Universal theory of complex SYK models and extremal charged black holes

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Jerusalem Winter School,
December 31, 2018

PHYSICS

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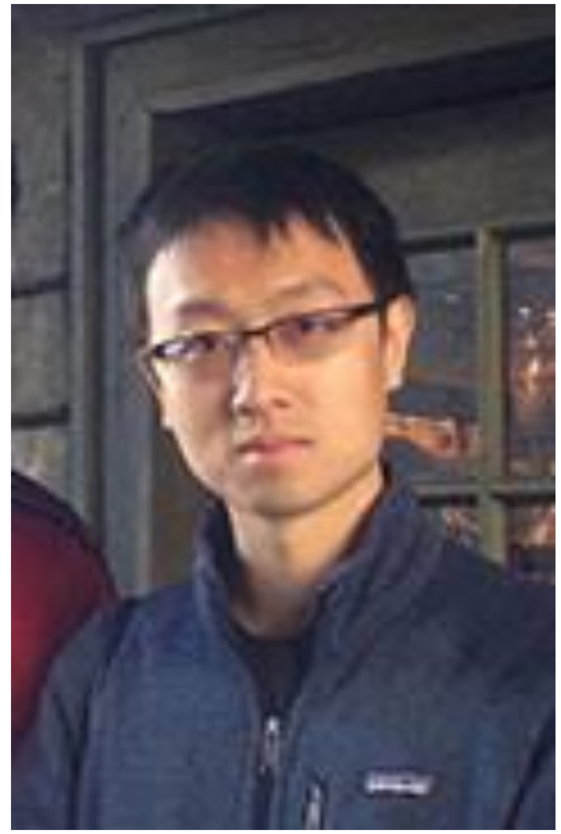
Wenbo Fu



Yingfei Gu

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DEPARTMENT OF ENERGY · UNITED STATES OF AMERICA

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1. Quantum matter with quasiparticles:
   random matrix model

2. Quantum matter without quasiparticles:
   the complex SYK model

3. Einstein-Maxwell theory of charged
   black holes in AdS space

4. Fluctuations, and the Schwarzian

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A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

Fermions occupying the eigenstates of a
N x *N* random matrix

A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

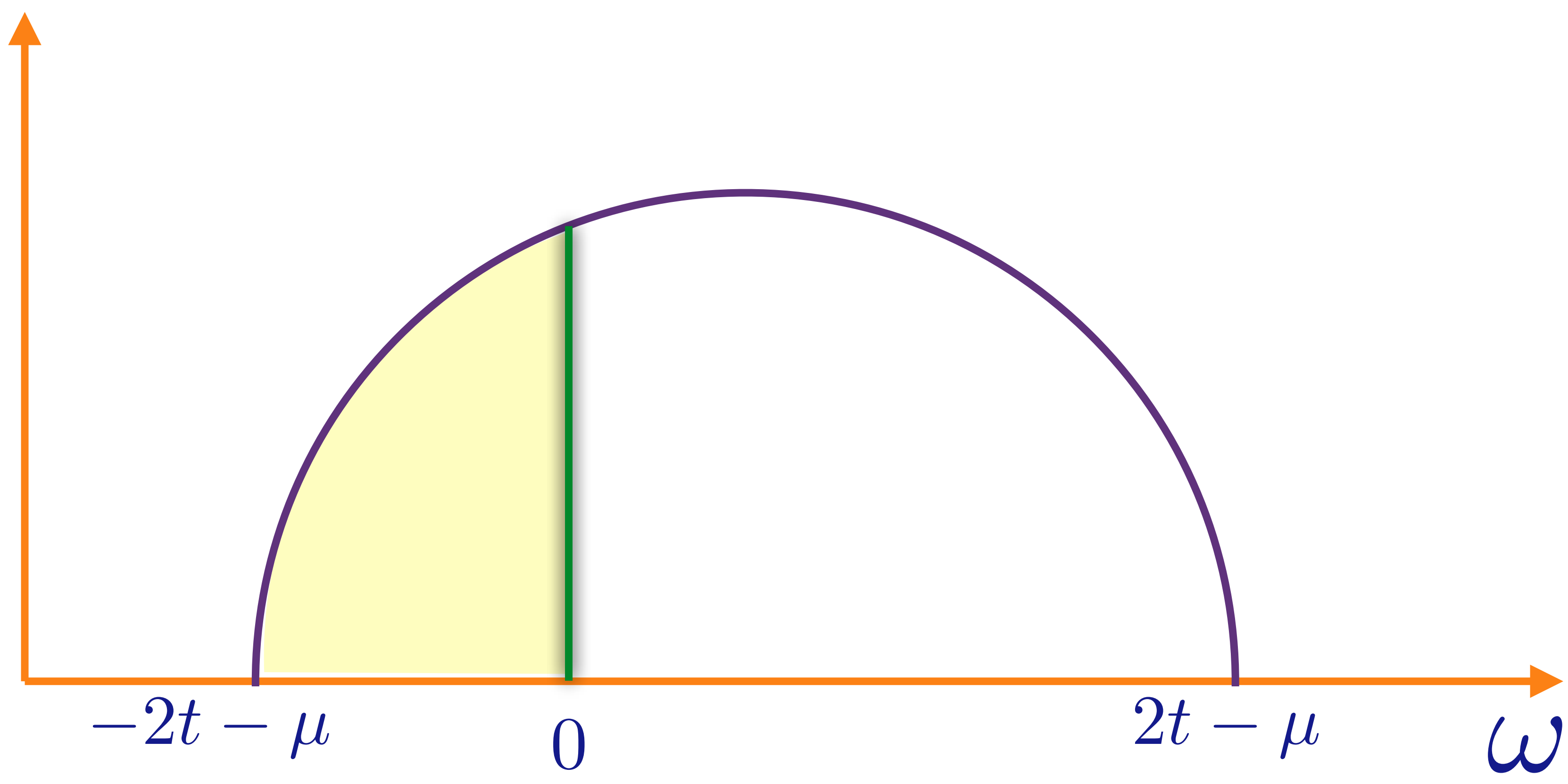
$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

$G(\omega)$ can be determined by solving a quadratic equation.

