

Solvable models of quantum matter without quasiparticles

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Quasiparticles are ubiquitous:

- Fermi liquid theory of metals, insulators, semiconductors
- Theory of superconductivity (pairing of quasiparticles)
- Theory of disordered metals and insulators (diffusion and localization of quasiparticles)
- Theory of metals in one dimension (collective modes as quasiparticles)
- Theory of the fractional quantum Hall effect (quasiparticles which are 'fractions' of an electron)

What are quasiparticles ?

- **Quasiparticles are additive excitations:**

The low-lying excitations of the many-body system can be identified as a set $\{n_\alpha\}$ of quasiparticles with energy ε_α

$$E = \sum_{\alpha} n_{\alpha} \varepsilon_{\alpha} + \sum_{\alpha, \beta} F_{\alpha\beta} n_{\alpha} n_{\beta} + \dots$$

In a lattice system of N sites, this parameterizes the energy of $\sim e^{\alpha N}$ states in terms of poly(N) numbers.

What are quasiparticles ?

- Quasiparticles eventually collide with each other. Such collisions eventually leads to thermal equilibration in a chaotic quantum state, but the equilibration takes a long time. In a Fermi liquid, this time diverges as

$$\tau_{\text{eq}} \sim \frac{\hbar E_F}{(k_B T)^2} \quad , \quad \text{as } T \rightarrow 0,$$

where E_F is the Fermi energy.

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where E_F is the Fermi energy.

- This time is much longer than the ‘Planckian time’ $\hbar/(k_B T)$, which we will find in systems without quasiparticle excitations.

$$\tau_{\text{eq}} \gg \frac{\hbar}{k_B T} \quad , \quad \text{as } T \rightarrow 0.$$

1. Random matrix quasiparticle model

$q=2$, complex SYK

2. Matter without quasiparticles

$q=4$, complex SYK

3. The Schwarzian theory

4. Connections to black holes

with AdS_2 horizons

5. Connections to strange metals

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$q=2$, complex SYK

2. Matter without quasiparticles

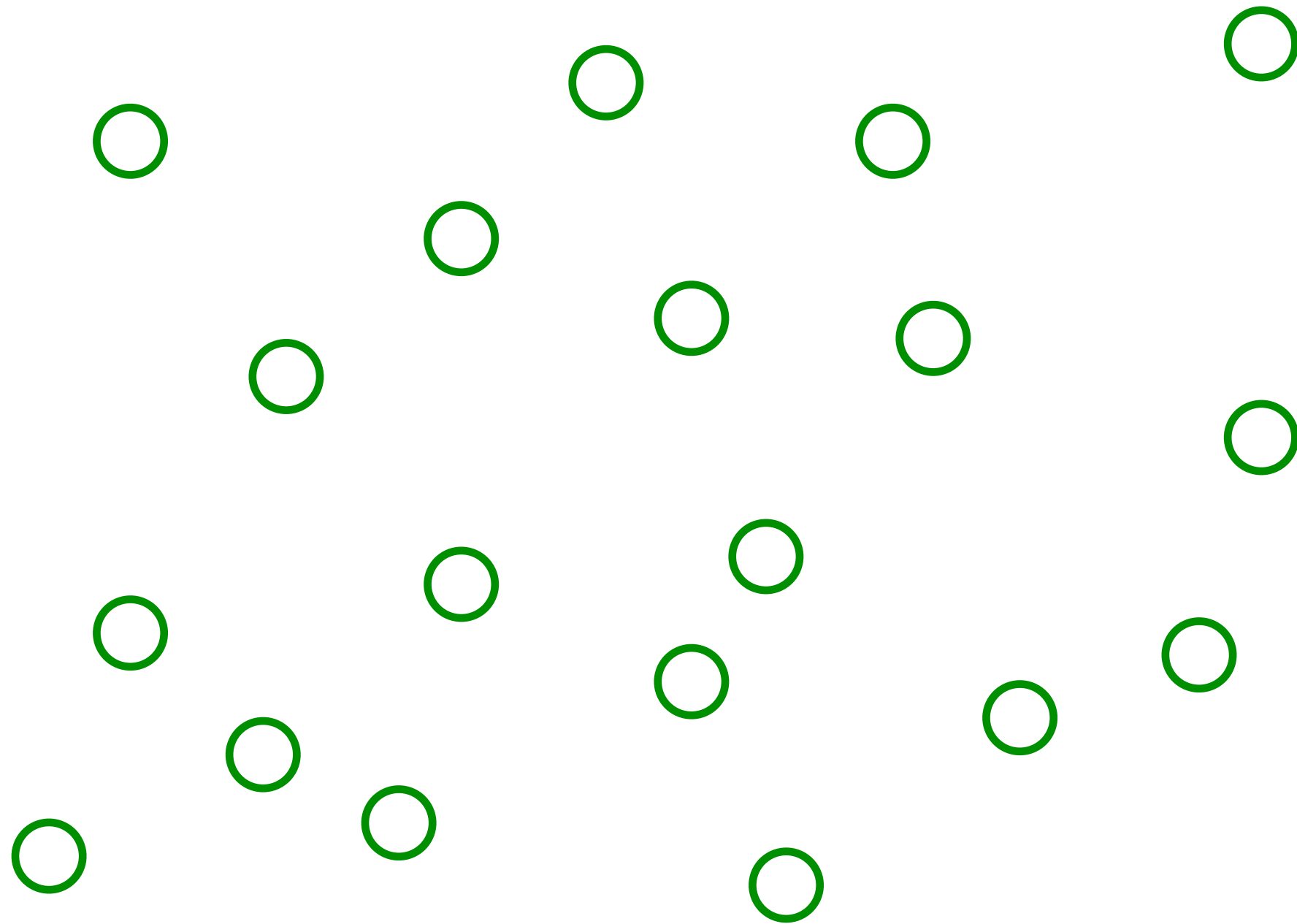
$q=4$, complex SYK

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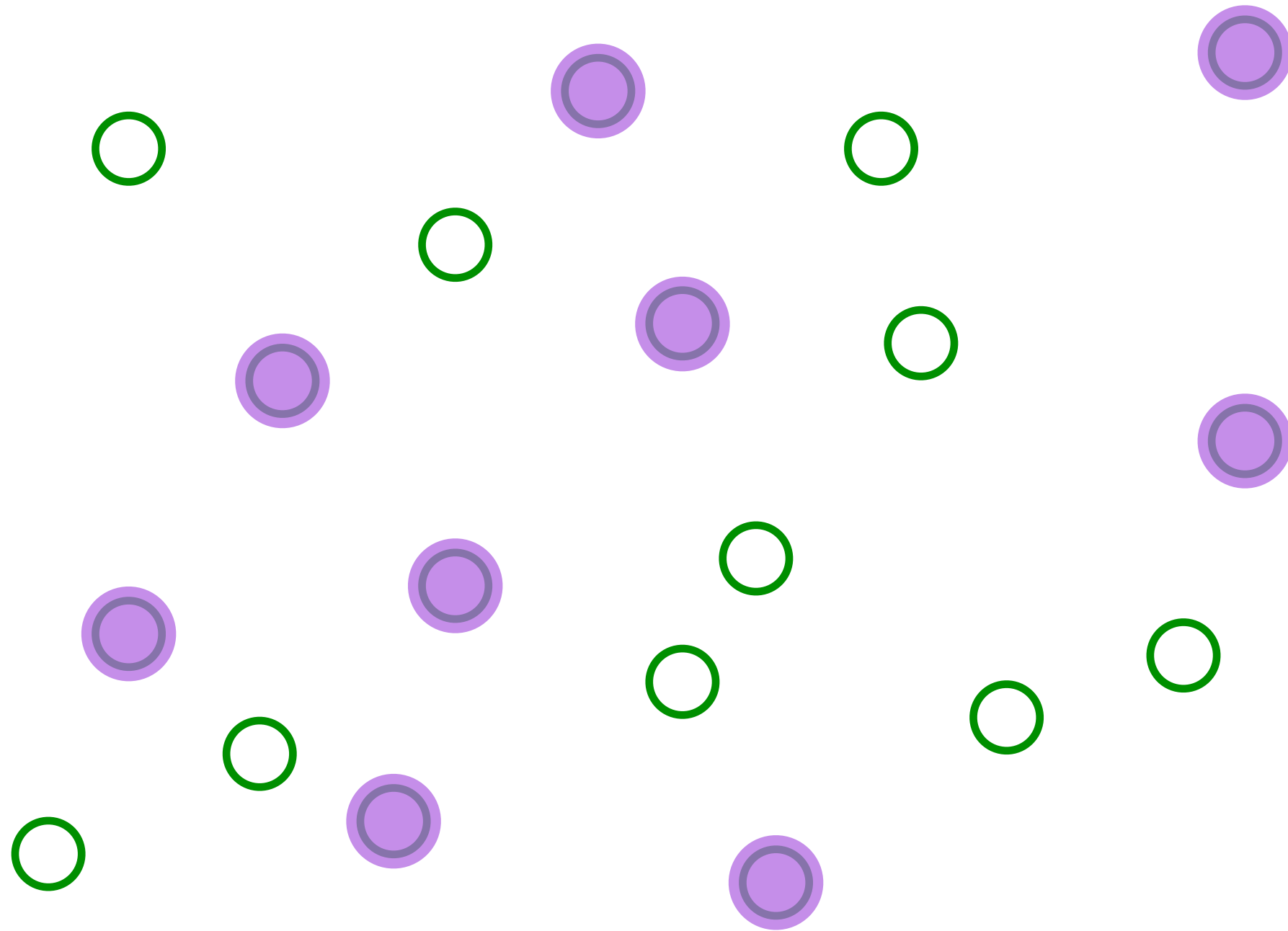
5. Connections to strange metals

