# Emergent gauge fields and topological order in the 3D XY model

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1. Review: XY model in 2 and 3 dimensions

2. Topological order in the XY model in 3 dimensions

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   in 2 and 3 dimensions

2. Topological order in the XY
   model in 3 dimensions

$$\mathcal{Z}_{XY} = \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp\left(-H_{XY}\right)$$

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

- Non-zero *T* (classical) phase transitions of superfluids, magnets with `easy-plane’ spins, …..in *D* spatial dimensions

- *T*=0 (quantum) phase transitions of bosons at integer filling between superfluid and insulator in *D-1* spatial dimensions

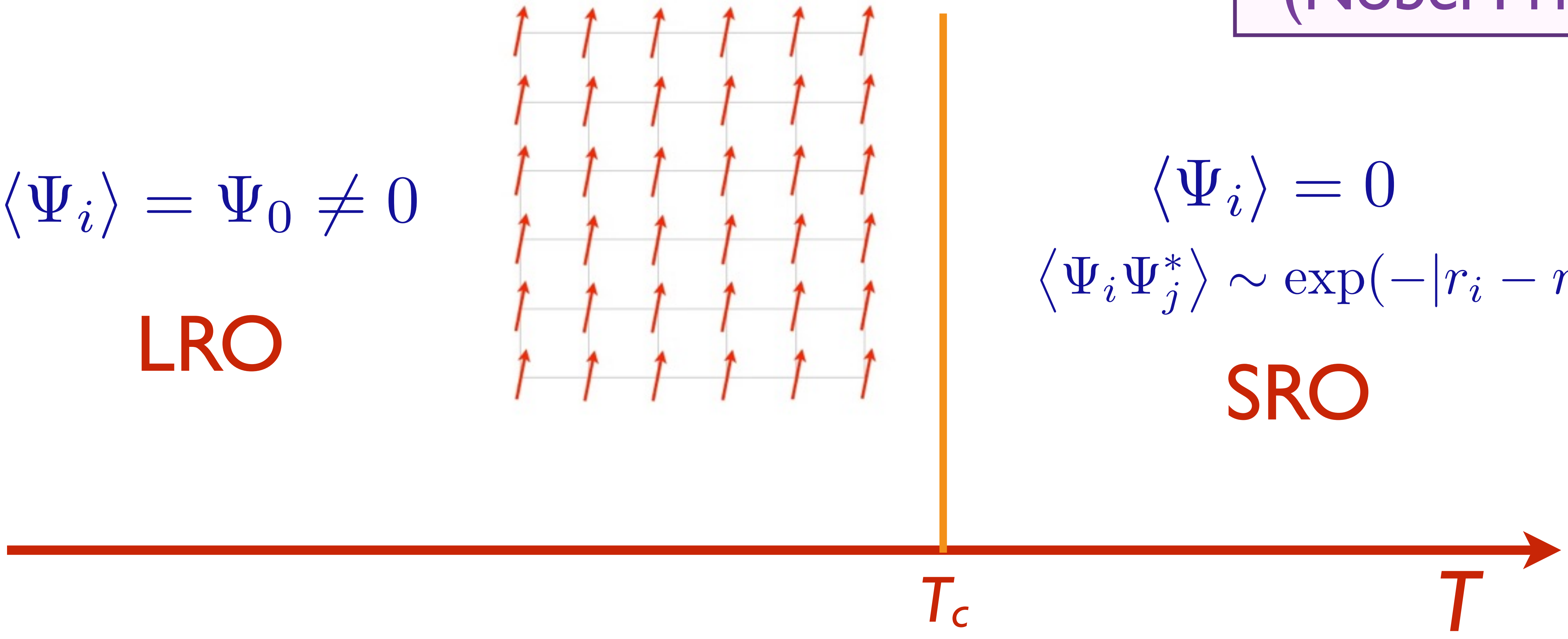
In dimension $\boxed{D = 3,}$ in the low $T$ phase, the symmetry $\theta_i \to \theta_i + c$ is "spontaneously broken". There is (off-diagonal) long-range order (LRO) characterized by ($\Psi_i \equiv e^{i\theta_i}$)

$$\lim_{|r_i - r_j| \to \infty} \left\langle \Psi_i \Psi_j^* \right\rangle = |\Psi_0|^2 \neq 0 \,.$$

We break the symmetry by choosing an overall phase so that

$$\langle \Psi_i \rangle = \Psi_0 \neq 0$$

Wilson-Fisher theory
(Nobel Prize, 1982)



$$\langle \Psi_i \rangle = \Psi_0 \neq 0$$

**LRO**

$$\langle \Psi_i \rangle = 0$$

$$\left\langle \Psi_i \Psi_j^* \right\rangle \sim \exp(-|r_i - r_j|/\xi)$$

**SRO**

# Kosterlitz-Thouless theory in *D*=2

In spatial dimension $\boxed{D = 2,}$ the symmetry $\theta_i \to \theta_i + c$ is preserved at all non-zero $T$. There is no LRO, and

$$\langle \Psi_i \rangle = 0 \text{ for all } T > 0.$$

Nevertheless, there is a phase transition at $T = T_{KT}$, where the nature of the correlations changes

$$\lim_{|r_i - r_j| \to \infty} \langle \Psi_i \Psi_j^* \rangle \sim \begin{cases} |r_i - r_j|^{-\alpha}, & \text{for } T < T_{KT}, \text{ (QLRO)} \\ \exp(-|r_i - r_j|/\xi), & \text{for } T > T_{KT}, \text{ (SRO)} \end{cases}$$

KT theory
(Nobel Prize, 2016)