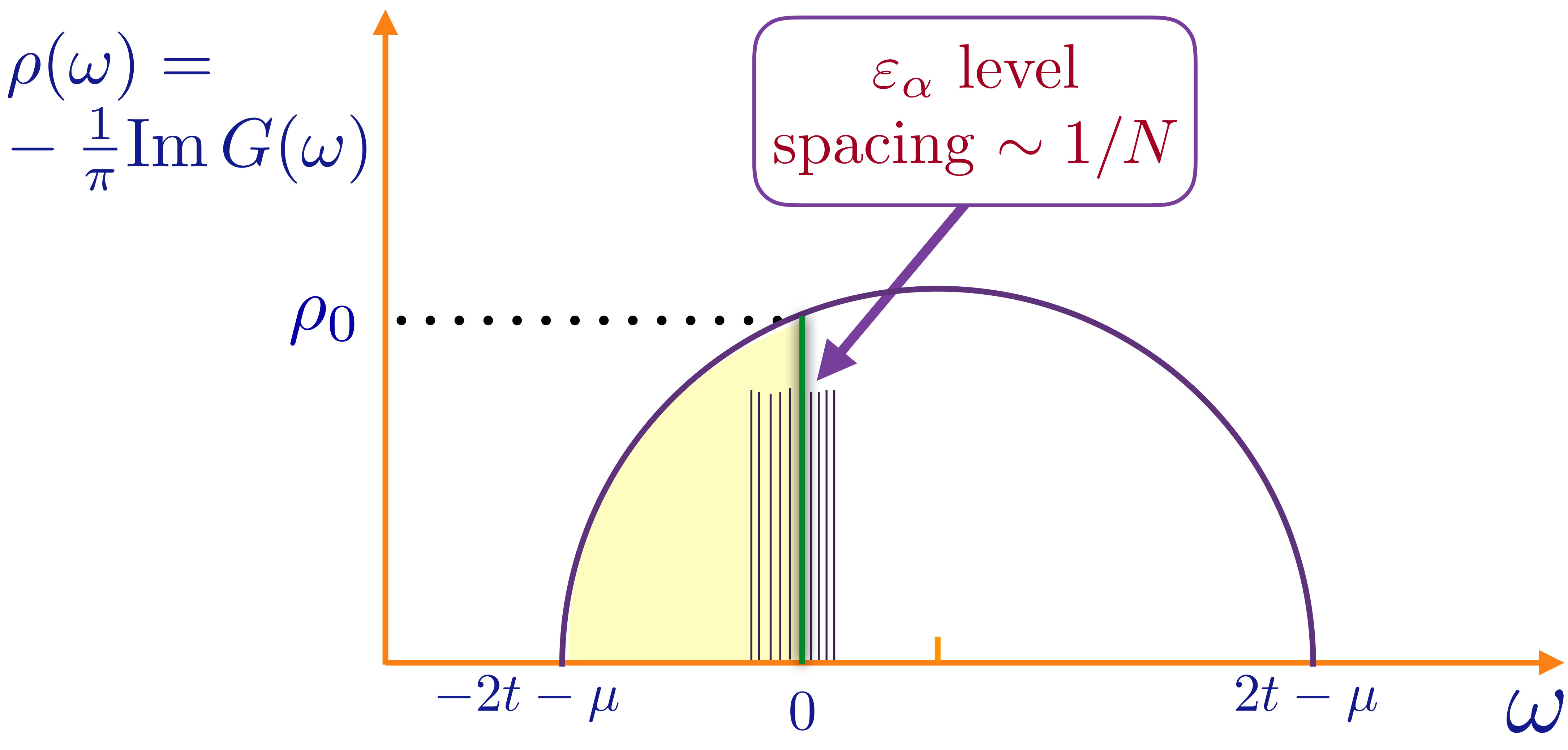
$$\rho(\omega) = -\frac{1}{\pi} \text{Im}\, G(\omega)$$



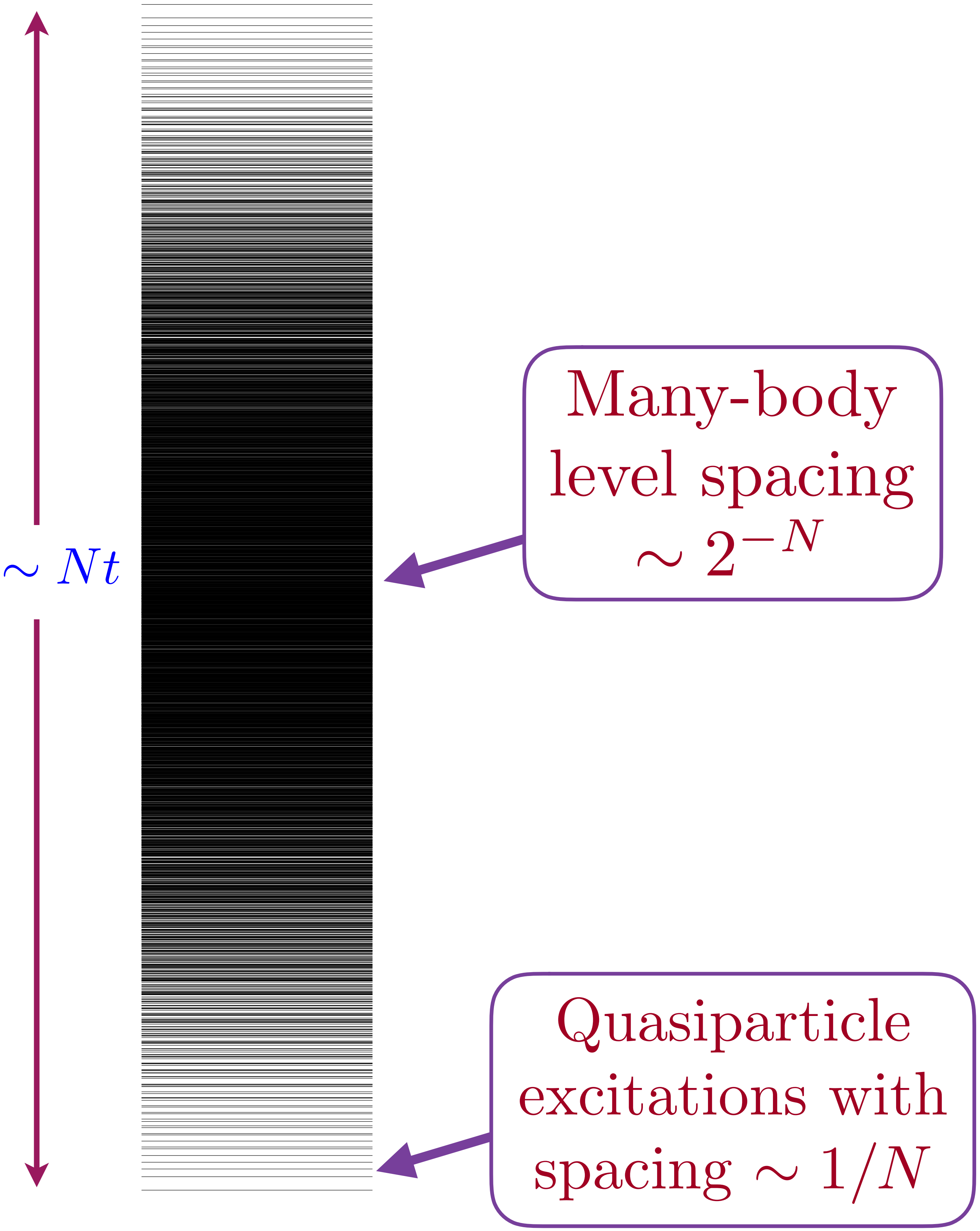
A simple model of a metal with quasiparticles

Let ε_α be the eigenvalues of the matrix $t_{ij}/\sqrt{N}$. The fermions will occupy the lowest NQ eigenvalues, upto the Fermi energy E_F. The single-particle density of states is

$\rho(\omega) = (1/N)\sum_\alpha \delta(\omega - \varepsilon_\alpha)$, and $\rho_0 \equiv \rho(\omega = 0)$.



