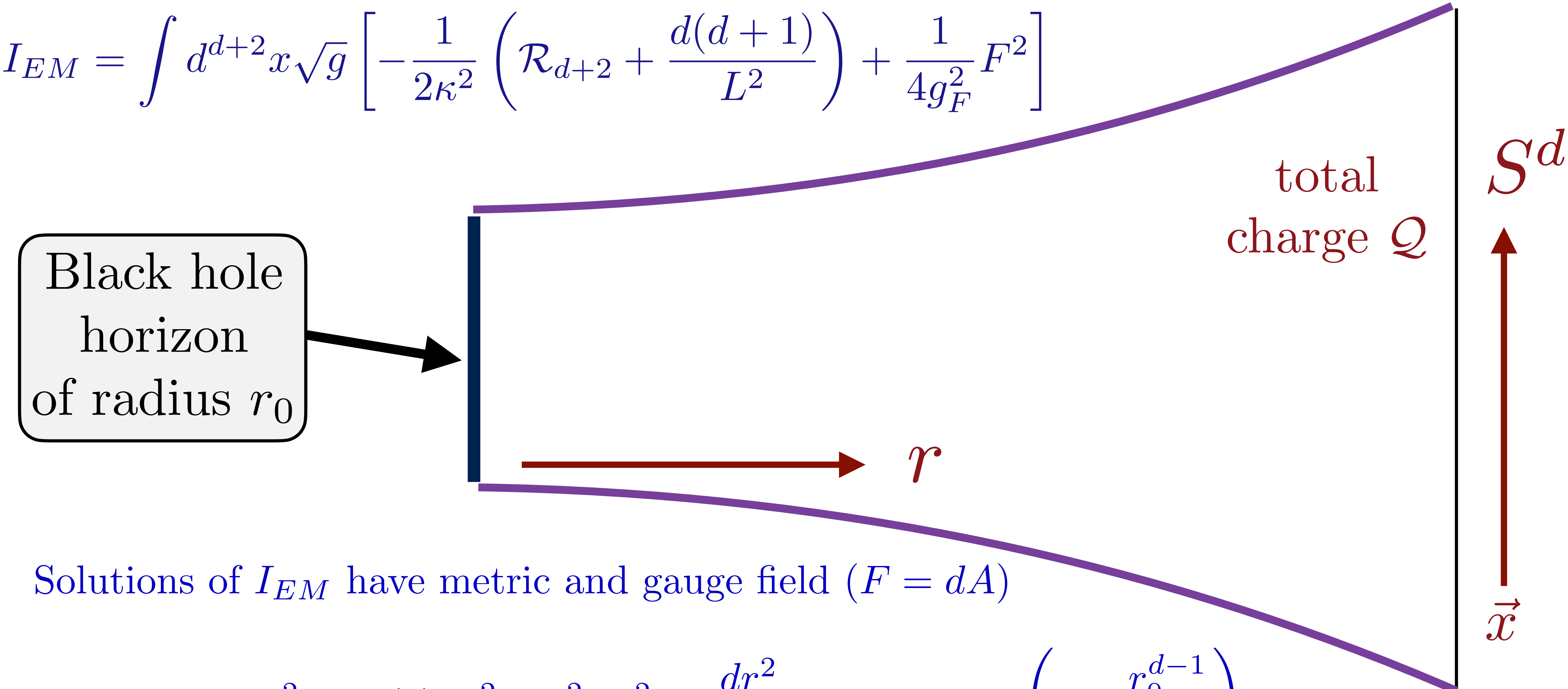



Solutions of $I_{EM}$ have metric and gauge field ($F = dA$)

$$ds^2 = V(r)d\tau^2 + r^2 d\Omega_d^2 + \frac{dr^2}{V(r)} \quad , \quad A = i\mu\left(1 - \frac{r_0^{d-1}}{r^{d-1}}\right)d\tau$$

$$V(r) = 1 + \frac{r^2}{L^2} + \frac{\Theta^2}{r^{2d-2}} - \frac{M}{r^{d-1}}.$$

where $d\Omega_d^2$ is the metric of the $d$-sphere. All parameters of the solution are determined in terms of the chemical potential $\mu$ (related to the charge $Q$), and the Hawking temperature of horizon, $T_H$ (related to the mass $M$).