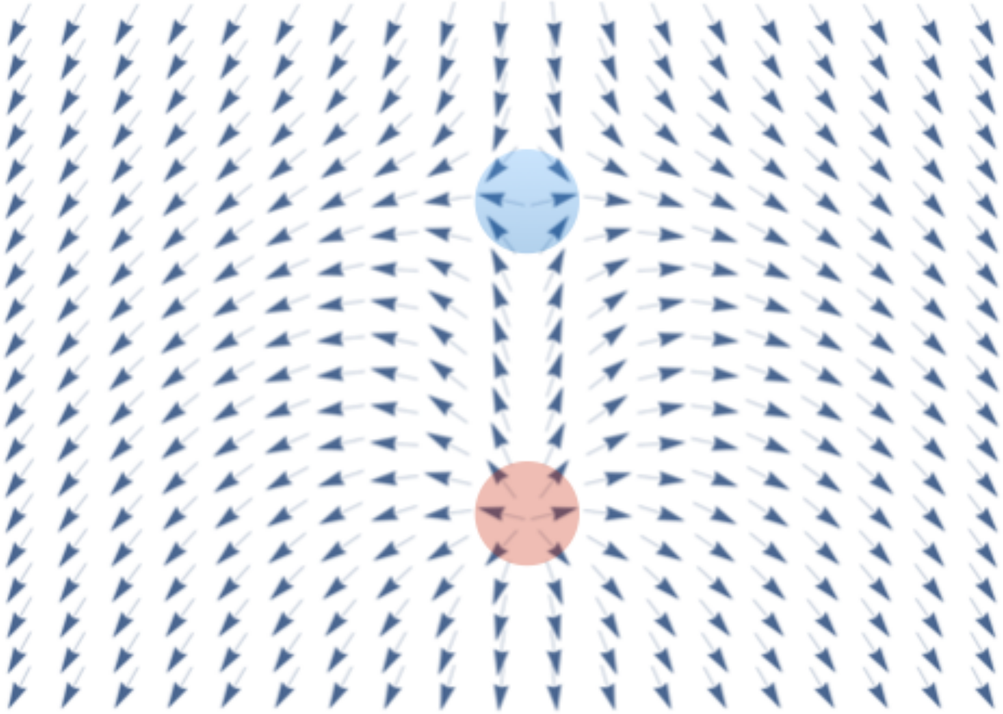
# Kosterlitz-Thouless theory in $D=2$

QLRO

Topological order

Vortices suppressed

SRO



Vortices proliferate

$T_{KT}$

$T$

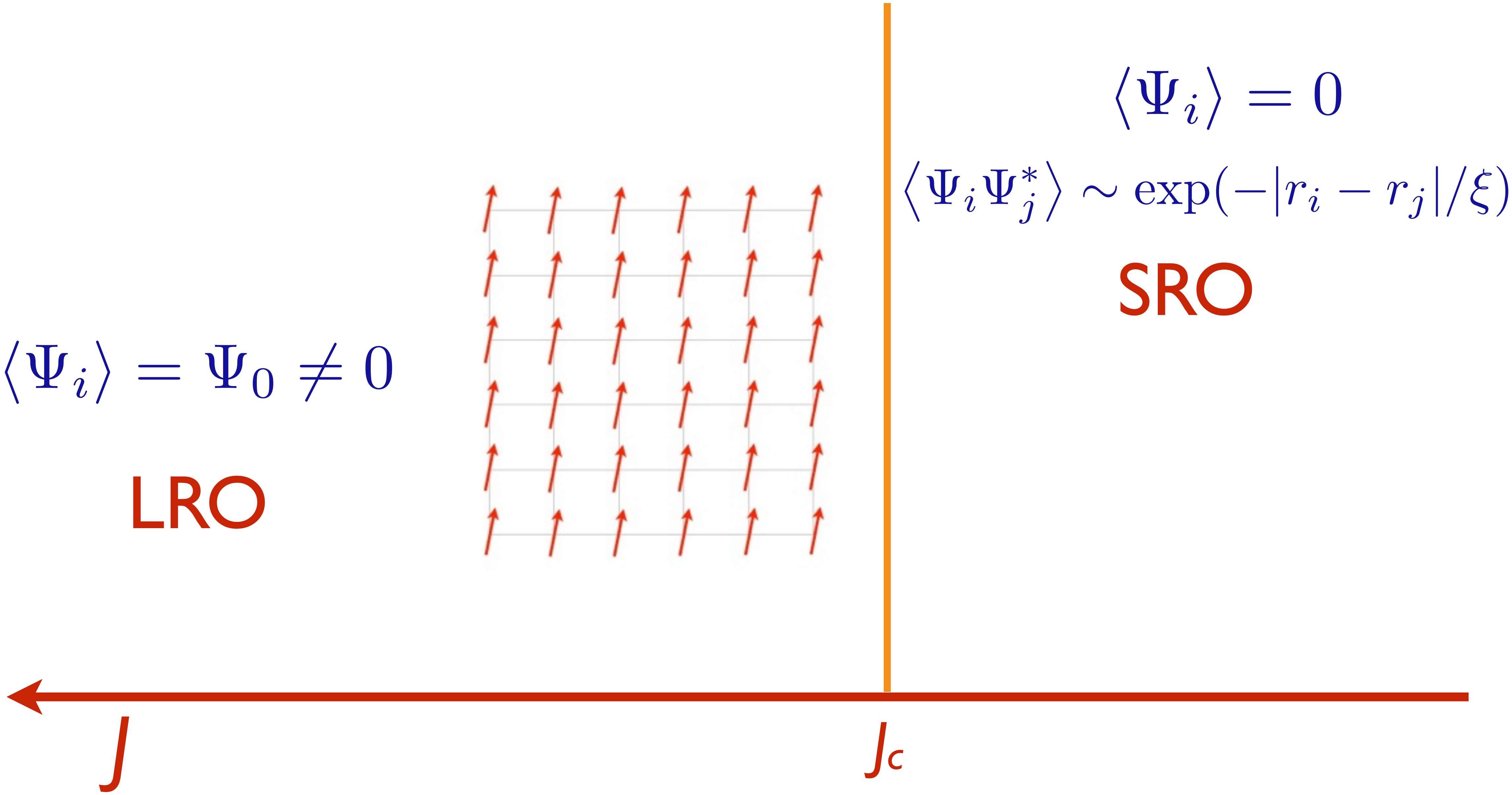
The low $T$ phase also has **topological order** associated with the suppression of vortices.

KT theory
(Nobel Prize, 2016)

1. Classical XY model in 2 and 3 dimensions

2. Topological order in the classical XY model in 3 dimensions

# Can we modify the XY model Hamiltonian to obtain a phase with "topological order" in *D=3* ?



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$\langle \Psi_i \Psi_j^* \rangle \sim \exp(-|r_i - r_j|/\xi)$

SRO

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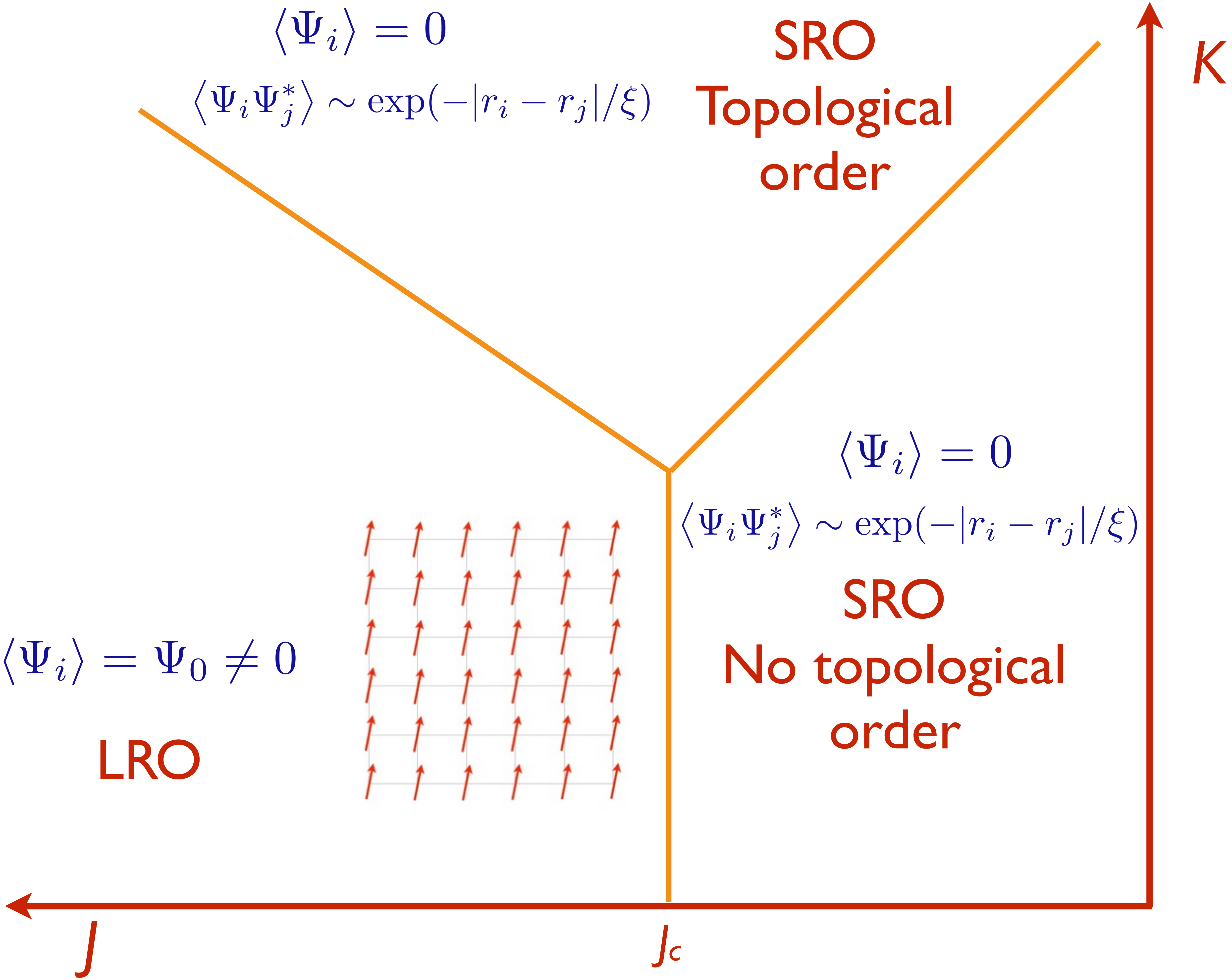
LRO