A simple model of a metal with quasiparticles



There are 2^N many body levels with energy

$$E = \sum_{\alpha=1}^{N} n_\alpha \varepsilon_\alpha,$$

where $n_\alpha = 0, 1$. Shown are all values of E for a single cluster of size $N = 12$. The ε_α have a level spacing $\sim 1/N$.

A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N\left(-\frac{\pi^2}{6}\rho_0 T^2 + \mathcal{O}(T^4)\right) + \ldots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.
We can also define the *many* body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$D(E) \sim \exp\left(\pi\sqrt{\frac{2N\rho_0(E-E_0)}{3}}\right) \quad , \quad E > E_0 \ , \ \frac{1}{N} \ll \rho_0(E-E_0) \ll N$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

A simple model of a metal with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;k\ell} \, c_i^\dagger c_j^\dagger c_k c_\ell$$

$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$. We compute the lifetime of a quasiparticle, τ_α, in an exact eigenstate $\psi_\alpha(i)$ of the free particle Hamitonian with energy ε_α. By Fermi's Golden rule, for ε_α at the Fermi energy

$$\begin{aligned} \frac{1}{\tau_\alpha} &= \pi U^2 \rho_0^2 \int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\delta f(\varepsilon_\beta)(1 - f(\varepsilon_\gamma))(1 - f(\varepsilon_\delta))\delta(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_\gamma - \varepsilon_\delta) \\ &= \frac{\pi^3 U^2 \rho_0^2}{4} T^2 \end{aligned}$$

where ρ_0 is the density of states at the Fermi energy, and $f(\epsilon) = 1/(e^{\epsilon/T} + 1)$ is the Fermi function.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as $\sim T^{-2}$ at the Fermi level.

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The complex SYK model

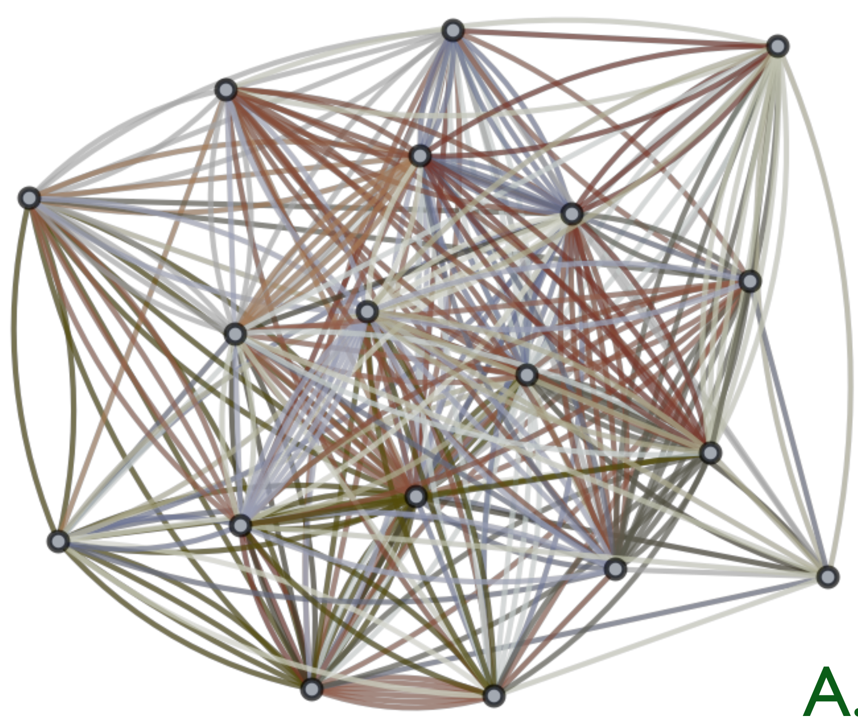
(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large N limit; T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\mathcal{Q} = \frac{1}{N} \sum_i c_i^\dagger c_i$$

$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$
$N \to \infty$ yields critical strange metal.



S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

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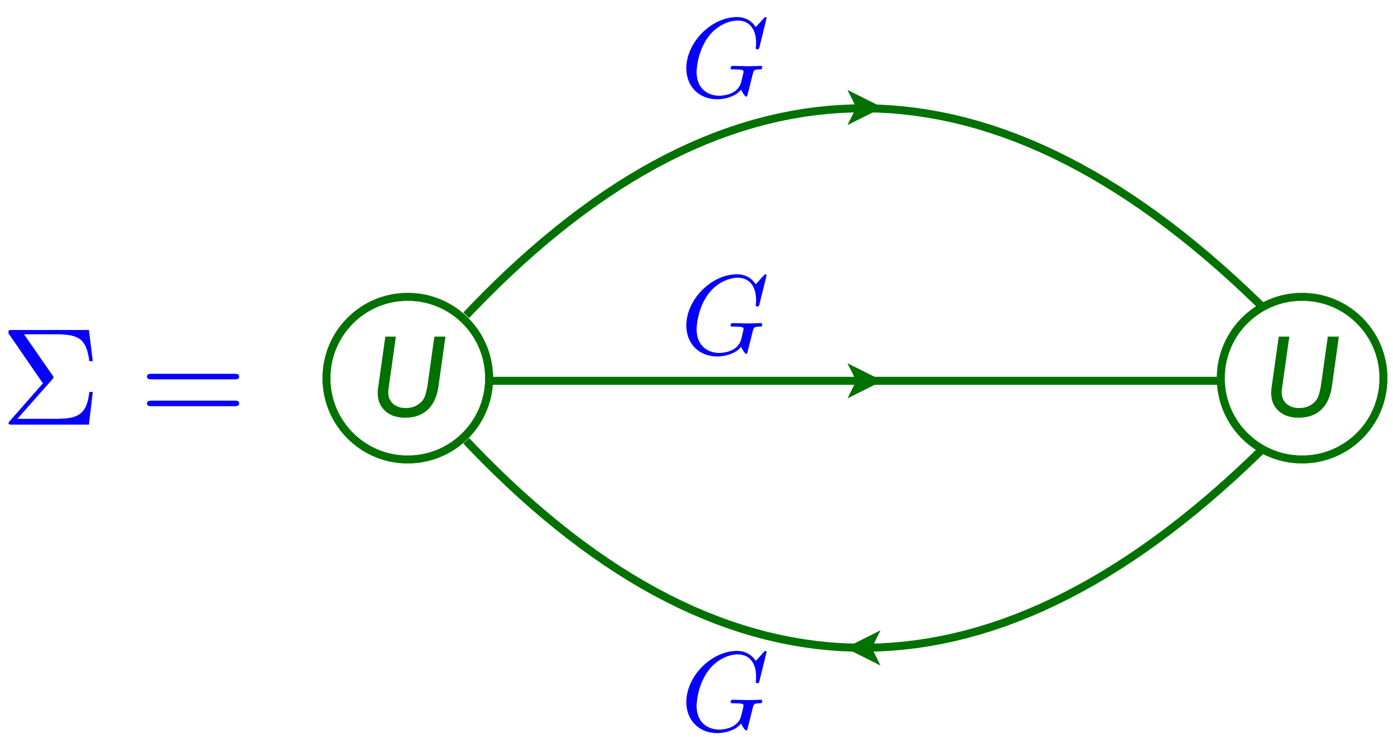