A simple model of a metal with quasiparticles



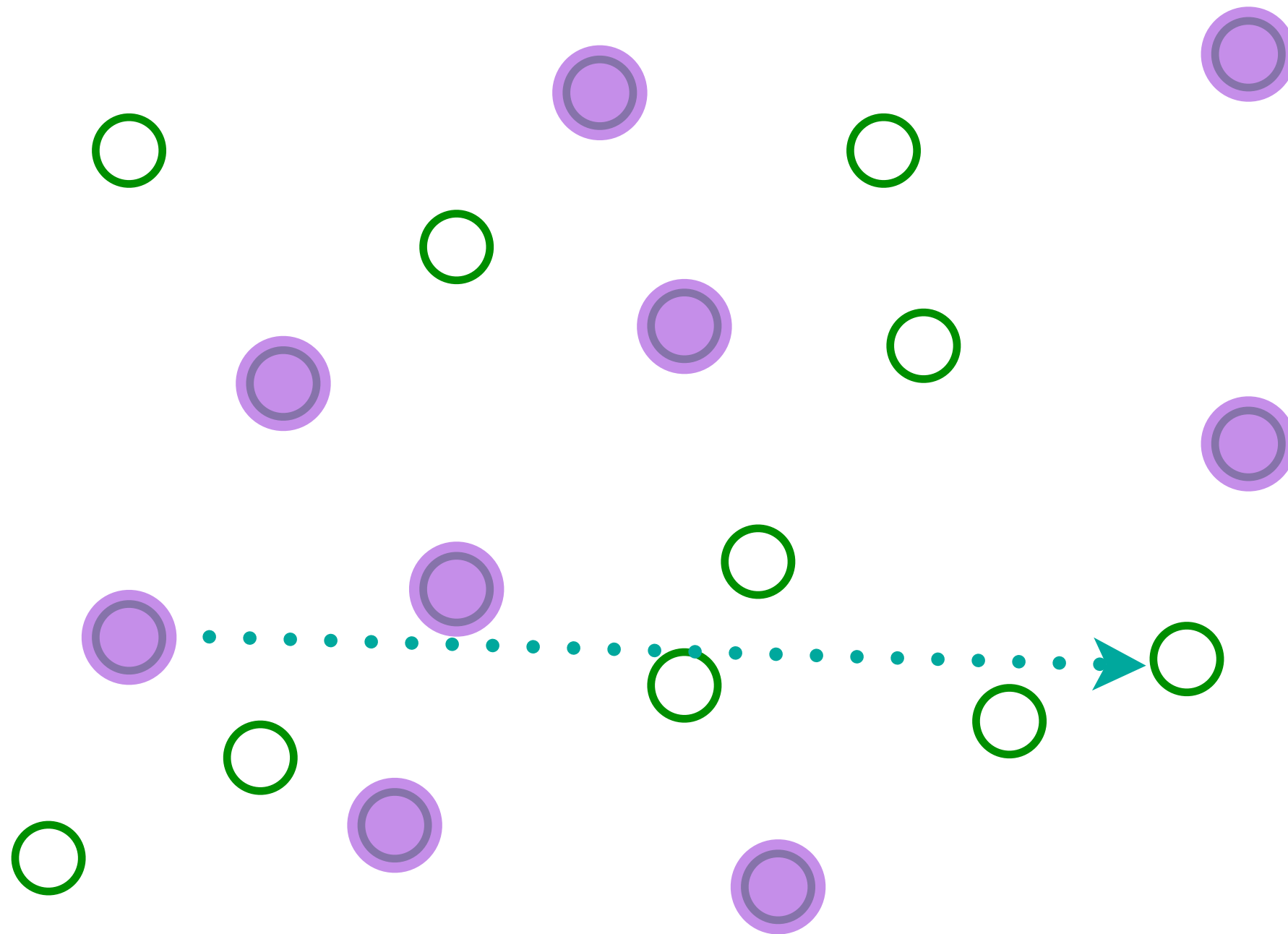
Pick a set of random positions

A simple model of a metal with quasiparticles



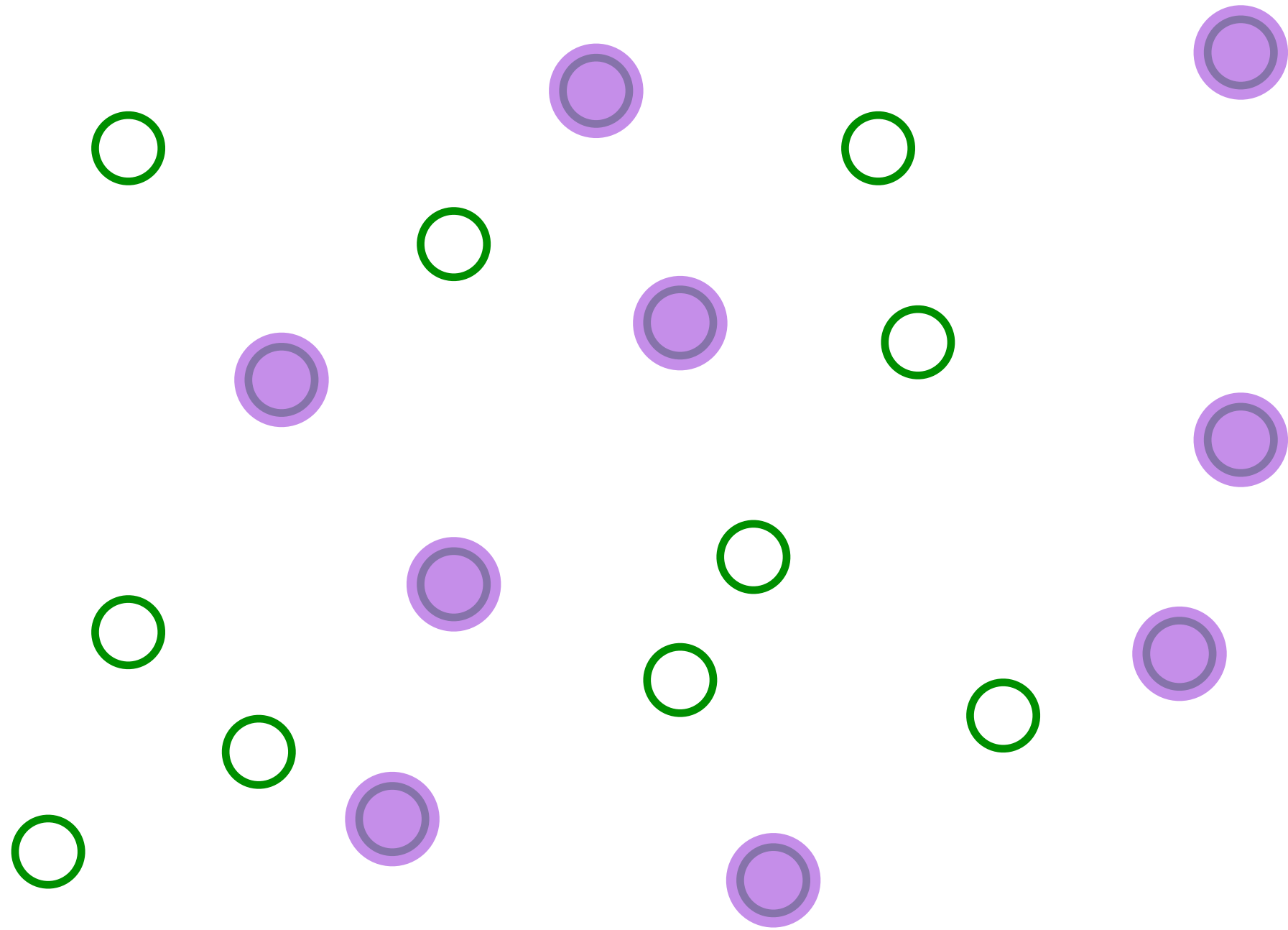
Place electrons randomly on some sites

A simple model of a metal with quasiparticles



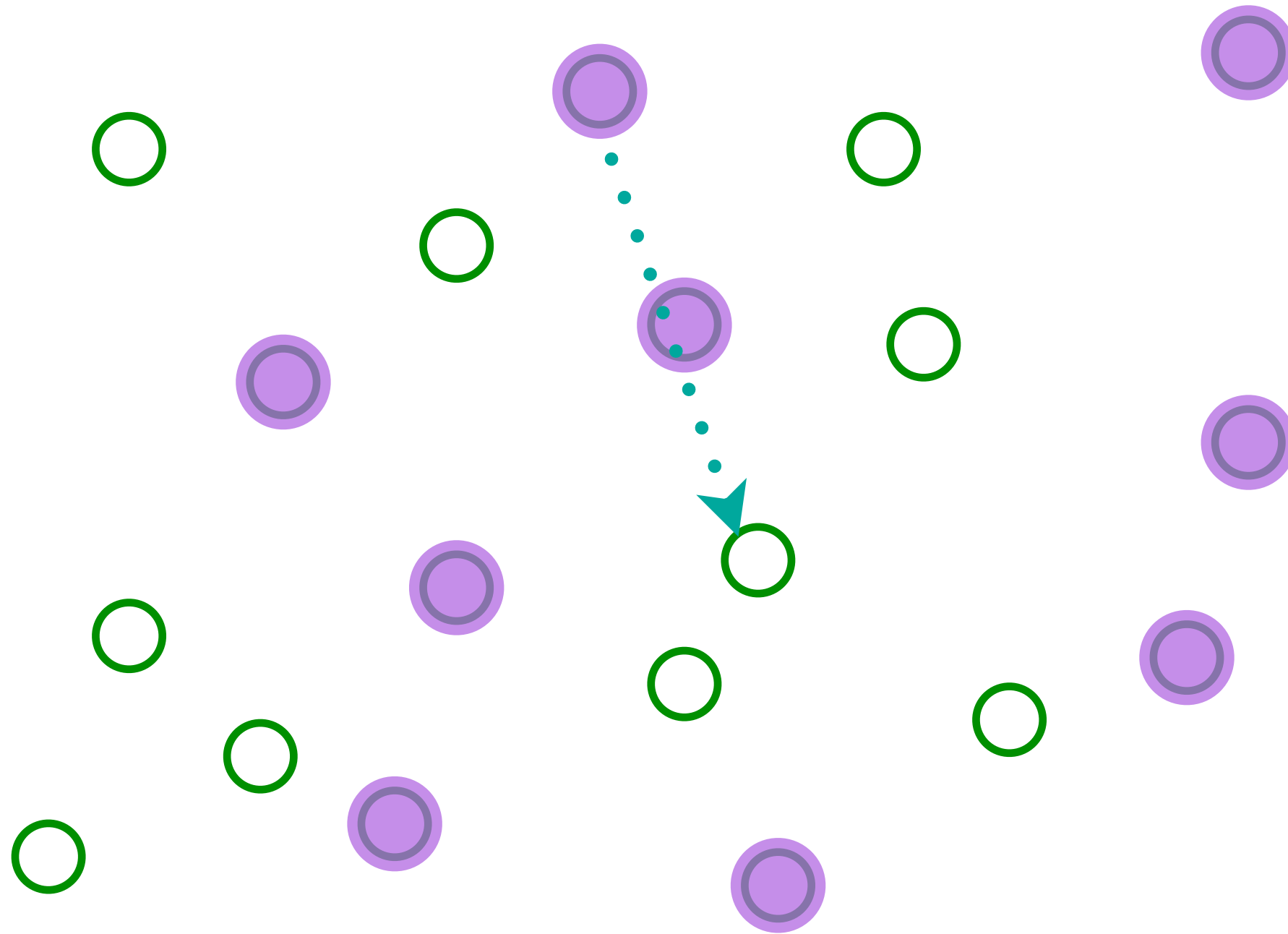
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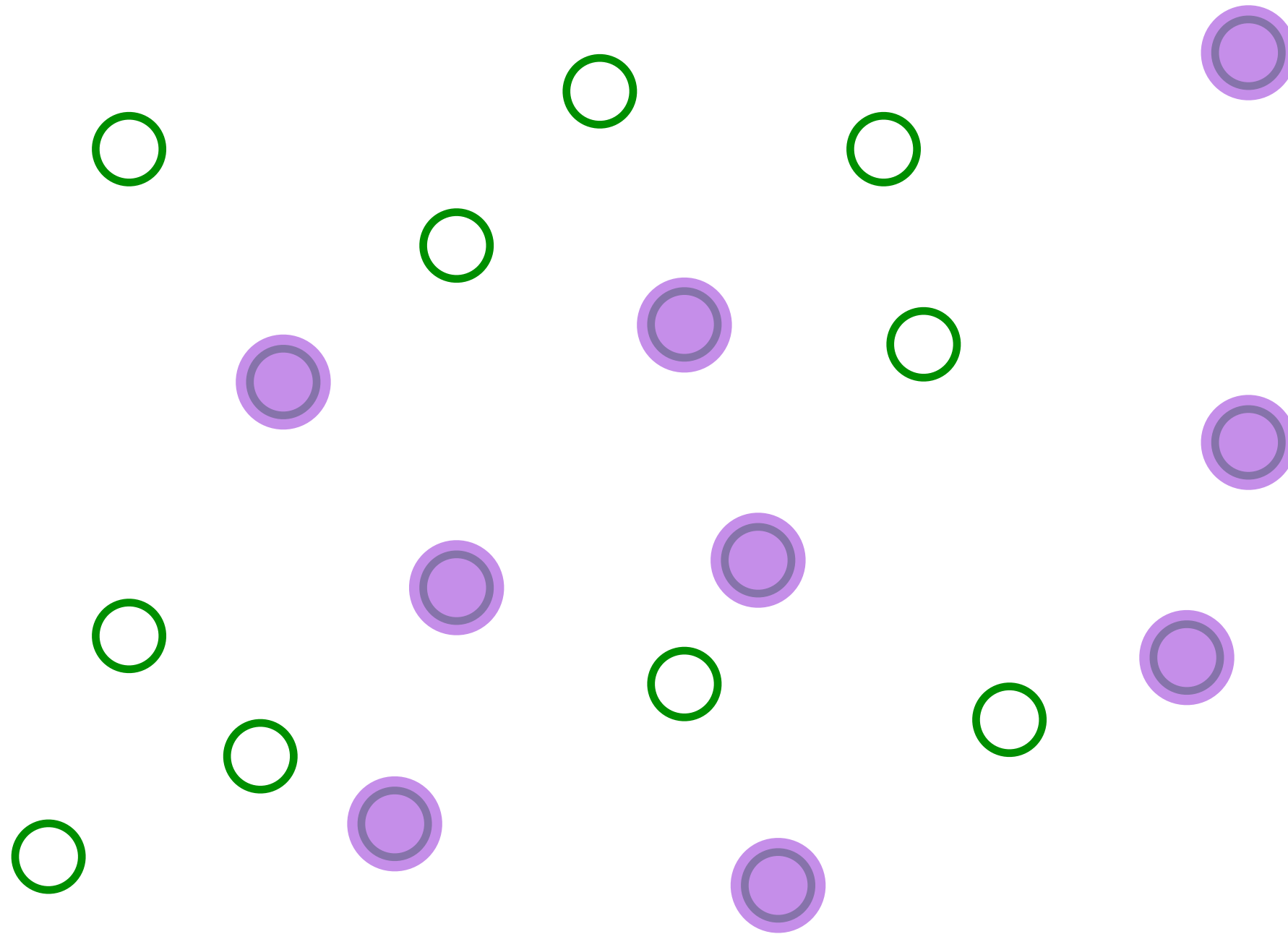
Electrons move one-by-one randomly