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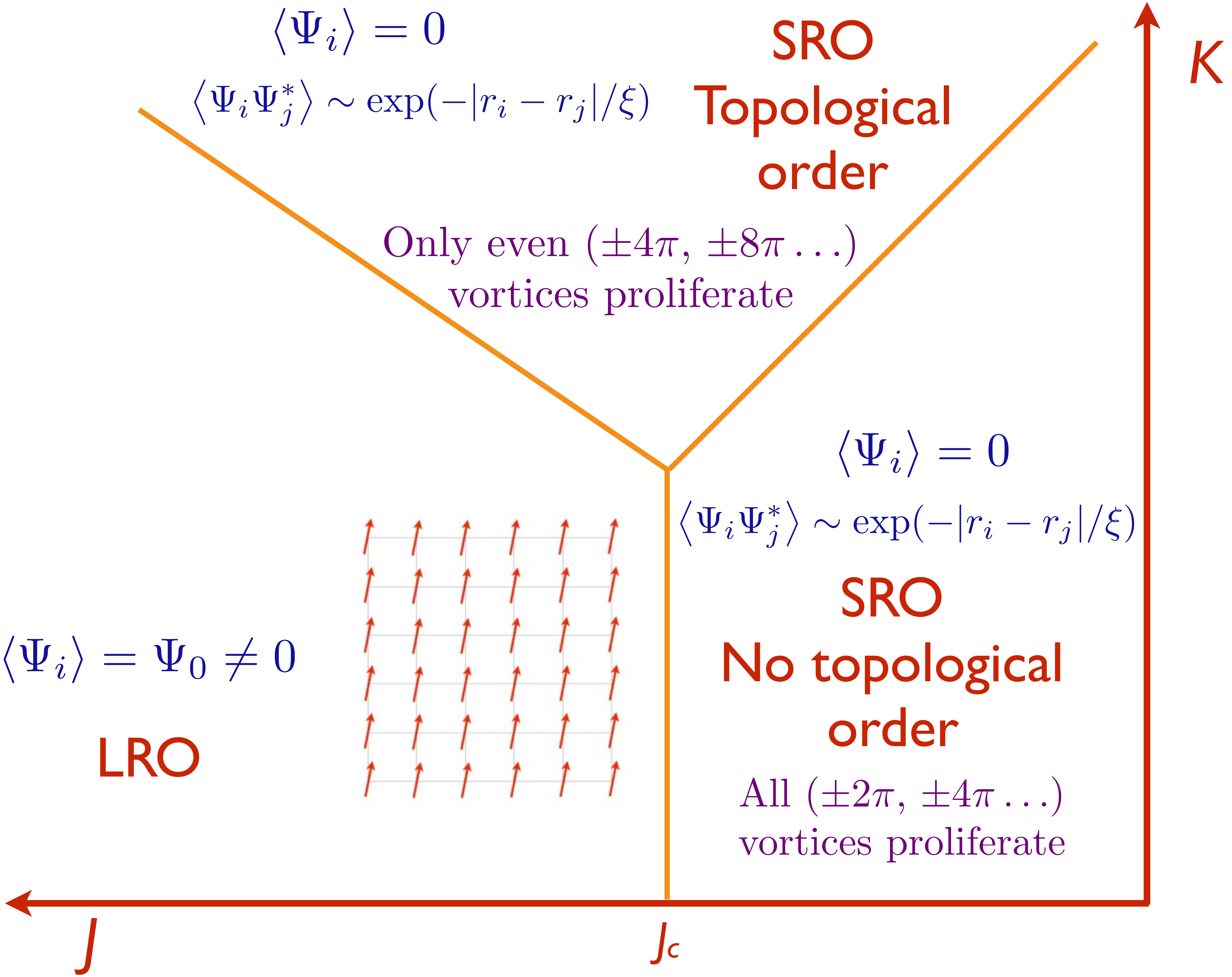

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**SRO**
**Topological**
**order**

Only even $(\pm 4\pi, \pm 8\pi \ldots)$
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**No topological**
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$$\begin{aligned}
\widetilde{\mathcal{Z}}_{XY} &= \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} \exp\left(-\widetilde{H}_{XY}[\theta]\right) \\
\widetilde{H}_{XY}[\theta] &= -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \\
&\quad + \sum_{ijk\ell} K_{ijk\ell} \cos(\theta_i + \theta_j - \theta_k - \theta_\ell) + \ldots\ldots
\end{aligned}$$

Add terms which suppress single but not double vortices…..

All allowed terms are invariant under a global U(1) symmetry ($\theta_i \to \theta_i + c$) and periodic in all the $\theta_i$ ($\theta_i \to \theta_i + 2\pi n_i$, $n_i$ integers)

We rewrite $\widetilde{\mathcal{Z}}_{XY}$ using the decomposition

$$\Psi_i = H_i \phi_i^2$$

where

$$H_i \equiv e^{i\vartheta_i} \text{ and } \phi_i \equiv e^{i\varphi_i}$$

The idea is that <u>single</u> vortices in $\Psi$ will appear as <u>single</u> vortices in $H$, while <u>double</u> vortices in $\Psi$ will appear as <u>single</u> vortices in $\phi$.