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

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Solution of these equations and corresponding evaluation of the free energy yields the following universal results (*i.e.* all results are *quantitatively* unchanged by adding additional higher q fermion terms):

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- At general charge $\mathcal{Q}$, there is a spectral symmetry determined by a parameter $\mathcal{E}$:

$$G(\tau) \sim \begin{cases} -\tau^{-2\Delta} & \tau > 0 \\ e^{-2\pi\mathcal{E}}(-\tau)^{-2\Delta} & \tau < 0 \end{cases} \quad , \quad T = 0$$

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- There is a universal 'Luttinger relation' between $-\infty < \mathcal{E} < \infty$ and the total charge $0 < \mathcal{Q} < 1$

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi\Delta + \theta)}{\sin(\pi\Delta - \theta)}$$

$$\mathcal{Q} = \frac{1}{2} - \frac{\theta}{\pi} + \left(\Delta - \frac{1}{2}\right)\frac{\sin(2\theta)}{\sin(2\pi\Delta)}$$

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

The complex SYK model

Solution of these equations and corresponding evaluation of the free energy yields the following universal results (*i.e.* all results are *quantitatively* unchanged by adding additional higher q fermion terms):

- At $T > 0$, we obtain a solution with a conformal structure

$$G(\tau) = -A \frac{e^{-2\pi \mathcal{E} T \tau}}{\sqrt{1 + e^{-4\pi \mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \,,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The complex SYK model

We now examine the behavior of the chemical potential, μ, as $T \to 0$ at fixed $\mathcal{Q}$. For this we relate the long-time 'conformal' Greens function, (valid for $\tau \gg 1/U$) to its short-time behavior. In particular at $|\omega_n| \gg U$ we have

$$G(i\omega_n) = \frac{1}{i\omega_n} - \frac{\mu}{(i\omega_n)^2} + \ldots$$

which implies for the spectral density of the Green's function, $\rho(\Omega)$

$$\mu = -\int_{-\infty}^{\infty} \frac{d\Omega}{\pi}\, \Omega \rho(\Omega),$$

which makes it evident that μ depends only upon the particle-hole asymmetric part of the spectral density. Next, we can relate the Ω integrals to the derivative of the imaginary time correlator

$$\mu = -\partial_\tau G(\tau = 0^+) - \partial_\tau G(\tau = (1/T)^-).$$

The complex SYK model

We pull out an explicitly particle-hole asymmetric part of $G(\tau)$ by defining

$$G(\tau) \equiv e^{-2\pi \mathcal{E} T \tau} G_c(\tau) \quad , \quad 0 < \sigma < \frac{1}{T}.$$

where G_c will be given by a particle-hole symmetric conformal form at low T and low ω. Then we obtain

$$\begin{aligned} \mu &= 2\pi \mathcal{E} T \left[G(\tau = 0^+) + G(\tau = (1/T)^-) \right] - \partial_\tau g(\tau = 0^+) \\ &\qquad + \text{terms dependent on } G_c \\ &= -2\pi \mathcal{E} T + \text{terms dependent on } G_c \end{aligned}$$

It can be shown that all the terms dependent upon G_c have a T dependence that is weaker than linear in T provided $\mathcal{Q}$ is held fixed. Hence we have

$$\mu = \mu_0 - 2\pi \mathcal{E} T + \text{terms vanishing as } T^p \text{ with } p > 1$$

with μ_0 a non-universal constant. From this relation we obtain

The complex SYK model

with μ_0 a non-universal constant. From this relation we obtain

$$\left(\frac{\partial \mu}{\partial T}\right)_{\mathcal{Q}} = -2\pi\mathcal{E} \quad , \quad T \to 0,$$

Using a Maxwell relation we then have

$$\frac{1}{N}\left(\frac{\partial S}{\partial \mathcal{Q}}\right)_{T} = 2\pi\mathcal{E} \neq 0 \;\; \text{as } T \to 0.$$

The complex SYK model

Solution of these equations and corresponding evaluation of the free energy yields the following universal results (*i.e.* all results are *quantitatively* unchanged by adding additional higher q fermion terms):

- There is a non-vanishing entropy in the zero temperature limit

$$S(T \to 0) = N s_0 + \ldots$$

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- There is a non-vanishing entropy in the zero temperature limit

$$S(T \to 0) = N s_0 + \ldots$$

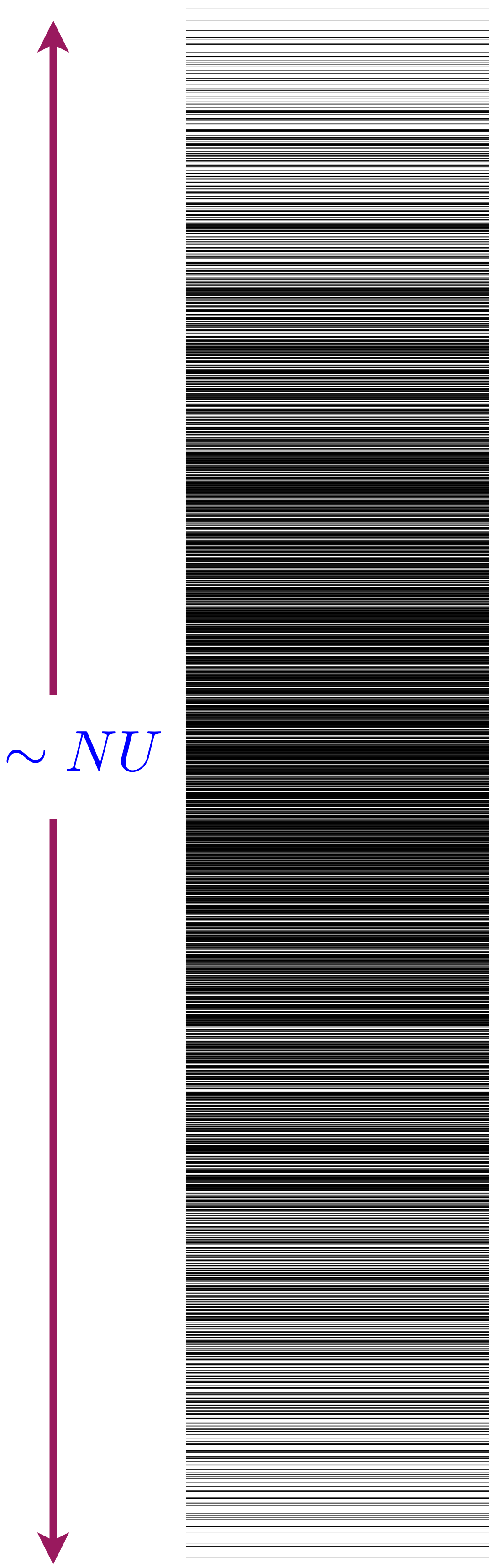
- The saddle point equations imply the relation