A simple model of a metal with quasiparticles



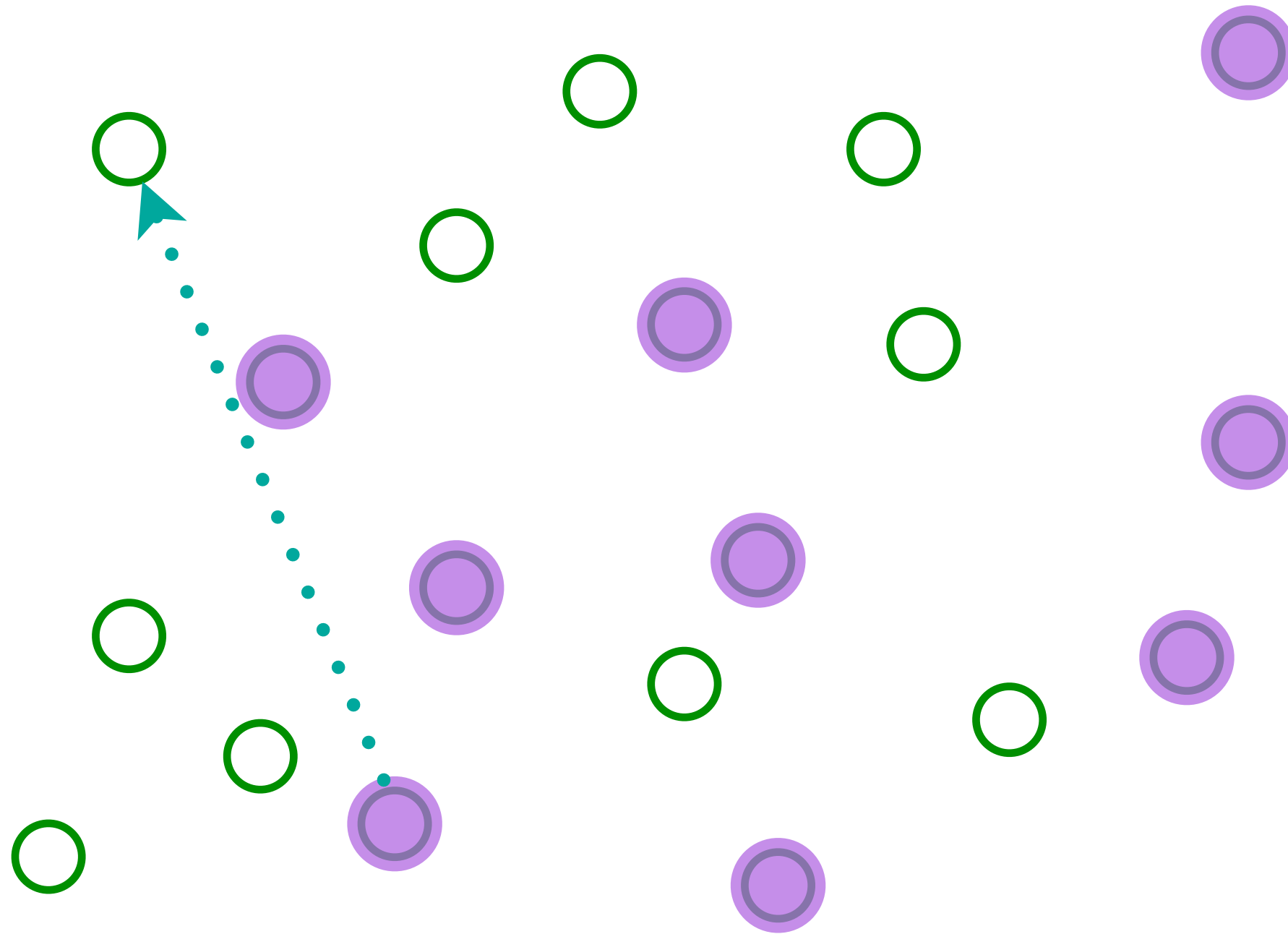
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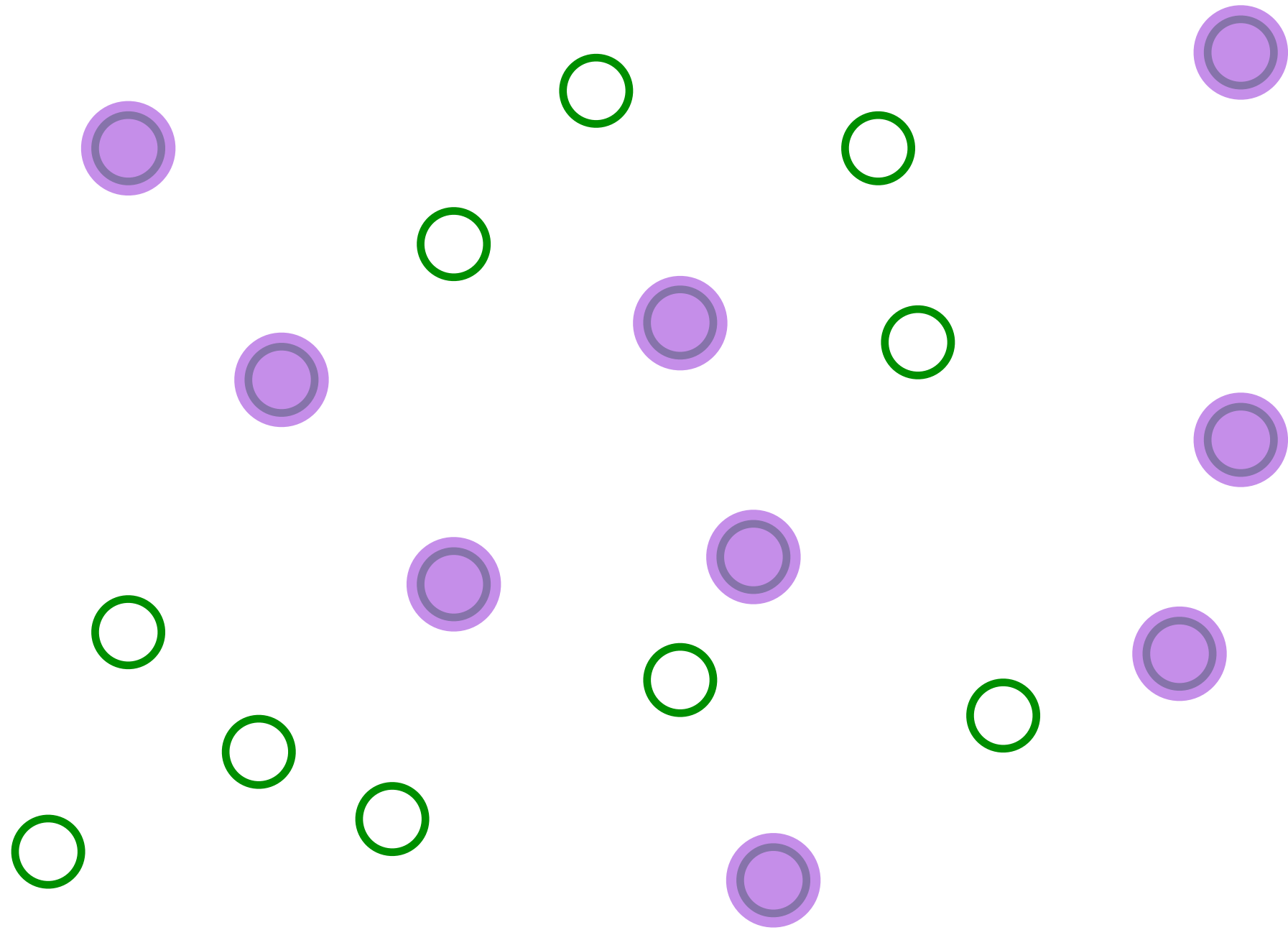
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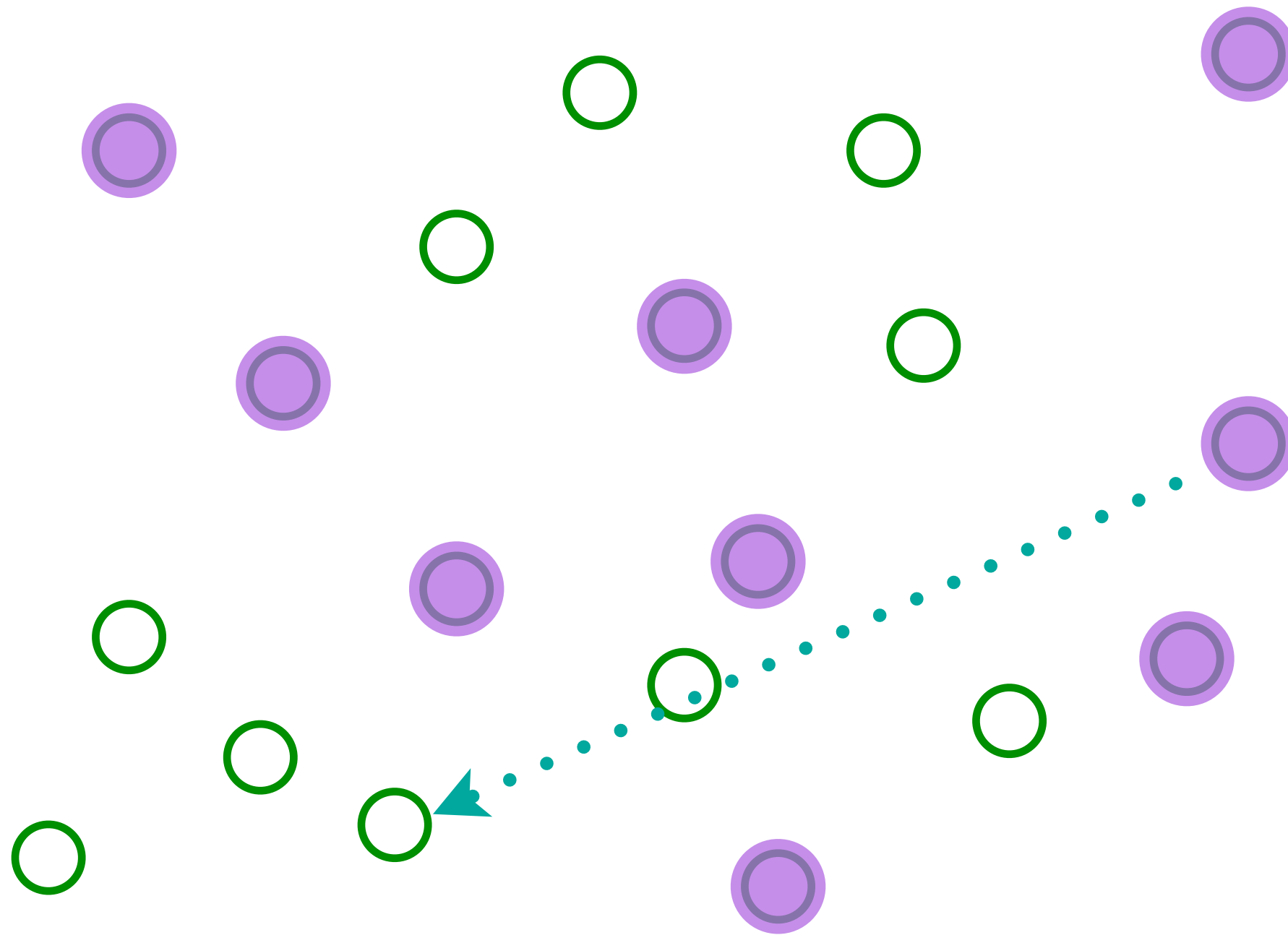
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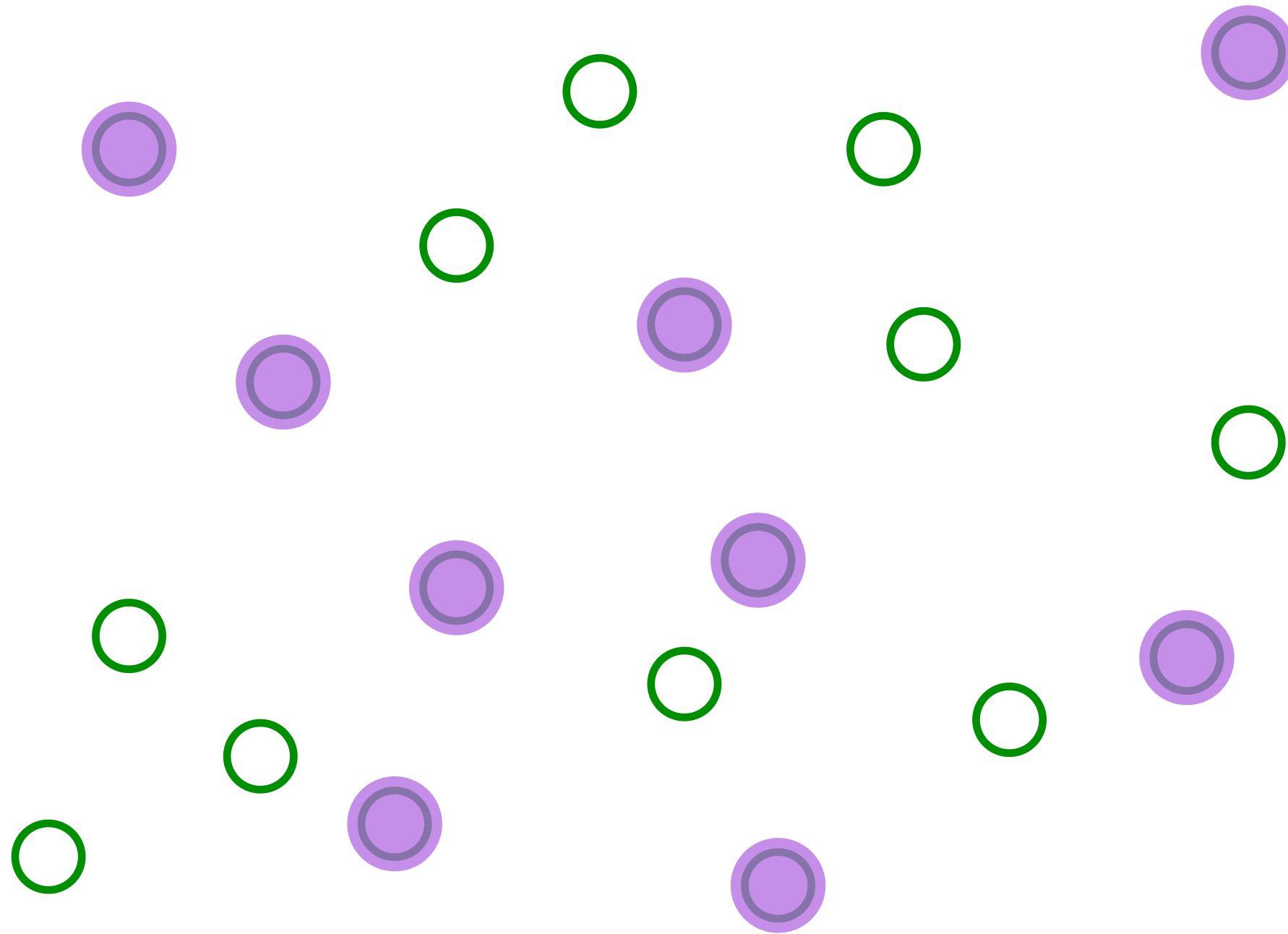