This decomposition now demands that any action be invariant under the U(1) gauge transformations

$$\vartheta_i \to \vartheta_i + 2\alpha_i \quad , \quad \varphi_i \to \varphi_i - \alpha_i$$

To obtain simple effective actions, we also introduce a U(1) gauge field $A_{i\mu}$ ($\mu = 1, 2, 3$) which transforms as

$$A_{i\mu} \to A_{i\mu} + \Delta_\mu \alpha_i$$

We now write down a U(1) gauge theory, $\mathcal{Z}_{U(1)}$ consistent the U(1) gauge invariance and the global symmetry

$$\mathcal{Z}_{U(1)} = \prod_i \int_0^{2\pi} \frac{d\vartheta_i}{2\pi} \frac{d\varphi_i}{2\pi} \prod_\mu \frac{dA_{i\mu}}{2\pi} \exp\left(-H_U[\vartheta, \varphi, A_\mu]\right)$$

$$\begin{aligned} H_U[\vartheta, \varphi, A_\mu] = & -J_1 \sum_{i,\mu} \cos(\Delta_\mu \vartheta_i - 2A_{i\mu}) \\ & -J_2 \sum_{i,\mu} \cos(\Delta_\mu \varphi_i + A_{i\mu}) \\ & -K \sum_{\square} \cos(\epsilon_{\mu\nu\lambda} \Delta_\nu A_{i\lambda}) \end{aligned}$$

Our claim is that this is the same theory as $\widetilde{\mathcal{Z}}_{XY}$; in particular

$$\prod_{i,\mu} \int_0^{2\pi} \frac{dA_{i\mu}}{2\pi} \exp\left(-H_U[\vartheta, \varphi, A_\mu]\right) \approx \exp(-\widetilde{H}_{XY}[\vartheta + 2\varphi])$$

This result follows from gauge invariance and the global U(1) symmetry, and can be explicitly established by performing the integrals over $A_{i\mu}$ order-by-order in $K$.

Villain mapping:

$$e^{J\cos(\theta)} \Rightarrow \sum_{n=-\infty}^{\infty} e^{(J/2)(\theta-2\pi n)^2} = \# \sum_{p=-\infty}^{\infty} e^{p^2/(2J)+ip\theta}$$

Villain action:

$$\begin{aligned}\mathcal{L}_V = &\frac{p_{1i\mu}^2}{2J_1} + ip_{1i\mu}(\Delta_\mu \vartheta_i - 2A_{i\mu}) + \frac{p_{2i\mu}^2}{2J_2} + ip_{2i\mu}(\Delta_\mu \varphi_i + A_{i\mu}) \\ &+ \frac{m_{\jmath\mu}^2}{2K} + im_{\jmath\mu}\epsilon_{\mu\nu\lambda}\Delta_\nu A_{i\lambda}\end{aligned}$$

Here $i$ labels sites on the direct lattice, and $\jmath$ labels sites on the dual lattice. Now we integrate over $A_{i\mu}$ and obtain $p_{2i\mu} = 2p_{1i\mu} + \epsilon_{\mu\nu\lambda}\Delta_\nu m_{\jmath\lambda}$. So

$$\mathcal{L}_V = p_{1i\mu}^2\left(\frac{1}{2J_1} + \frac{2}{J_2}\right) + ip_{1i\mu}\Delta_\mu\theta_i - \frac{2}{J_2}m_{\jmath\mu}\epsilon_{\mu\nu\lambda}\Delta_\nu p_{1i\lambda} + \frac{(\epsilon_{\mu\nu\lambda}\Delta_\nu m_{\jmath\lambda})^2}{2J_2} + \frac{m_{\jmath\mu}^2}{2K}$$

Note that the expression now only depends upon the gauge-invariant $\theta_i = \vartheta_i + 2\varphi_i$, and the first two terms generate the nearest-neighbor XY term with

$$\frac{1}{J} = \frac{1}{J_1} + \frac{4}{J_2}$$

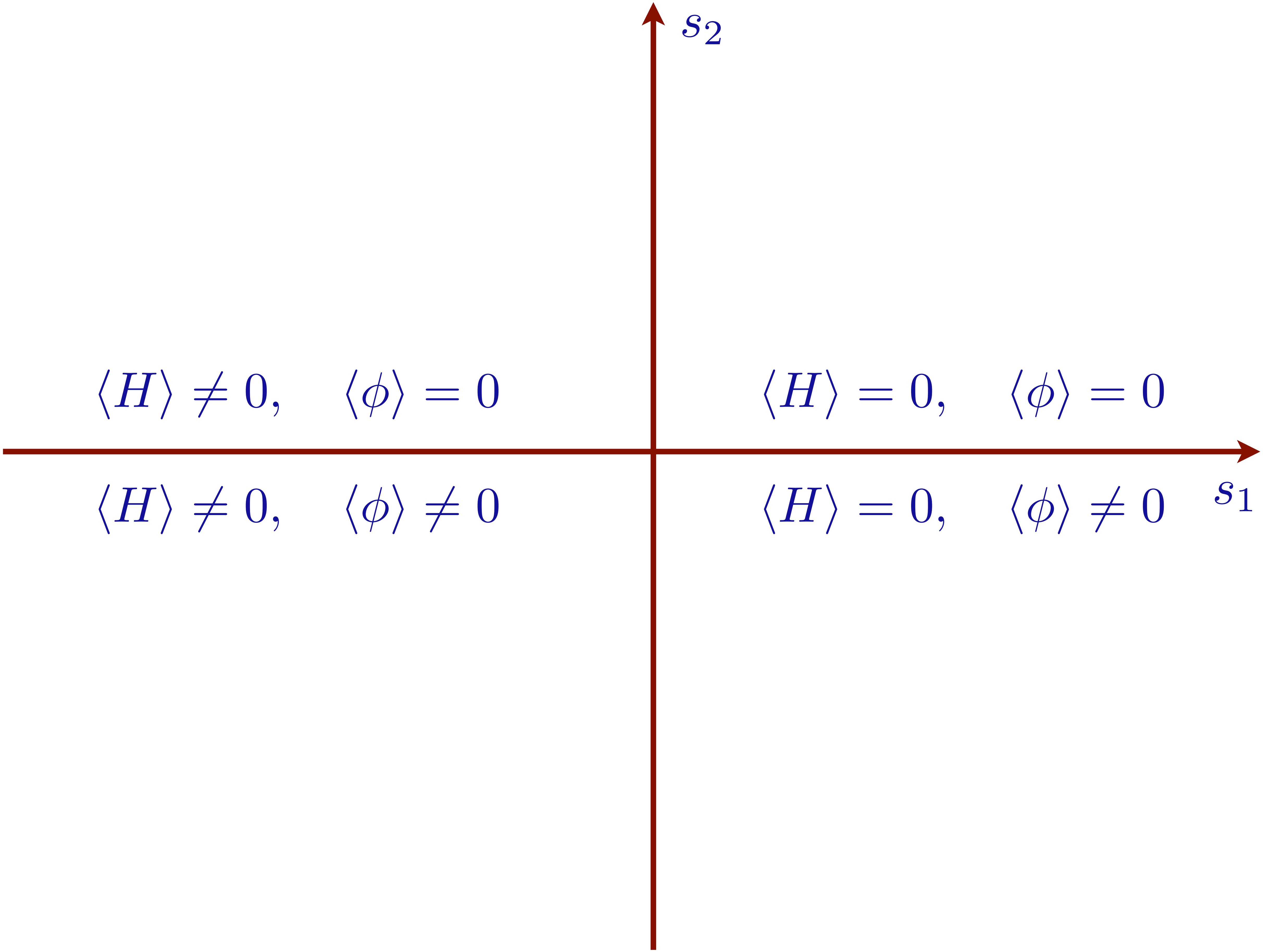
The others generate interactions around a plaquette similar to $K_{ijk\ell}$. This can be seen in an expansion in $K$: at small $K$ we sum over $m_{\jmath\mu} = \pm 1$ only to obtain the leading terms of order $e^{-1/K}$. This term involves $(\epsilon_{\mu\nu\lambda}\Delta_\nu p_{1i\lambda})^2$ and couples $\theta_i$ around a plaquette.