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}}$$

Integrating this relation from $s_0 = 0$, $\mathcal{Q} = 0$, allows us to compute s_0 as a function of $\mathcal{Q}$.

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

The complex SYK model



Many-body level spacing $\sim 2^{-N} = e^{-N \ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-N s_0}$

There are 2^N many body levels with energy E. Shown are all values of E for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}\,.$$

At $\mathcal{Q} = 1/2$,

$$s_0 = \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where G is Catalan's constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

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S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)
S. Sachdev, PRX **5**, 041025 (2015)

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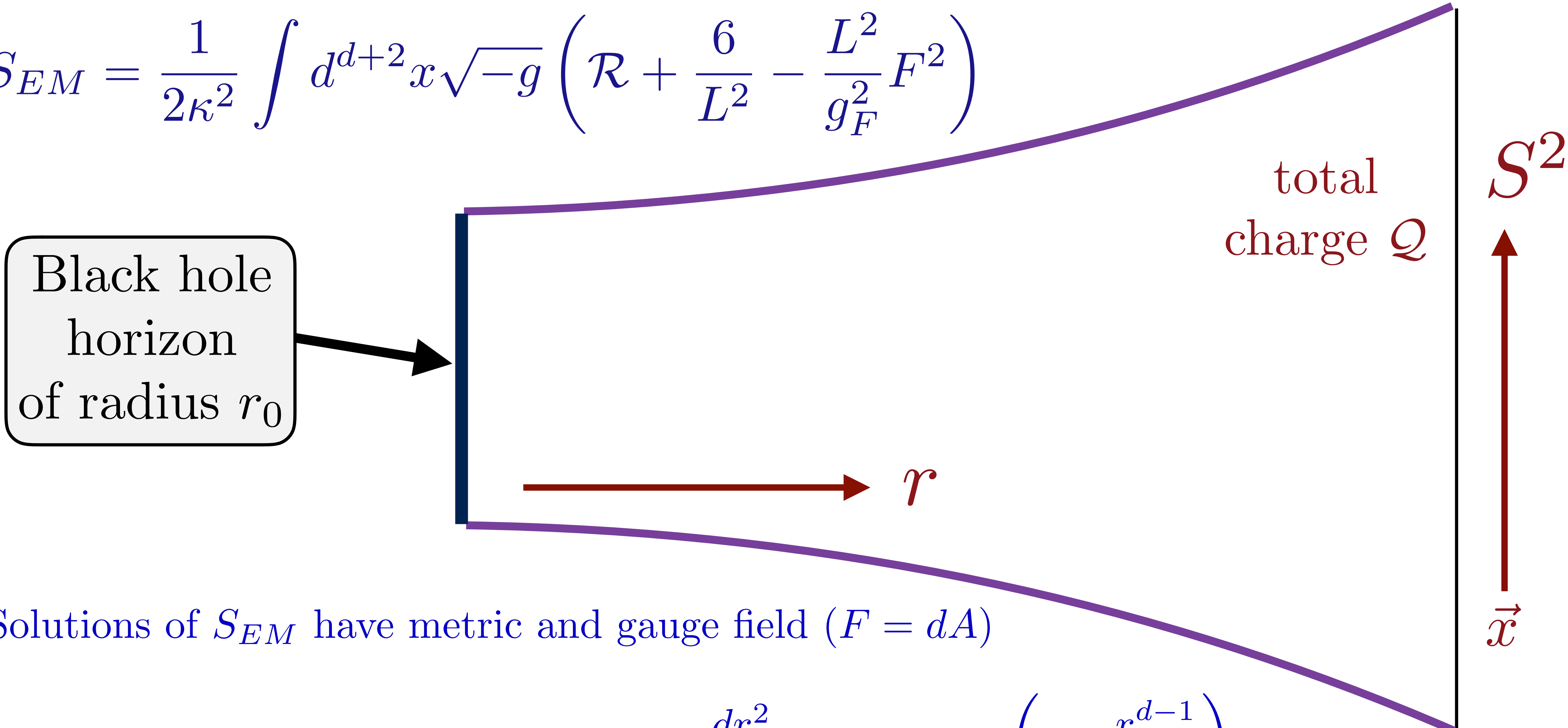
Charged black holes

$$S_{EM} = \frac{1}{2\kappa^2}\int d^{d+2}x\sqrt{-g}\left(\mathcal{R} + \frac{6}{L^2} - \frac{L^2}{g_F^2}F^2\right)$$



Solutions of S_{EM} have metric and gauge field ($F = dA$)

$$ds^2 = -V(r)dt^2 + r^2 d\Omega_d^2 + \frac{dr^2}{V(r)} \quad , \quad A = \mu\left(1 - \frac{r_0^{d-1}}{r^{d-1}}\right)dt$$

$$V(r) = 1 + \frac{r^2}{L^2} + \frac{\Theta^2}{r^{2d-2}} - \frac{M}{r^{d-1}}.$$

where $d\Omega_d^2$ is the metric of the d-sphere. All parameters of the solution are determined in terms of the chemical potential μ, and the Hawking temperature of horizon, T.