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A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^N t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = Q$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

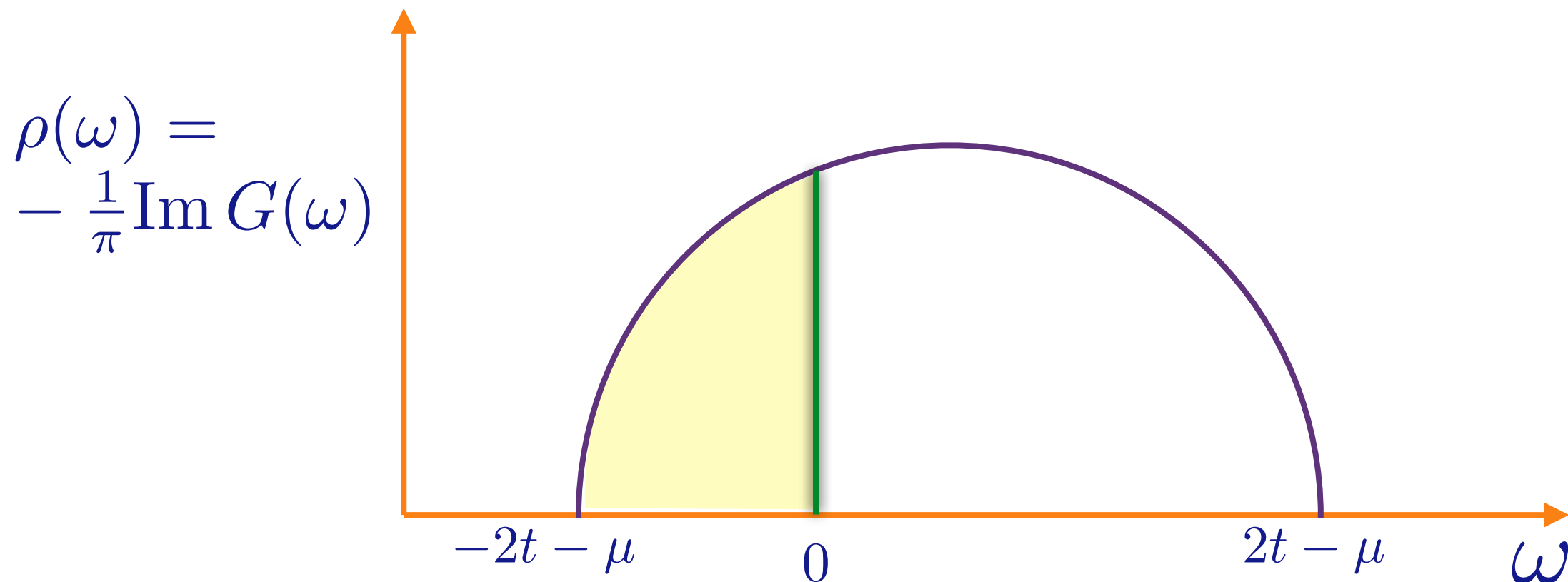
**Fermions occupying the eigenstates of a
 $N \times N$ random matrix**