First we examine the phase diagram by taking a naive continuum limit of $H_U$, and studying the resulting mean-field theory

$$\begin{aligned}\mathcal{L} = &\left|(\partial_\mu - 2iA_\mu - ia_\mu^{\mathrm{ext}})H\right|^2 + s_1|H|^2 + u_1|H|^4 \\ &+ \left|(\partial_\mu + iA_\mu)\phi\right|^2 + s_2|\phi|^2 + u_2|\phi|^4 \\ &+ K(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 + \mathcal{L}_{\mathrm{monopoles}}\end{aligned}$$

We have included a fixed external field $a_\mu^{\mathrm{ext}}$ which couples to the current of the global U(1) charge. The monopoles play a crucial role, similar to those of vortices in the 2D XY model, and they will strongly modify the mean-field phase diagram.

# Mean field phase diagram



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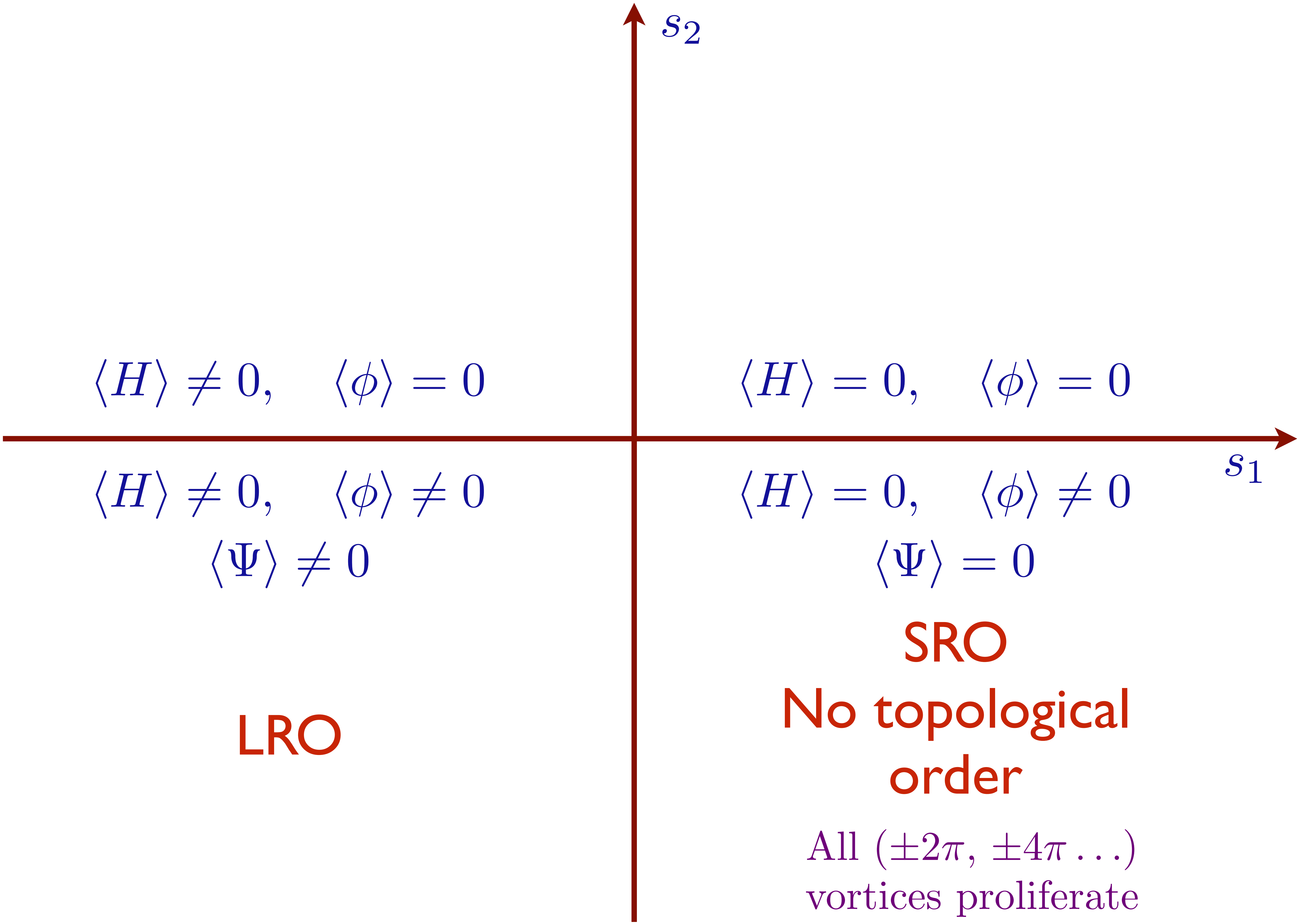