A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, PRD **60**, 064018 (1999)

Charged black holes

In the $T \to 0$ limit, at fixed μ, we obtain a charged black hole solution with radius $r_0(T \to 0, \mu) = R_h$. All properties of this black hole can be expressed in terms of R_h

- The total charge in the black hole is

$$\mathcal{Q} = \frac{R_h^{d-1}\sqrt{2d\left[(d+1)R_h^2 + (d-1)L^2\right]}}{\kappa^2 g_F}$$

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$$\mathcal{Q} = \frac{R_h^{d-1}\sqrt{2d\left[(d+1)R_h^2 + (d-1)L^2\right]}}{\kappa^2 g_F}$$

- The Bekenstein-Hawking entropy remains finite as $T \to 0$ (s_d is the area of the d-dimensional surface of a unit sphere)

$$S(T \to 0) = s_0 + \ldots \quad , \quad s_0 = \frac{2\pi s_d}{\kappa^2} R_h^d$$

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In the $T \to 0$ limit, at fixed μ, we obtain a charged black hole solution with radius $r_0(T \to 0, \mu) = R_h$. All properties of this black hole can be expressed in terms of R_h

- In the near-horizon region, we change co-ordinates from r to ζ so that

$$r - R_h = \frac{R_2^2}{\zeta} \quad , \quad R_2 = \frac{L R_h}{\sqrt{d(d+1)R_h^2 + (d-1)^2 L^2}}.$$

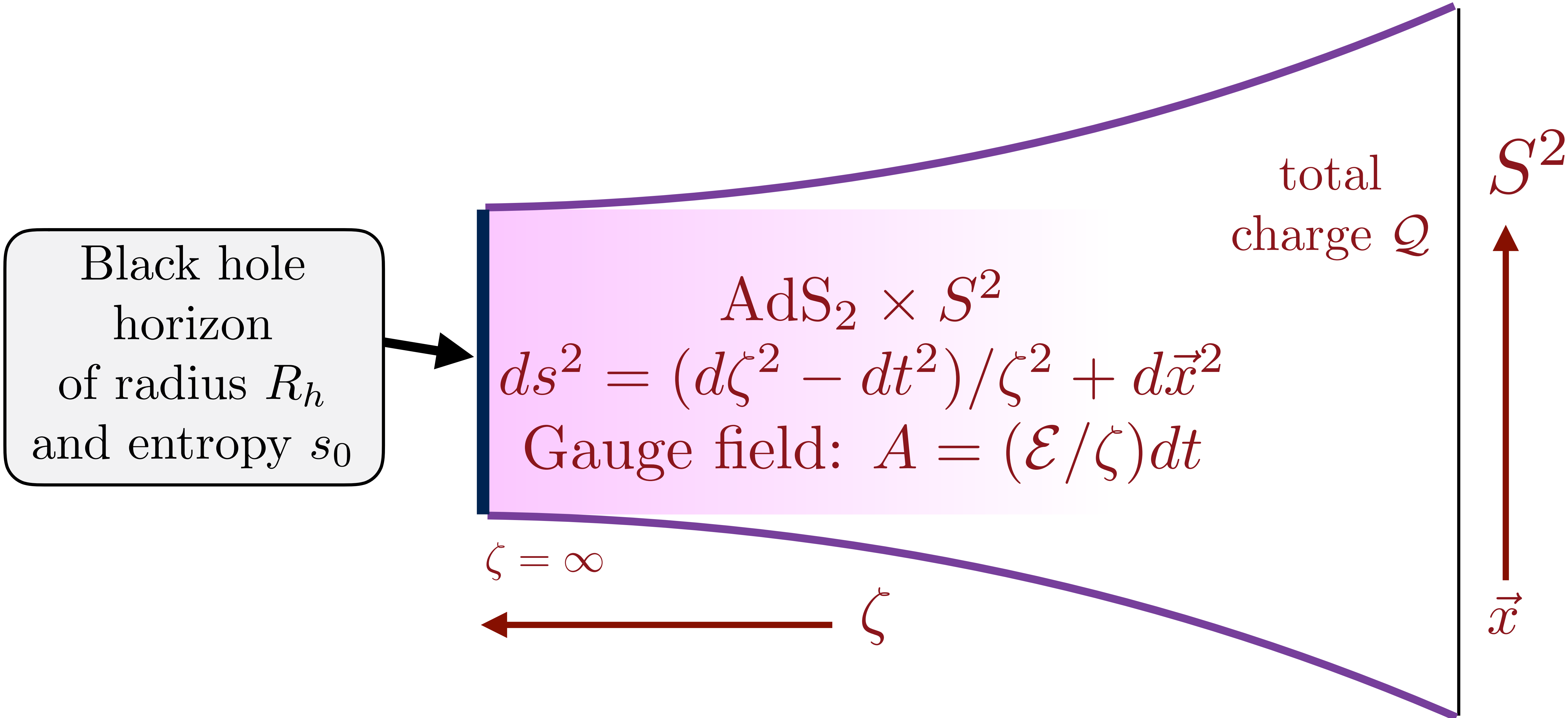
Then the near-horizon metric becomes $AdS_2 \times S_d$, with

$$ds^2 = R_2^2 \left[\frac{-dt^2 + d\zeta^2}{\zeta^2} \right] + R_h^2 d\Omega_d^2 \quad , \quad A = \frac{\mathcal{E}}{\zeta} dt \, .$$

where the dimensionless electric field $\mathcal{E}$ is

$$\mathcal{E} = \frac{g_F R_h \sqrt{2d\left[(d+1)R_h^2 + (d-1)L^2\right]}}{2\left[d(d+1)R_h^2 + (d-1)^2 L^2\right]}.$$

Charged black holes



- The entropy s_0, the charge $\mathcal{Q}$, and the dimensionless electric field $\mathcal{E}$ obey

$$\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}$$

A. Sen, JHEP **0509**, 038 (2005)

Charged black holes

Dimensional reduction to 2D gravity

We neglect all dependence of the metric on angular co-ordinates, introduce a scalar field $\phi(r,\tau)$, and write the 4D metric as

$$ds^2 = \frac{1}{\sqrt{\phi}} h_{ab} dx^a dx^b + \phi \, d\Omega_2^2$$

where $a, b = x, \tau$ and h_{ab} is a 2D metric. Then the 4D action S_{EM} reduces to an action for 2D gravity

$$S_{JT} = \frac{2\pi}{\kappa^2} \int d^2x \sqrt{-h} \left(\phi \, \mathcal{R}_{2D} + V(\phi) - \frac{Z(\phi) L^2}{g_F^2} F^2 \right)$$

with

$$V(\phi) = \frac{2}{\sqrt{\phi}} + \frac{6\sqrt{\phi}}{L^2} \quad , \quad Z(\phi) = \phi^{3/2}$$

This describes Jackiw-Teitelbaum 2D gravity, along with a 2D electromagnetic field.

Charged black holes

Probe fermion in the AdS_2 near horizon

- A probe fermion has a near-horizon Green's function with a conformal structure

$$G(\tau) = -A \frac{e^{-2\pi \mathcal{E} T \tau}}{\sqrt{1 + e^{-4\pi \mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \,,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$. This is identical to the complex SYK model.