A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$
$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$
$$G(\tau = 0^-) = Q.$$

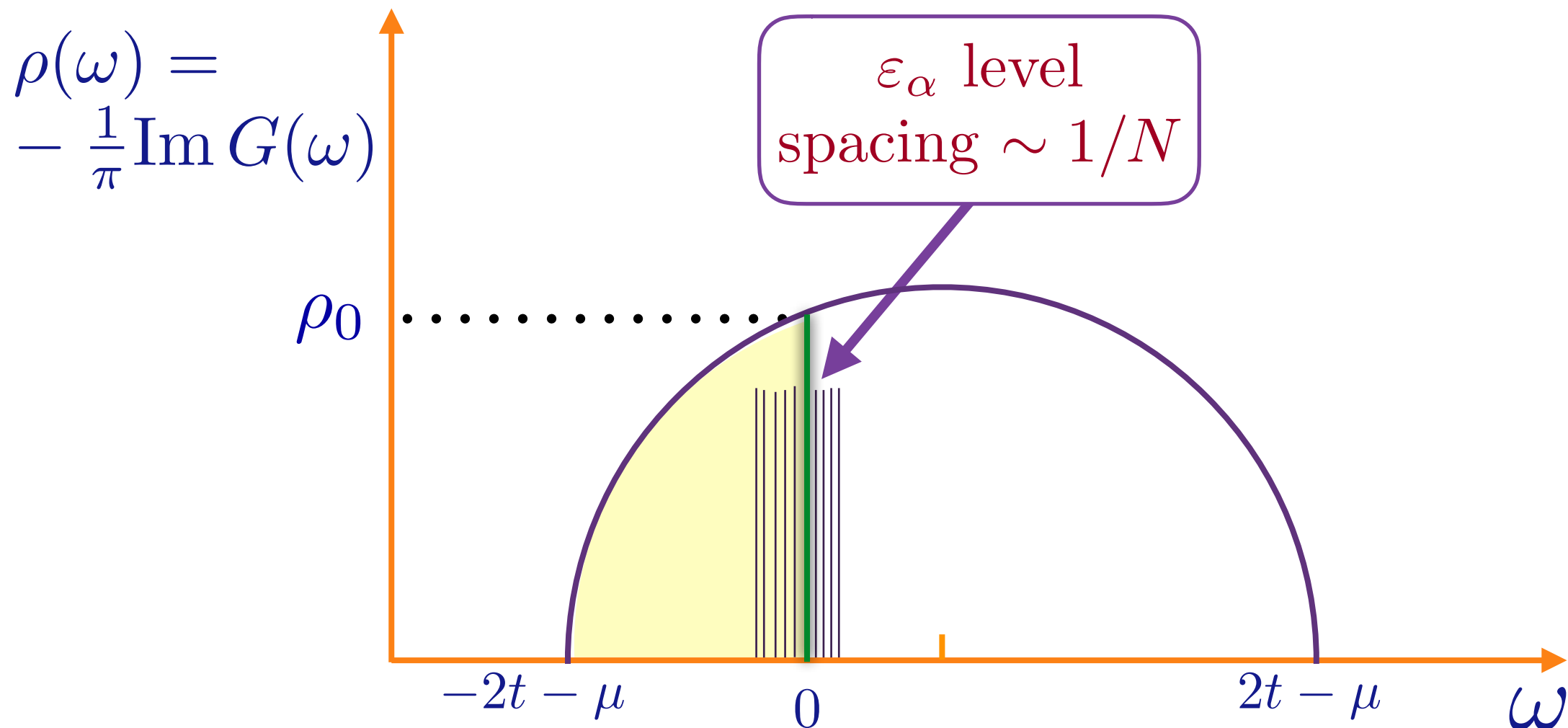
$G(\omega)$ can be determined by solving a quadratic equation.



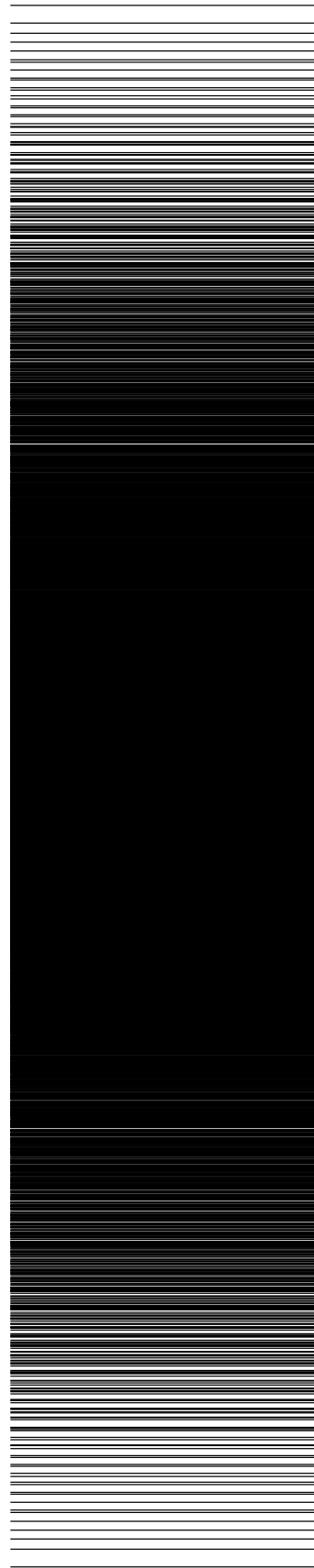
A simple model of a metal with quasiparticles

Let ε_α be the eigenvalues of the matrix t_{ij}/\sqrt{N} . The fermions will occupy the lowest NQ eigenvalues, upto the Fermi energy E_F . The single-particle density of states is

$$\rho(\omega) = (1/N) \sum_\alpha \delta(\omega - \varepsilon_\alpha), \text{ and } \rho_0 \equiv \rho(\omega = 0).$$



A simple model of a metal with quasiparticles



Many-body
level spacing
 $\sim 2^{-N}$

Quasiparticle
excitations with
spacing $\sim 1/N$

There are 2^N many
body levels with energy

$$E = \sum_{\alpha=1}^N n_{\alpha} \varepsilon_{\alpha},$$

where $n_{\alpha} = 0, 1$. Shown
are all values of E for a
single cluster of size
 $N = 12$. The ε_{α} have a
level spacing $\sim 1/N$.

A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N \left(-\frac{\pi^2}{6} \rho_0 T^2 + \mathcal{O}(T^4) \right) + \dots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.

We can also define the *many* body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to be performed with care (it does not commute with the $1/N$ expansion), and we obtain

$$D(E) \sim \exp \left(\pi \sqrt{\frac{2N\rho_0(E - E_0)}{3}} \right), \quad E > E_0, \quad \frac{1}{N} \ll \rho_0(E - E_0) \ll N$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

A simple model of a metal with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^N t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^N U_{ij;kl} c_i^\dagger c_j^\dagger c_k c_\ell$$

$U_{ij;kl}$ are independent random variables with $\overline{U_{ij;kl}} = 0$ and $|\overline{U_{ij;kl}}|^2 = U^2$. We compute the lifetime of a quasiparticle, τ_α , in an exact eigenstate $\psi_\alpha(i)$ of the free particle Hamiltonian with energy ε_α . By Fermi's Golden rule, for ε_α at the Fermi energy

$$\begin{aligned} \frac{1}{\tau_\alpha} &= \pi U^2 \rho_0^2 \int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\delta f(\varepsilon_\beta)(1 - f(\varepsilon_\gamma))(1 - f(\varepsilon_\delta)) \delta(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_\gamma - \varepsilon_\delta) \\ &= \frac{\pi^3 U^2 \rho_0^2}{4} T^2 \end{aligned}$$

where ρ_0 is the density of states at the Fermi energy, and $f(\varepsilon) = 1/(e^{\varepsilon/T} + 1)$ is the Fermi function.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as $\sim T^{-2}$ at the Fermi level.

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$q=2$, complex SYK

2. Matter without quasiparticles

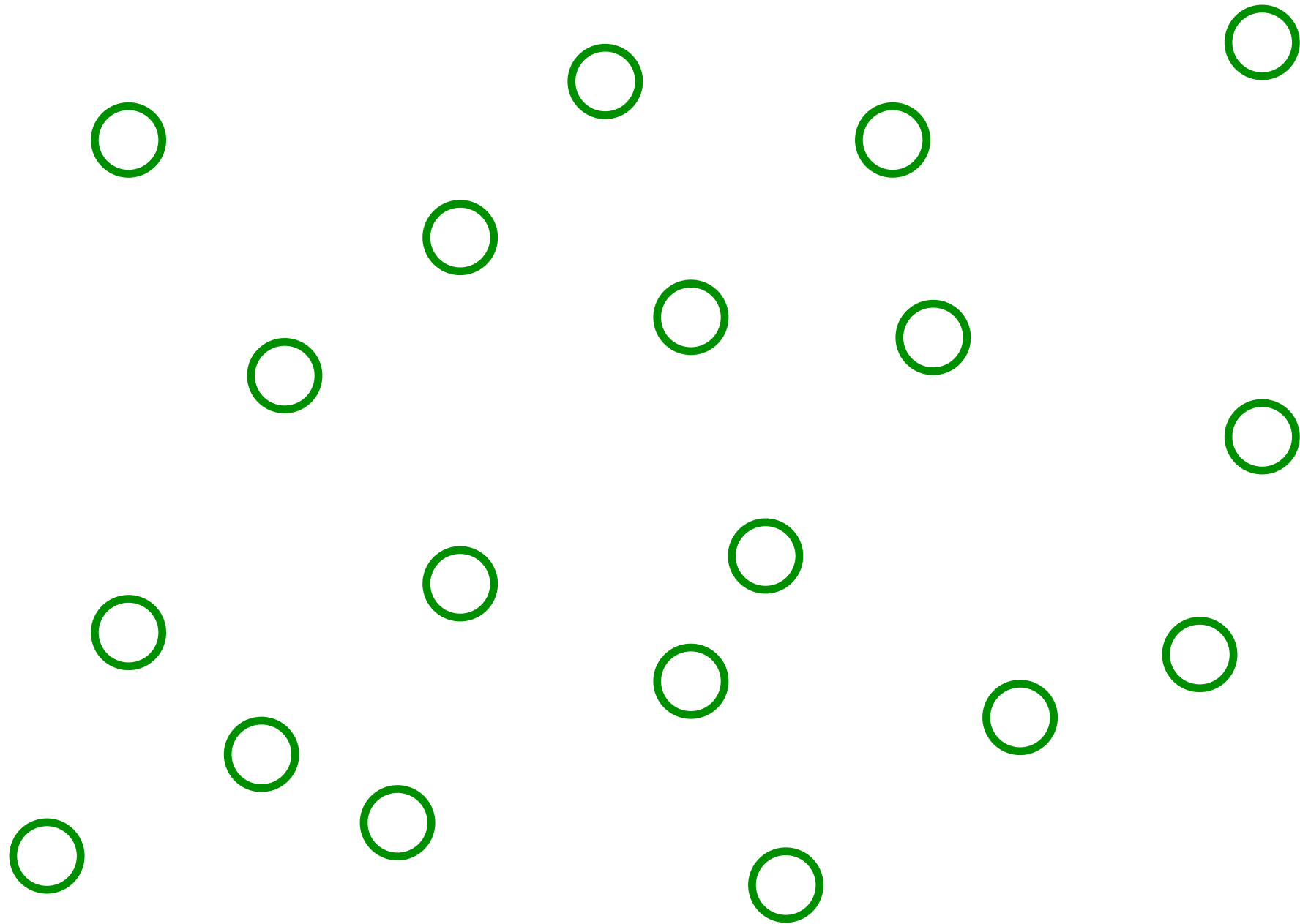
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3. The Schwarzian theory