A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, PRD **60**, 064018 (1999)

# Charged black holes

In the $T \to 0$ limit, at fixed $\mu$, we obtain a charged black hole solution with radius $r_0(T \to 0, \mu) = R_h$. All properties of this black hole can be expressed in terms of $R_h$

- In the near-horizon region, we change co-ordinates from $r$ to $\zeta$ so that

$$r - R_h = \frac{R_2^2}{\zeta} \quad , \quad R_2 = \frac{L R_h}{\sqrt{d(d+1)R_h^2 + (d-1)^2 L^2}}.$$

Then the near-horizon metric becomes AdS$_2\times$ S$_d$, with

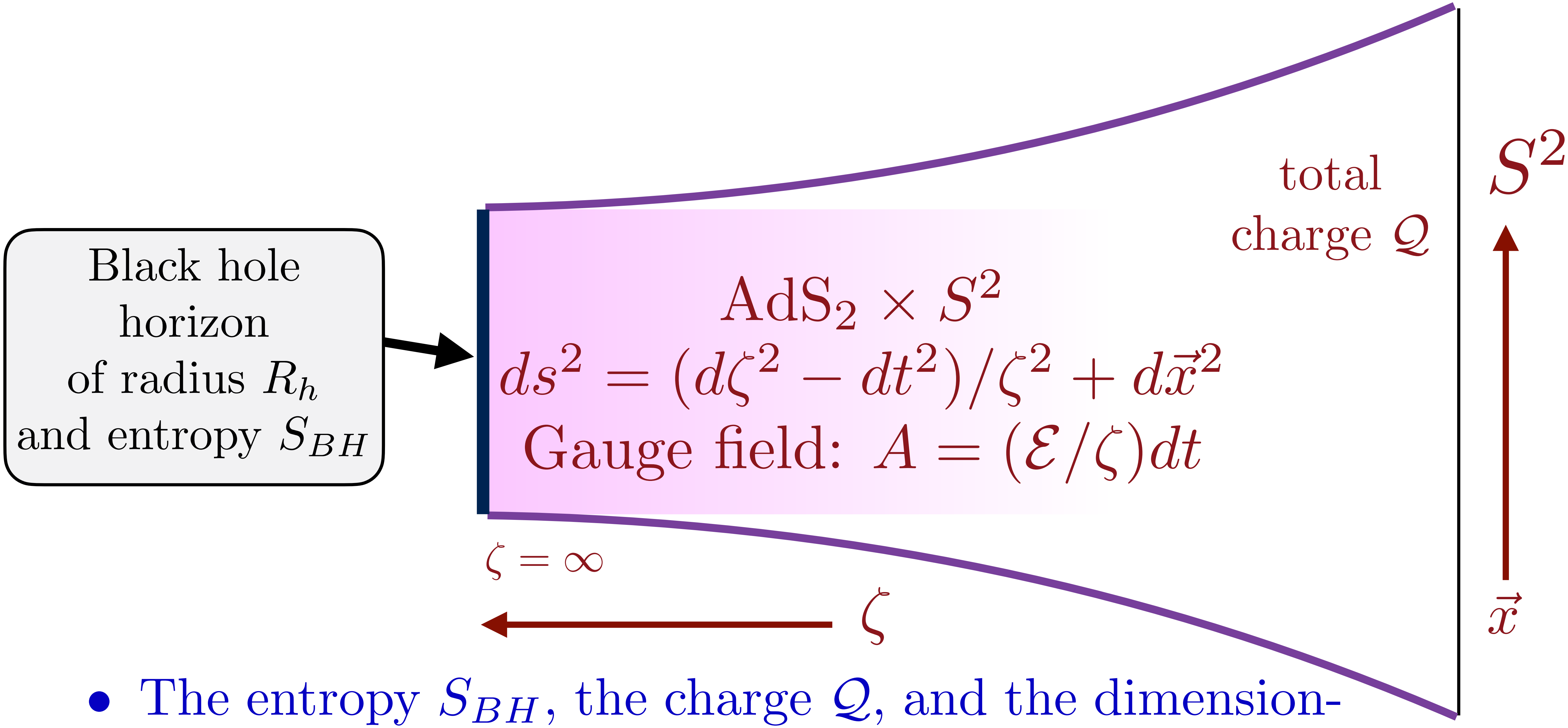
$$ds^2 = R_2^2 \left[\frac{-dt^2 + d\zeta^2}{\zeta^2}\right] + R_h^2 d\Omega_d^2 \quad , \quad A = \frac{\mathcal{E}}{\zeta} dt \,.$$