T. Faulkner, Hong Liu, J. McGreevy, D. Vegh, PRD **83**, 125002 (2011)

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$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

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$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

At frequencies $\ll U$, the $i\omega + \mu$ can be dropped, and without it equations are invariant under the reparametrization and gauge transformations.
The singular part of the self-energy and the Green's function obey

$$\int_0^\beta d\tau_2 \, \Sigma_{\text{sing}}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma_{\text{sing}}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

A. Kitaev, 2015

S. Sachdev, PRX **5,** 041025 (2015)

The SYK model

$$\int_0^\beta d\tau_2 \, \Sigma_(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

These equations are invariant under

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
By using $f(\sigma) = \tan(\pi T \sigma)/(\pi T)$ we can
now obtain the $T > 0$ solution from the $T = 0$ solution.

A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

The SYK model

Let us write the large N saddle point solutions of S as

$$\begin{aligned} G_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-1/2} \\ \Sigma_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-3/2}. \end{aligned}$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for Σ). It turns out this is true only for the SL(2, R) transformations under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

A. Kitaev

Fluctuations

- The saddle-point

$$G(\tau_1 - \tau_2) = -A \frac{e^{-2\pi \mathcal{E} T(\tau_1 - \tau_2)}}{\sqrt{1 + e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T(\tau_1 - \tau_2))} \right)^{2\Delta}$$

is invariant only under PSL(2, R) transformations which map the thermal circle onto itself, and an associated gauge transformation

$$\frac{\tan(\pi T f(\tau))}{\pi T} = \frac{a \dfrac{\tan(\pi T \tau)}{\pi T} + b}{c \dfrac{\tan(\pi T \tau)}{\pi T} + d} \quad , \quad ad - bc = 1,$$

$$-i\phi(\tau) = -i\phi_0 + 2\pi \mathcal{E} T (\tau - f(\tau))$$

A. Kitaev, 2015

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

Infinite-range (SYK) model without quasiparticles

After introducing replicas $a = 1 \dots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp \left[-\sum_{ia} \int_0^\beta d\tau \, c_{ia}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{ia} \right.$$
$$\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) \right. \right.$$
$$\left. \left. + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \right) \right] .$$

Infinite-range (SYK) model without quasiparticles

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

$$S = \ln \det \left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right]$$

$$+ \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left[G(\tau_2, \tau_1) + (U^2/2) G^2(\tau_2, \tau_1) G^2(\tau_1, \tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

The SYK model

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_1)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1)-i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-NS_{\text{eff}}[f,\phi]}.$$

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Fluctuations

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\text{eff}}[f, \phi] = \frac{NK}{2} \int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T) \partial_\tau f)^2 - \frac{N\gamma}{4\pi^2} \int_0^{1/T} d\tau \, \{\tan(\pi T f(\tau)), \tau\},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g, \tau\} \equiv \frac{g'''}{g'} - \frac{3}{2} \left(\frac{g''}{g'} \right)^2 .$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\text{eff}} \left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0 \right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

Fluctuations

An *exact* path integral over the effective action leads to the following physical consequences

- The ground state energy with fermion number $N\mathcal{Q} + p$ (p integer) varies as

  $$E_p = E_0 + \frac{p^2}{2NK}$$

  This identifies K with the compressibility $K = d\mathcal{Q}/d\mu$ at $T = 0$.

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- The low temperature corrections to the entropy at *fixed* $\mathcal{Q}$ are

$$S(T \to 0, \mathcal{Q}) = N\Big[s_0 + \gamma T + \ldots\Big] + 2\ln(U/T)\ldots$$

This defines γ as the co-efficient of the linear-in-T specific heat (at fixed $\mathcal{Q}$)

D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S. Josephine Suh, arXiv:1711.08467

Fluctuations

An *exact* path integral over the effective action leads to the following physical consequences

- The *many*-body density of states, $D(E)$, is related to the grand potential, $\Omega(T)$ by

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

We obtain

$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} \, d\left(E - E_p\right)$$

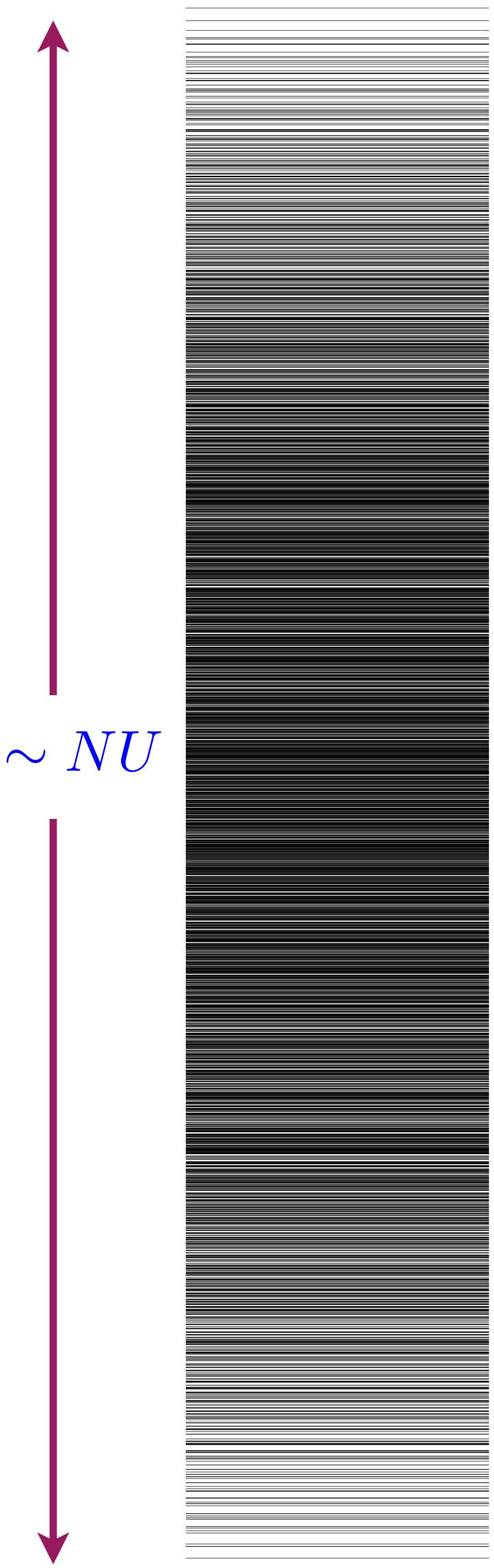
where $N\mathcal{Q} + p$ is the integer fermion number,

$$d(E) \sim \exp\left(N s_0\right) \sinh\left(\sqrt{2N\gamma E}\right) \quad , \quad E > 0 \, , \; e^{-cN} \ll \gamma E \ll N$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N.