4. Connections to black holes
with AdS_2 horizons

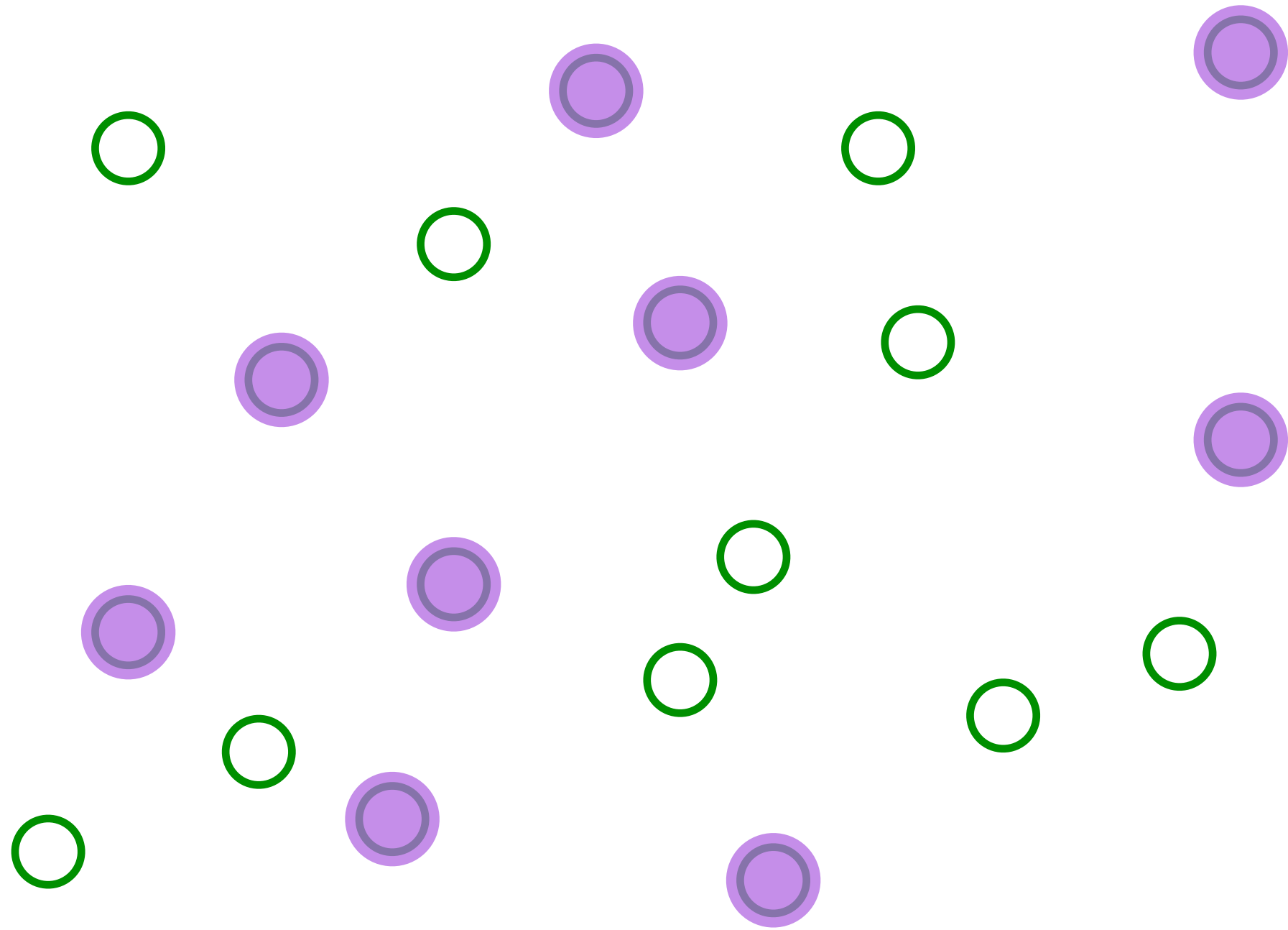
5. Connections to strange metals

The Sachdev-Ye-Kitaev (SYK) model



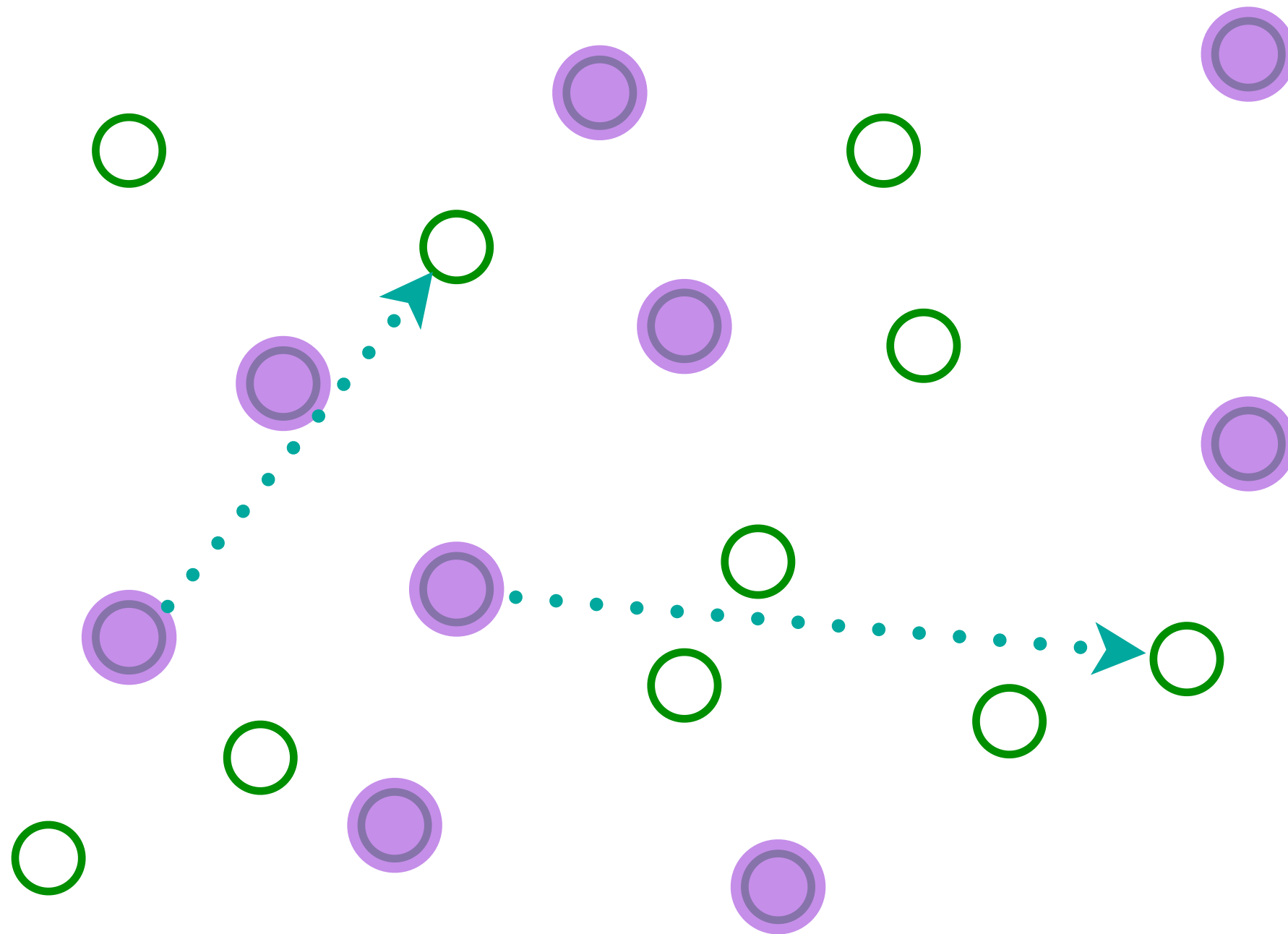
Pick a set of random positions

The SYK model



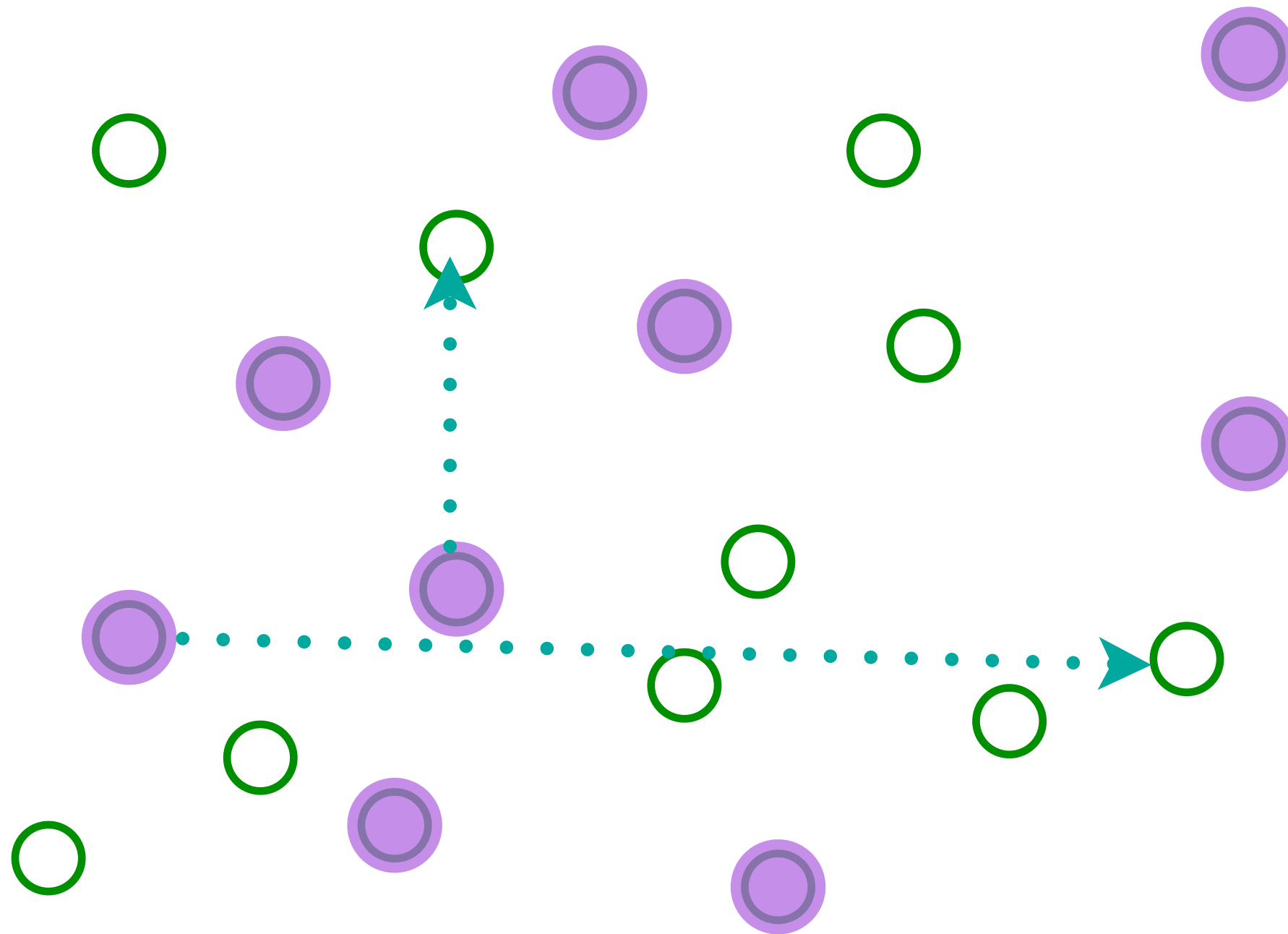