where the dimensionless electric field $\mathcal{E}$ is

$$\mathcal{E} = \frac{g_F R_h \sqrt{2d\left[(d+1)R_h^2 + (d-1)L^2\right]}}{2\left[d(d+1)R_h^2 + (d-1)^2 L^2\right]}.$$

S. Sachdev, Journal of Mathematical Physics **60**, 052303 (2019)

# Charged black holes




- The entropy $S_{BH}$, the charge $\mathcal{Q}$, and the dimensionless electric field $\mathcal{E}$ obey the same thermodynamic relation as the SYK model

$$\frac{dS_{BH}}{d\mathcal{Q}} = 2\pi\mathcal{E}$$

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010); PRX **5,** 041025 (2015)

1. The complex SYK model

   *A. Large N theory*

   *B. Fluctuations and the Schwarzian*

2. Charged black holes

   *A. Einstein-Maxwell-Hawking theory*

   *B. Semiclassical quantum gravity*

# Symmetries of the saddle point

- In imaginary time, $AdS_2$ is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under SL(2,R)
- The time reparameterization invariance of general relativity is broken down to SL(2,R) by the $AdS_2$ saddle point.

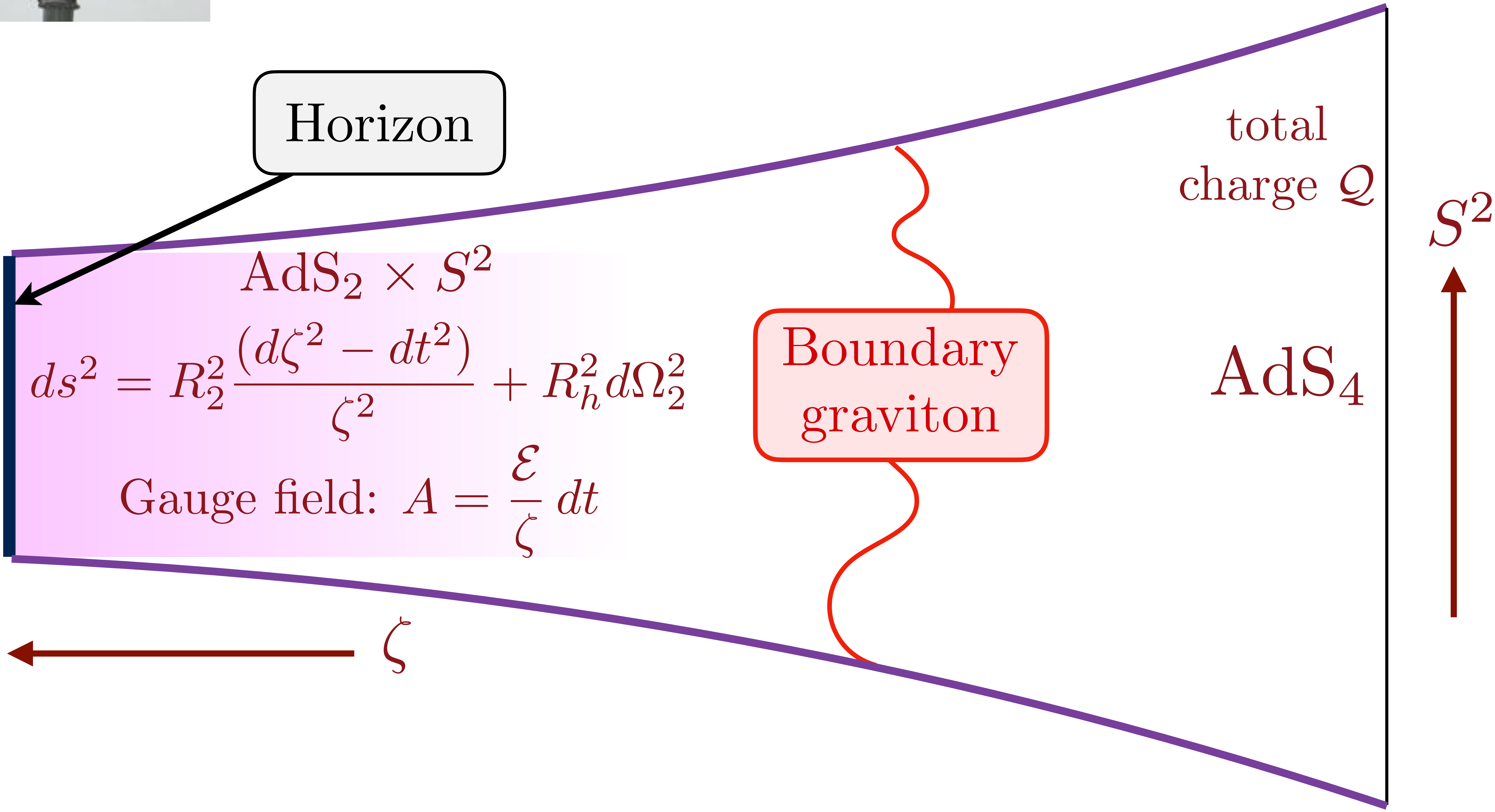
$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \text{ with } ad - bc = 1.$$