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, arXiv:1611.04650; A.M. Garcia-Garcia and J.J.M. Verbaarschot, arXiv:1701.06593; D. Bagrets, A. Altland, and A. Kamenev, arXiv:1702.08902; D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467

The complex SYK model



There are 2^N many body levels with energy E. Shown are all values of E for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}\,.$$

Many-body level spacing $\sim 2^{-N} = e^{-N \ln 2}$

At $\mathcal{Q} = 1/2$,

$$s_0 = \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where G is Catalan's constant.

Non-quasiparticle excitations with spacing $\sim e^{-N s_0}$

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

Fluctuations

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D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467

Fluctuations

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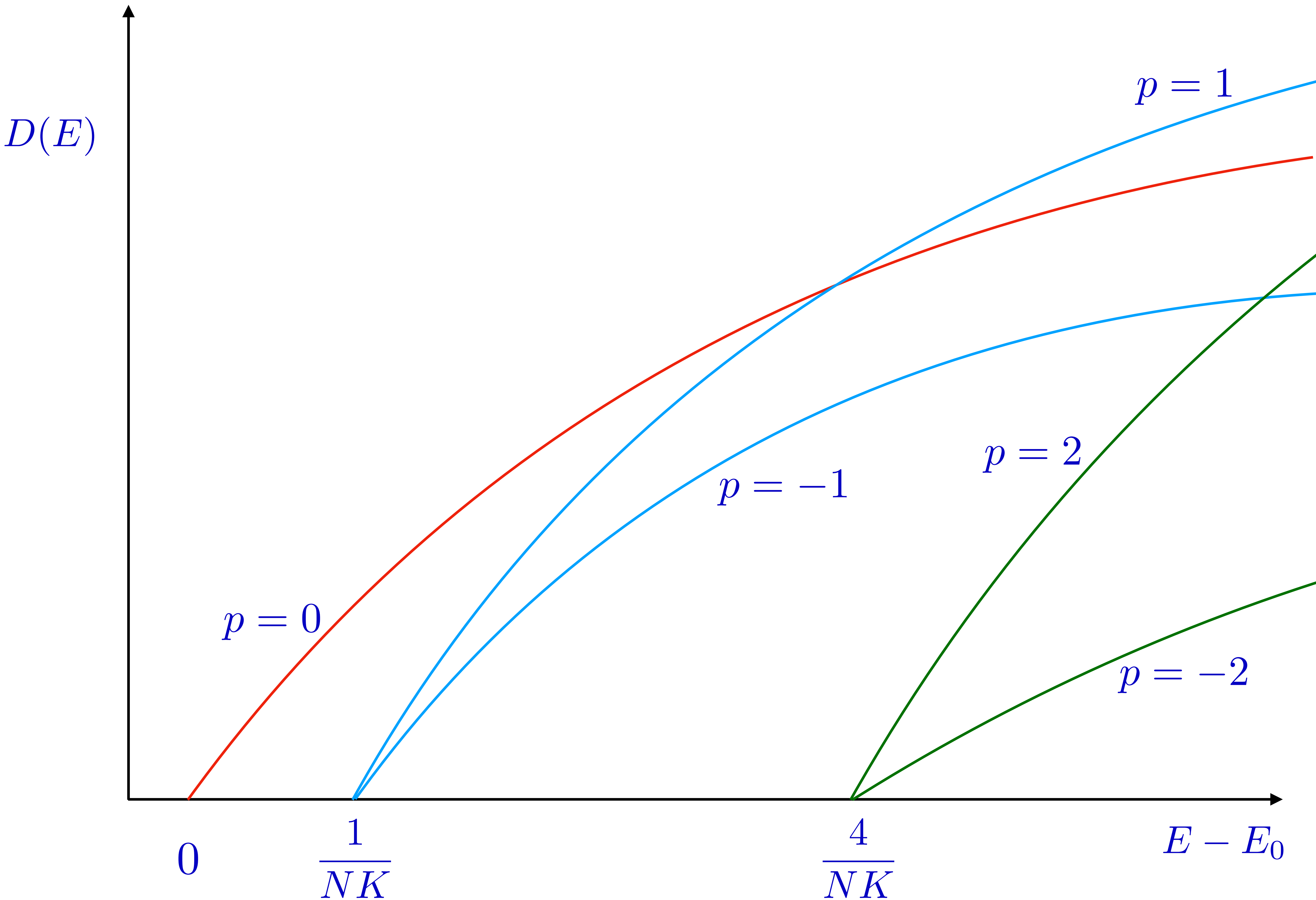
- At charge $N\mathcal{Q} + p$, the prefactor of the $\sinh(\sqrt{2N\gamma(E - E_p)})$ term is

$$\exp\left[N s_0(\mathcal{Q}) + 2\pi p \mathcal{E}\right] \approx \exp\left[N s_0(\mathcal{Q} + p/N)\right]$$

using

$$\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}$$



$D(E)$
$p = 1$
$p = 0$
$p = -1$
$p = 2$
$p = -2$
0
$\frac{1}{NK}$
$\frac{4}{NK}$
$E - E_0$

The Schwarzian theory and black holes

- Reparameterization invariance is a defining property of Einstein's theory of gravity

- In imaginary time, AdS_2 is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under SL(2,R)

$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \text{ with } ad - bc = 1.$$

A. Kitaev, 2015
J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857

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Semiclassical fluctuations about the saddle-point of Einstein-Maxwell theory of a charged black holes in $d \geq 2$ spatial dimensions lead to the same Schwarzian+phase theory of fluctuations.

P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal, arXiv:1802.09547
U. Moitra, S. P. Trivedi, and V. Vishal, arXiv:1808.08239
P. Chaturvedi, Yingfei Gu, Wei Song, Boyang Yu, arXiv:1808.08062
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

The Schwarzian theory and black holes

- The Einstein-Maxwell theory leads to the following parameters for the Schwarzian+phase theory

$$K = \left.\frac{d\mathcal{Q}}{d\mu}\right|_{T=0} = \frac{2(d-1)L^2 s_d R_h^{d-3}\left[d(d+1)R_h^2 + (d-1)^2 L^2\right]}{(d+1)g_F^2 \kappa^2}$$

$$S(T \to 0, \mathcal{Q}) = s_0 + \gamma T + \ldots$$

$$\gamma = \frac{4\pi^2 d s_d L^2 R_h^{d+1}}{\kappa^2 (d(d+1)R_h^2 + (d-1)^2 L^2)} \, .$$

Quantum matter without quasiparticles

- Planckian dynamics is realized in the 'solvable' SYK models
- Black holes thermalize in a time $\sim \hbar/(k_B T_H)$, where T_H is the Hawking temperature.
- A Schwarzian theory of a time reparameterization mode, with SL(2,R) symmetry, describes the quantum dynamics of
  - the SYK models
  - black holes with near-extremal AdS_2 horizons