Place electrons randomly on some sites

The SYK model



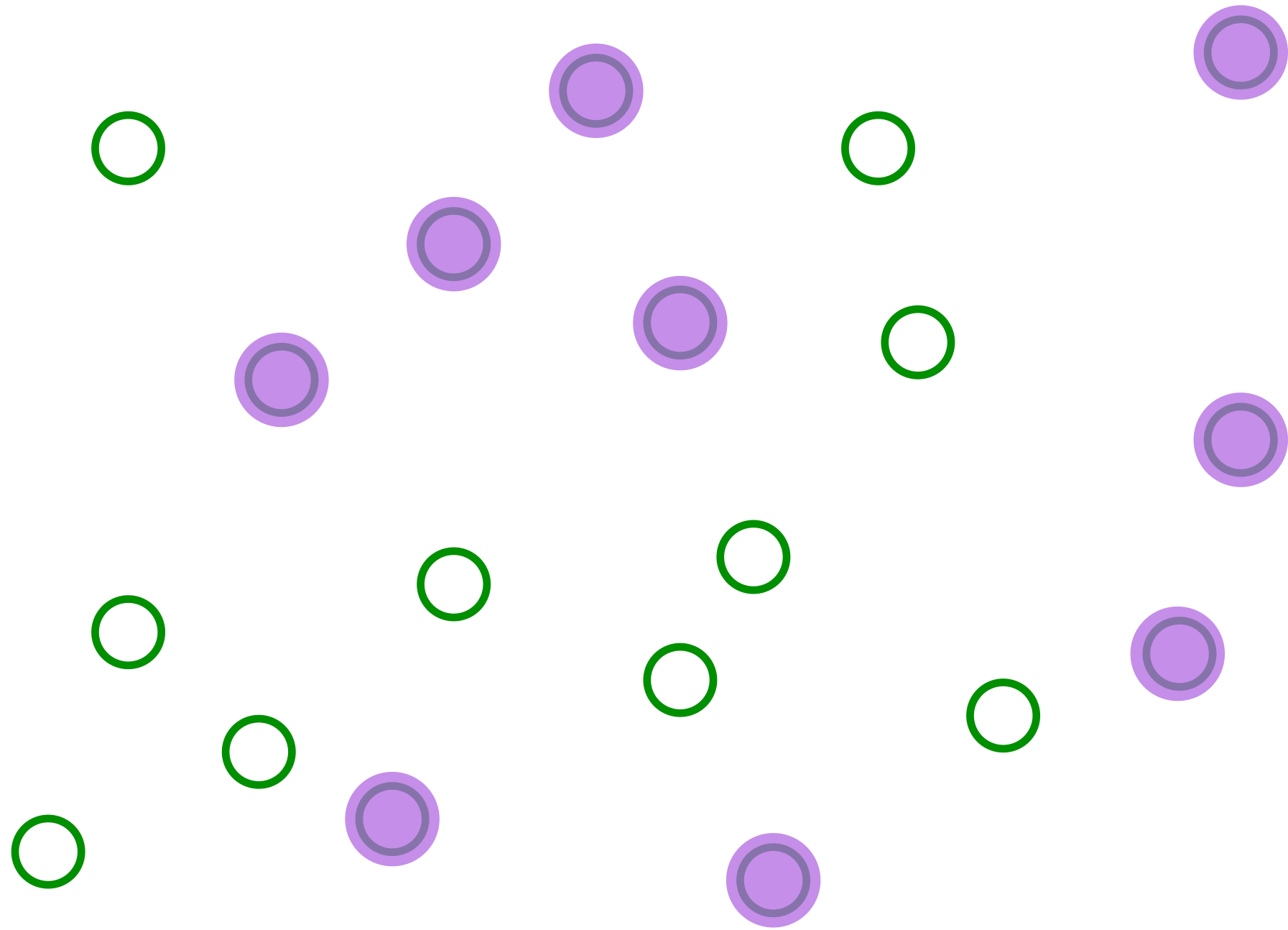
Place electrons randomly on some sites

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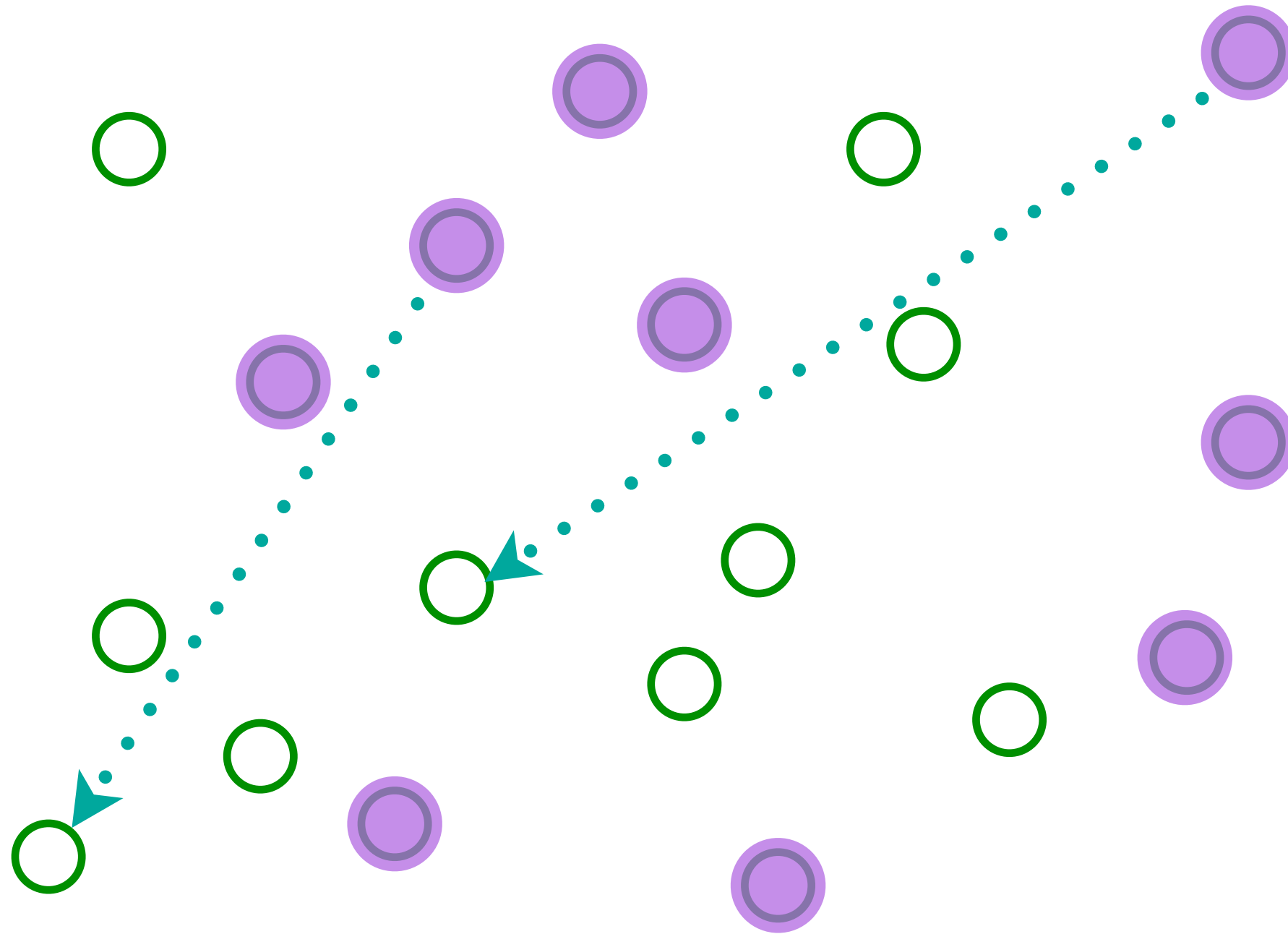
Place electrons randomly on some sites