Euclidean $AdS_2$

A. Kitaev, 2015
J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857


# Fluctuations about the saddle point



# Dimensional reduction

We write the $(d+2)$-dimensional metric $g$ of $I_{EM}$ in terms of a two-dimensional metric $h$ and a scalar field $\Phi$:

$$ds^2 = \frac{ds_2^2}{\Phi^{d-1}} + \Phi^2 \, d\Omega_d^2 \,.$$

The Einstein-Maxwell and Gibbons-Hawking actions reduce to and extension of Jackiw-Tietelbaum gravity ($x \equiv (\tau, \zeta)$)

$$\begin{aligned} I_{EM} &= \int d^2x \sqrt{h} \left[ -\frac{s_d}{2\kappa^2} \Phi^d \mathcal{R}_2 + U(\Phi) + \frac{Z(\Phi)}{4g_F^2} F^2 \right] \\ I_{GH} &= -\frac{s_d}{\kappa^2} \int_\partial dx \sqrt{h_b} \Phi^d \mathcal{K}_1 \end{aligned}$$

The explicit forms of the potentials $U(\Phi)$ and $Z(\Phi)$ are,

$$U(\Phi) = -\frac{s_d}{2\kappa^2} \left( \frac{d(d-1)}{\Phi} + \frac{d(d+1)\Phi}{L^2} \right) \quad , \quad Z(\Phi) = s_d \Phi^{2d-1} \,.$$

S. Sachdev, Journal of Mathematical Physics **60**, 052303 (2019)
L.V. Iliesiu, G.J. Turiaci, arXiv:2003.02860; M. Heydeman, L.V. Iliesiu, G.J. Turiaci, Wenli Zhao, arXiv:2011.01953