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LRO

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SRO

No topological order

All $(\pm 2\pi, \pm 4\pi \ldots)$ vortices proliferate

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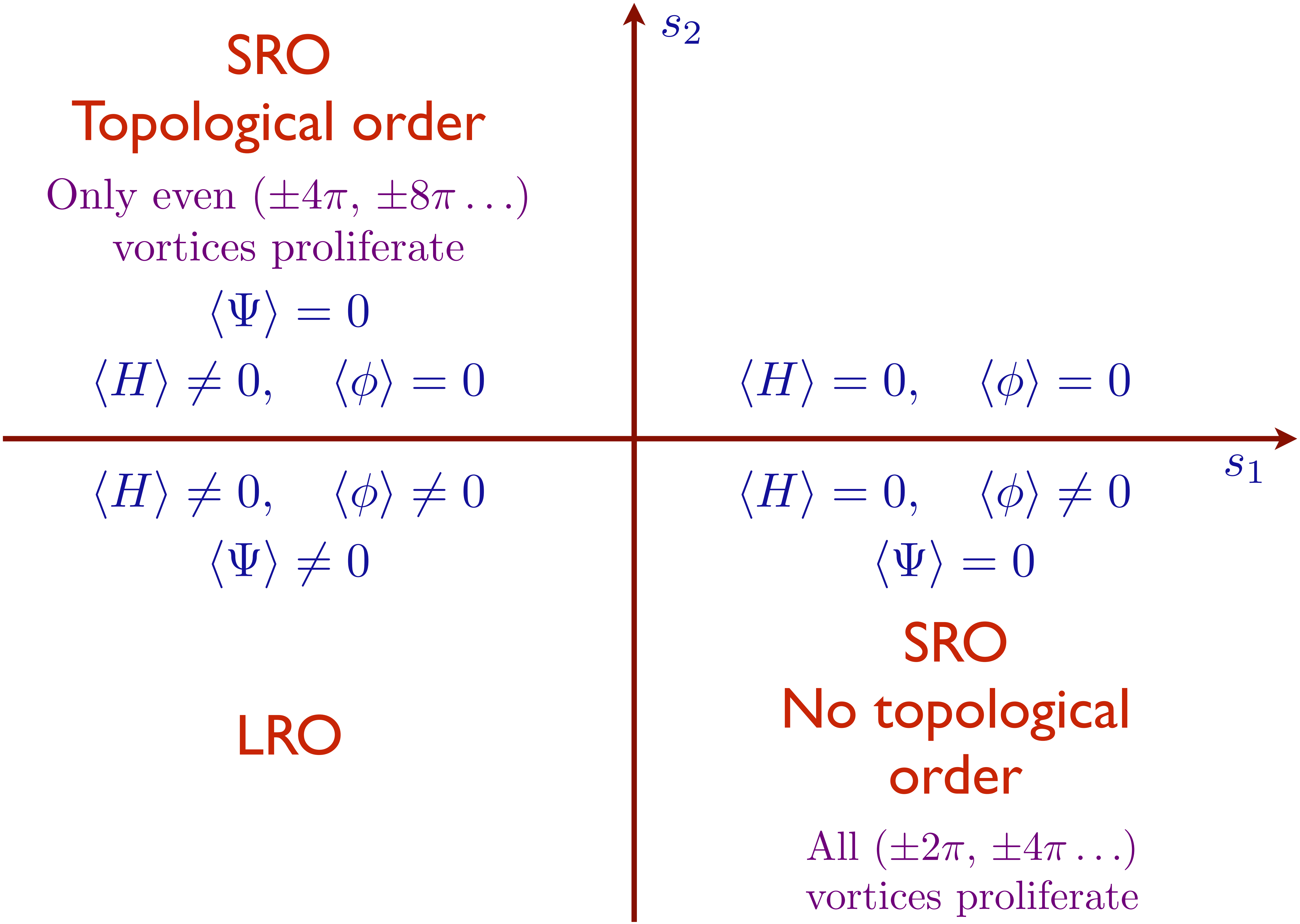

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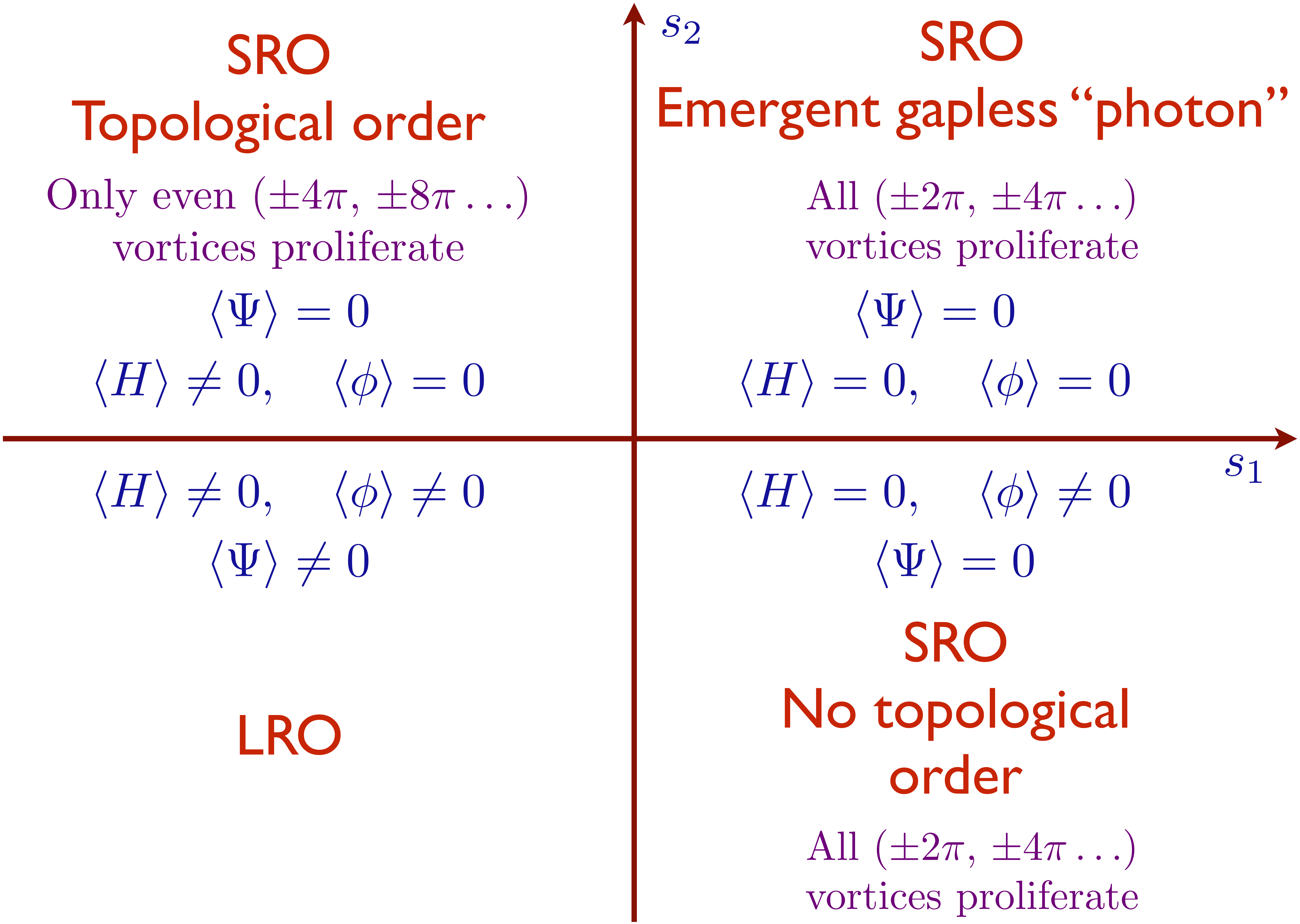

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The emergent “photon” phase is unstable to the proliferation of monopoles. The monopoles form a Coulomb plasma with $1/r$ interactions in 3D, very similar to the Coulomb plasma of vortices with $\ln(r)$ interactions in 2D. However, unlike 2D, in 3D there is never a state where monopoles are bound to antimonopoles. The $1/r$ interactions are always Debye screened, and the monopoles are effectively free. This proliferation of monopoles implies that there is no emergent gapless photon.