# Dimensional reduction

The exact saddle point of $\Phi$ relates to $R_h$ the horizon radius at $T = 0$

$$\Phi(\zeta) = R_h + \frac{R_2^2}{\zeta} \quad , \quad R_h \equiv \frac{L}{g_F}\left[\frac{(d-1)(\mu_0^2\kappa^2(d-1) - dg_F^2)}{d(d+1)}\right]^{1/2} ,$$

while the near-horizon, low $T \ll 1/R_h$ metric is $\mathrm{AdS}_2$

$$ds_2^2 = \frac{R_2^2 R_h^{d-1}}{\zeta^2}\left[(1 - 4\pi^2T^2\zeta^2)d\tau^2 + \frac{d\zeta^2}{1 - 4\pi^2T^2\zeta^2}\right] ,$$

where

$$R_2 = \frac{LR_h}{\sqrt{d(d+1)R_h^2 + (d-1)^2L^2}}$$

The field coupling to $\mathcal{R}_2$ is $\Phi^d$

$$[\Phi(\zeta)]^d = R_h^d + \frac{\Phi_1}{\zeta} + \ldots \quad , \quad \Phi_1 = dR_h^{d-1}R_2^2,$$

S. Sachdev, Journal of Mathematical Physics **60**, 052303 (2019)
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# Dimensional reduction

The field coupling to $\mathcal{R}_2$ is $\Phi^d$

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We choose the boundary of the AdS$_2$ region at bulk co-ordinates $(f(\tau), \zeta(\tau))$ with the induced boundary metric fixed at $(R_2^2 R_h^{d-1}/\zeta_b^2) d\tau^2$ by choosing

$$\zeta(\tau) = \zeta_b f'(\tau) + \zeta_b^3 \left( \frac{[f''(\tau)]^2}{2f'(\tau)} - 2\pi^2 T^2 [f'(\tau)]^3 \right) + \ldots .$$

Finally, we evaluate $I_{GH}$ along this boundary curve

$$I_1[f] = -\frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \, \{\tan(\pi T f(\tau)), \tau\} \,,$$

where

$$\gamma = \frac{4\pi^2 s_d \Phi_1}{\kappa^2} \,,$$

matches the linear-in-$T$ co-efficient of the specific heat of the full Reissner-Nördstorm solution in $d+2$ dimensions.

S. Sachdev, Journal of Mathematical Physics **60**, 052303 (2019)
L.V. Iliesiu, G.J. Turiaci, arXiv:2003.02860; M. Heydeman, L.V. Iliesiu, G.J. Turiaci, Wenli Zhao, arXiv:2011.01953

# Main result II

Low temperature thermodynamics: for $T \ll 1/R_h$

$$\mathcal{Z}_{\text{charged black hole in EM theory}} = \exp\left(\frac{S_{BH}}{k_B}\right) \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) \exp\left(-\frac{1}{\hbar} S_{\text{eff}}\left[f(\tau), \phi(\tau)\right]\right)$$

- The effective action $S_{\text{eff}}[f(\tau), \phi(\tau)]$ is the same as that for the SYK model
- The particle-hole asymmetry of the SYK model, $\mathcal{E}$, is identified as the electric field on the black horizon.
- The entropy depends upon the charge $\mathcal{Q}$ in an identical manner:

$$\frac{dS_0}{d\mathcal{Q}} = 2\pi\mathcal{E} \quad , \quad \frac{dS_{BH}}{d\mathcal{Q}} = 2\pi\mathcal{E}$$

# Main result I

Low temperature thermodynamics: for $k_B T \ll U$

$$\mathcal{Z} = \mathrm{Tr}\exp\left(-\frac{\mathcal{H}}{k_B T}\right)$$
$$= \exp\left(N\frac{S_0}{k_B}\right)\int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)\exp\left(-\frac{1}{\hbar}S_{\rm eff}\left[f(\tau),\phi(\tau)\right]\right)$$

- Feynman path integral over $f(\tau)$, the reparameterization of the time of the SYK model, and $\phi(\tau)$ a phase conjugate to the total charge $\mathcal{Q}$.


S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)
A. Kitaev (2015)
J. Maldacena and D. Stanford, Phys. Rev. D **94,** 106002 (2016)

# $G$-$\Sigma$ path integral

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\rm eff}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau \left\{\tan(\pi T f(\tau)), \tau\right\},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, the couplings $K$, $\gamma$, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g, \tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746