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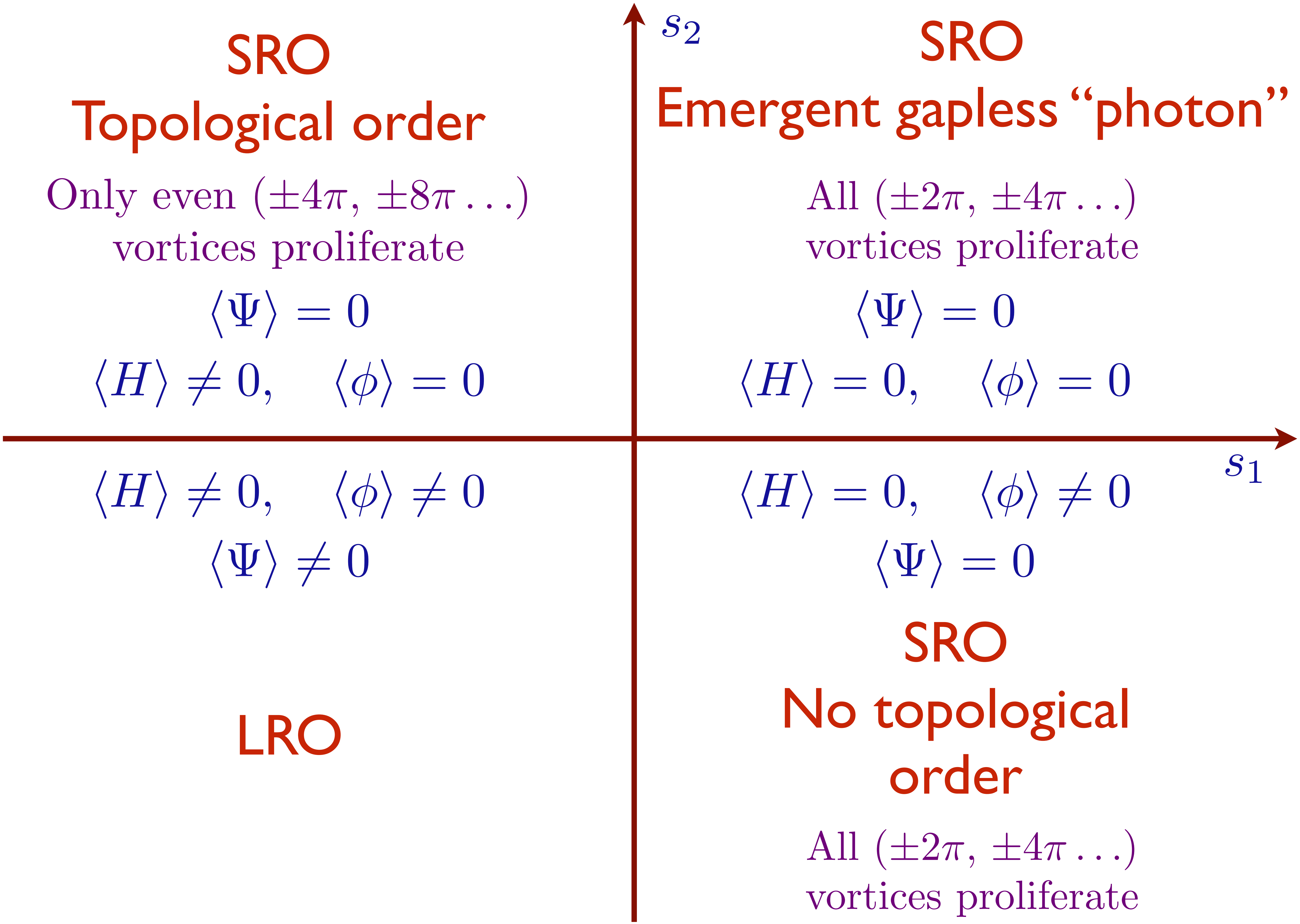

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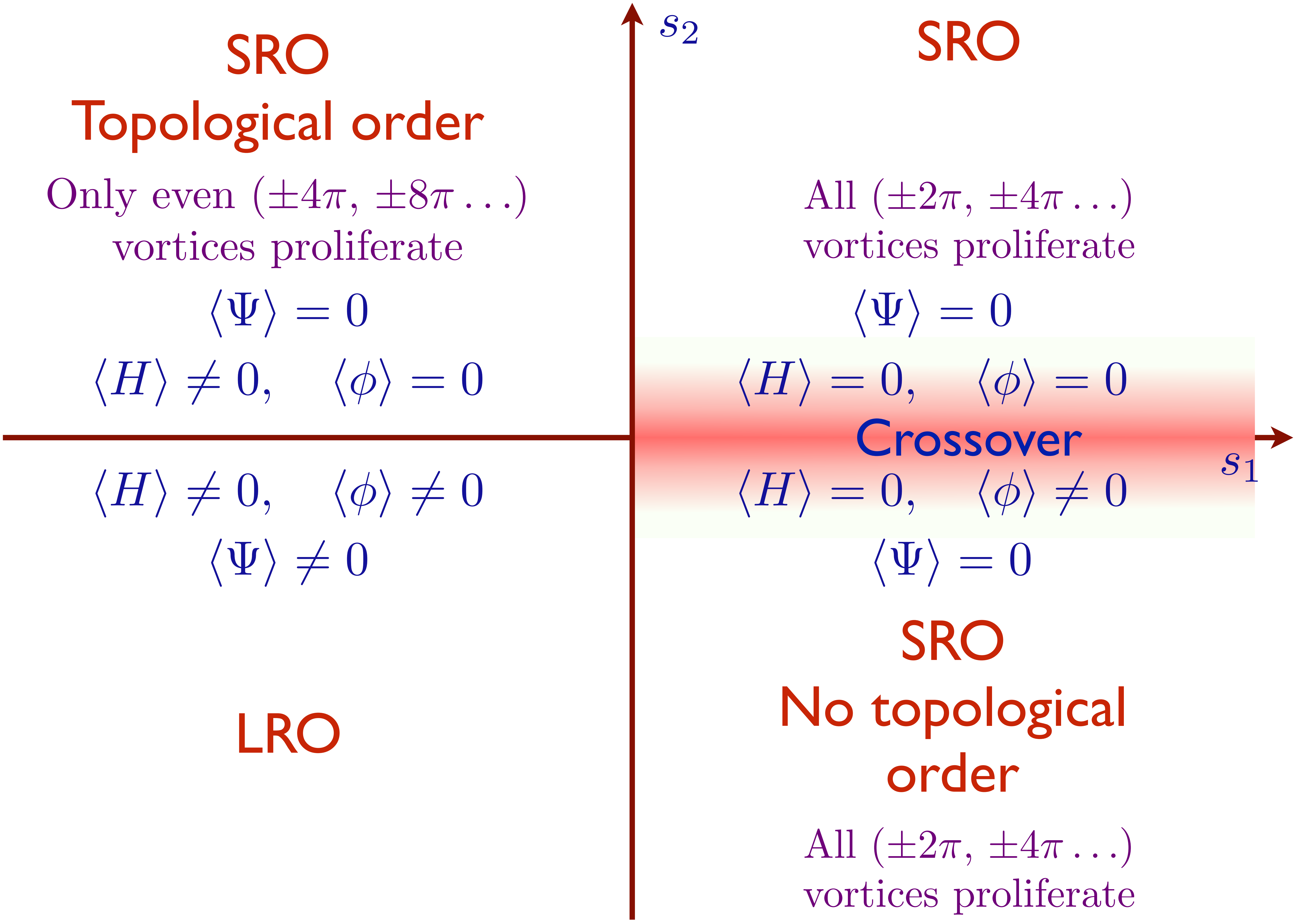

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# Structure of topological phase

The topological phase is in the regime $s_1 < 0$ and $s_2 > 0$ in the field theory

$$\mathcal{L} = |(\partial_\mu - 2iA_\mu - ia_\mu^{\text{ext}})H|^2 + s_1|H|^2 + u_1|H|^4 \\ + |(\partial_\mu + iA_\mu)\phi|^2 + s_2|\phi|^2 + u_2|\phi|^4 + K(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 + \mathcal{L}_{\text{monopoles}}$$

Perform a boson-boson (*i.e.* particle-vortex) duality on the boson $H$, while (temporarily) treating $A_\mu$ as a background field. This leads to a theory of a dual boson (vortex) $\psi$ coupled to a dual emergent gauge field $B_\mu$

$$\boxed{\begin{aligned}\mathcal{L}_{\text{dual}} = |(\partial_\mu - iB_\mu)\psi|^2 + \widetilde{s}_1|\psi|^2 + \widetilde{u}_1|\psi|^4 + \frac{i}{\pi}\epsilon_{\mu\nu\lambda}B_\mu\partial_\nu A_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}B_\mu\partial_\nu a_\lambda^{\text{ext}} \\ + |(\partial_\mu + iA_\mu)\phi|^2 + s_2|\phi|^2 + u_2|\phi|^4 + K(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 + \mathcal{L}_{\text{monopoles}}\end{aligned}}$$

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Note that when $s_1 < 0$, then $\widetilde{s}_1 > 0$: so both the $\psi$ and $\phi$ bosons are massive. Also, a monopole changes U(1) flux by $2\pi$ and this corresponds to inserting two $\psi$ bosons (each is a vortex carrying $\pi$ flux); therefore

$$\boxed{\mathcal{L}_{\rm monopoles} = \lambda\left(\psi^2 + \psi^{*2}\right)}$$

# Structure of the topological phase

- The topological phase is described by a TQFT:

$$\mathcal{L}_{TQFT} = \frac{i}{\pi}\epsilon_{\mu\nu\lambda}B_\mu\partial_\nu A_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}B_\mu a_\lambda^{\text{ext}}$$

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- This topological order is the same as that of the 'toric code', the $\phi$ are $e$ particles, and the $\psi$ are $m$ particles, and $\phi$-$\psi$ bound state is the $\epsilon$ particle. There is 4-fold ground state degeneracy on a large torus.

# Structure of the topological phase

- The topological phase can also be identified with the deconfined phase of an emergent $\mathbb{Z}_2$ gauge theory. The $\phi$ particle has a $\mathbb{Z}_2$ electric charge, the $\psi$ particle carries $\mathbb{Z}_2$ magnetic flux, and $\phi$-$\psi$ bound state is a fermionic dyon.

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- The topological phase also describes Anderson's 'RVB' for spin $S = 1$ antiferromagnets, identified here as a '$\mathbb{Z}_2$ spin liquid'. The $S = 1/2$ case is similar, but not identical. An important difference is the presence of an 't-Hooft anomaly between translations and the global U(1) symmetry, which prevents the existence of a trivial SRO phase.

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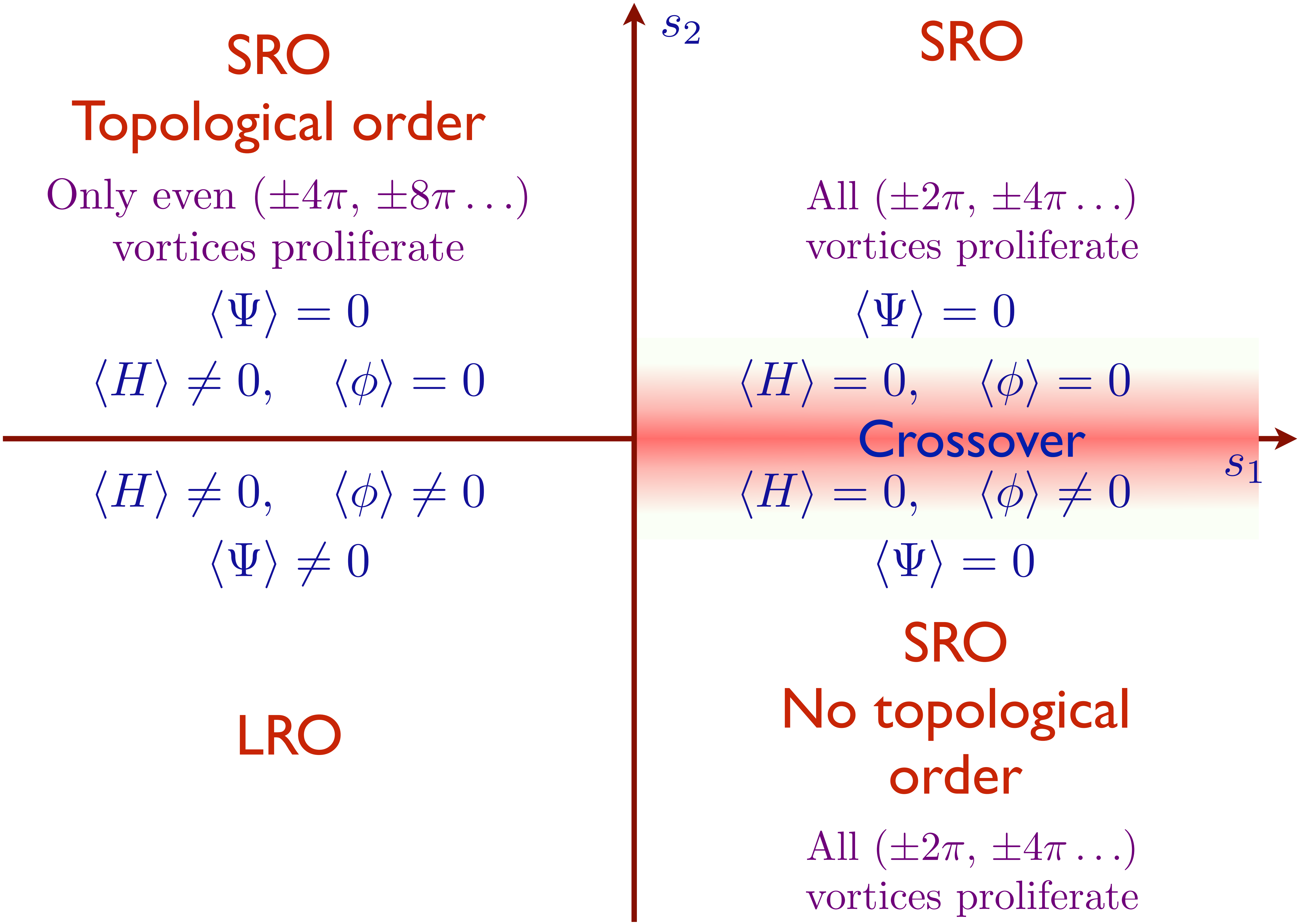
- The transition between the topological phase and SRO phase is described by a Ising* theory for $\psi$ alone. (The * implies that critical point CFT only includes operators invariant under $\psi \to -\psi$.) Ignoring the massive $\phi$ near the critical point, and integrating $A_\mu$ to Higgs $B_\mu$, we obtain the Ising Wilson-Fisher field theory

$$\mathcal{L}_I = |\partial_\mu \psi|^2 + \widetilde{s}_1 |\psi|^2 + \widetilde{u}_1 |\psi|^4 + \lambda \left( \psi^2 + \psi^{*2} \right)$$

# Phase diagram



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