The SYK model



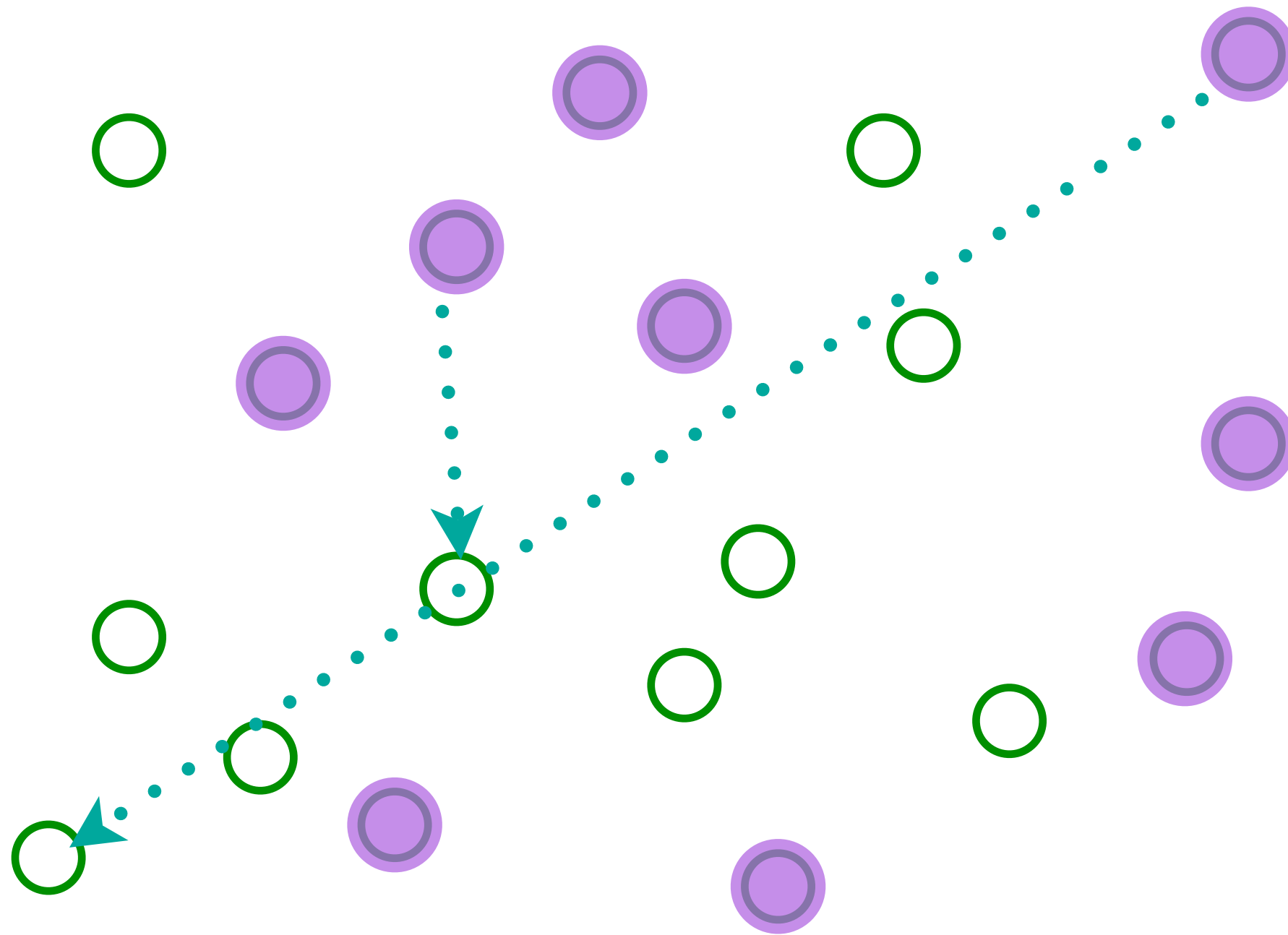
Entangle electrons pairwise randomly

The SYK model



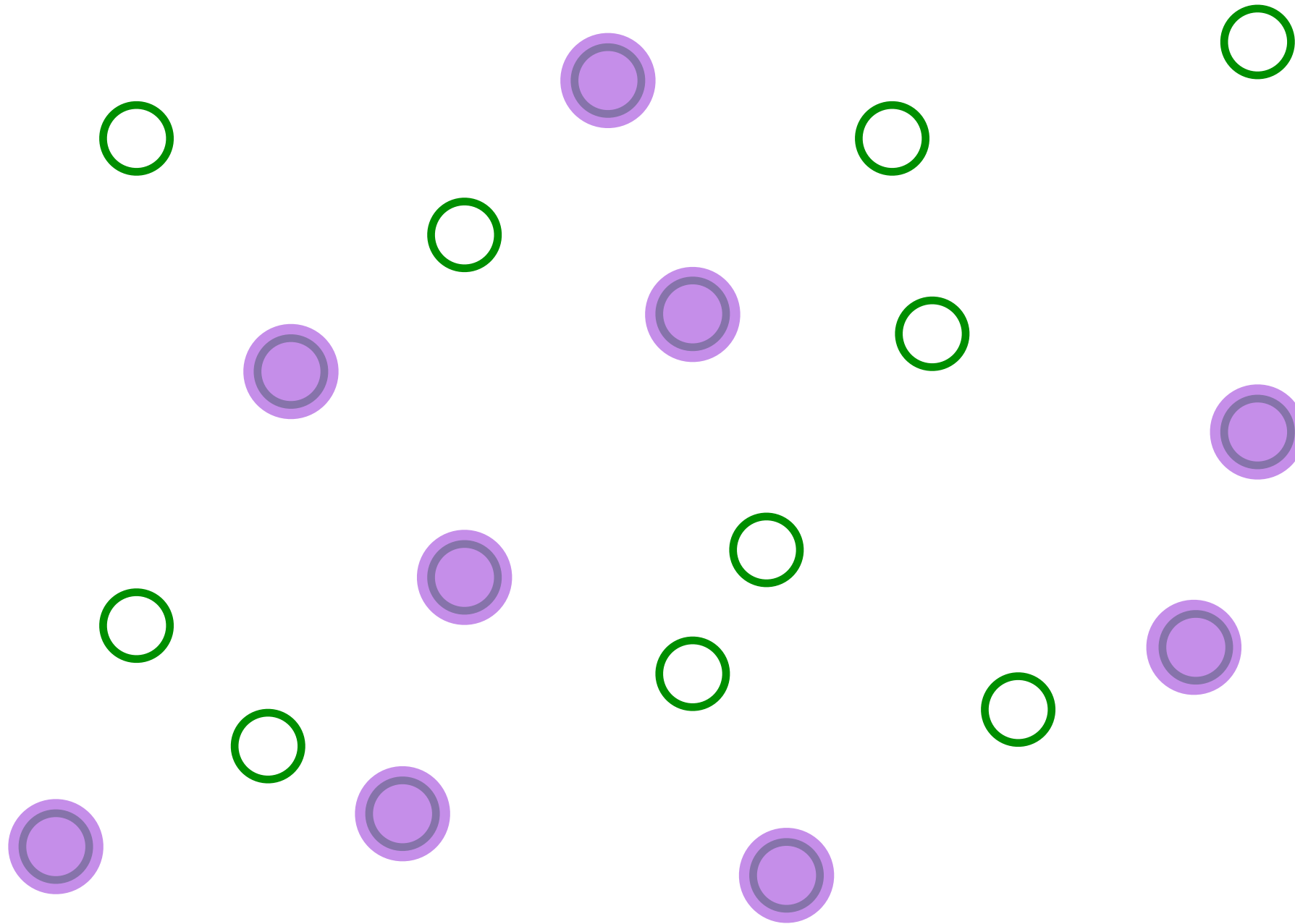
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The SYK model



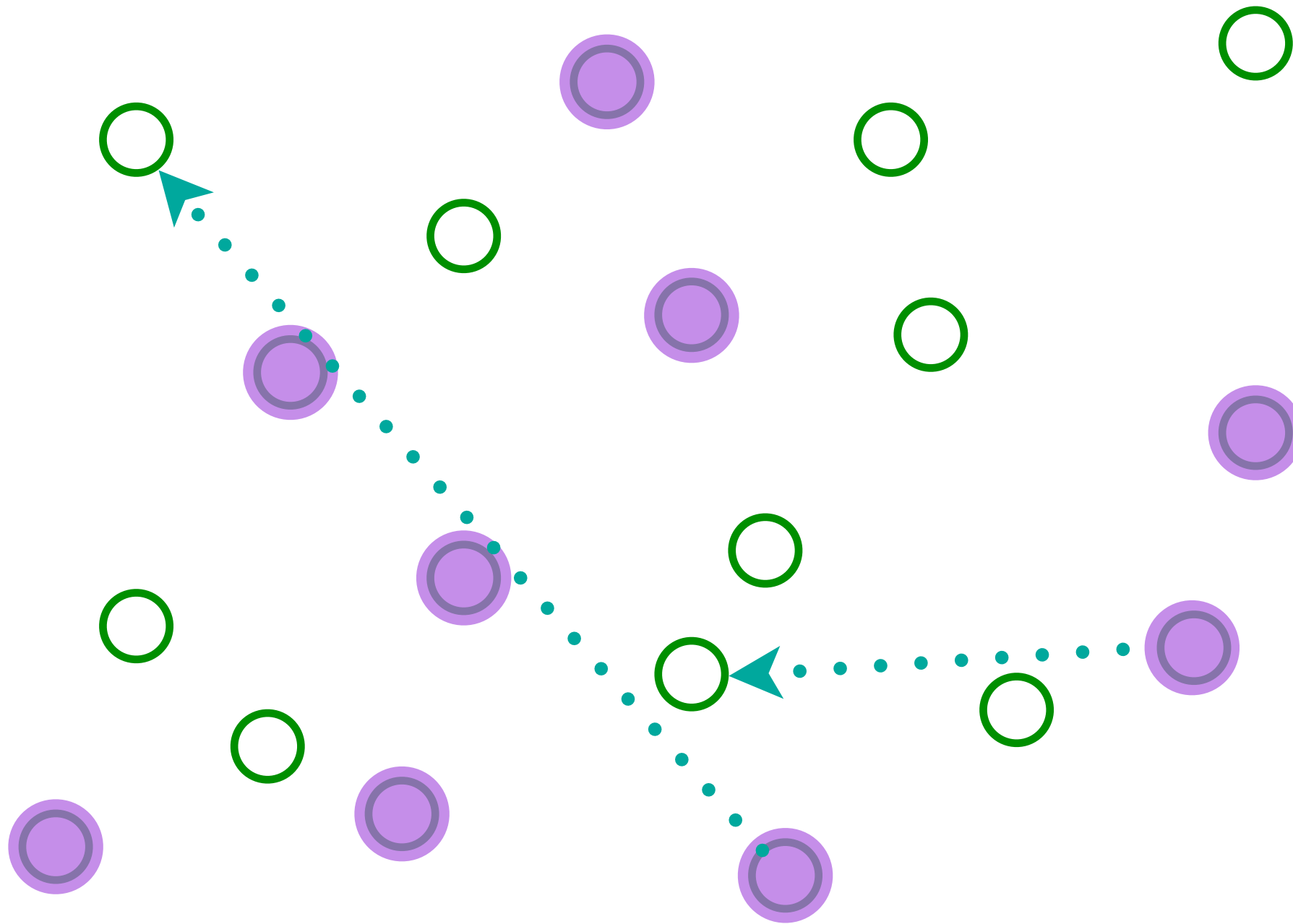
Entangle electrons pairwise randomly

The SYK model



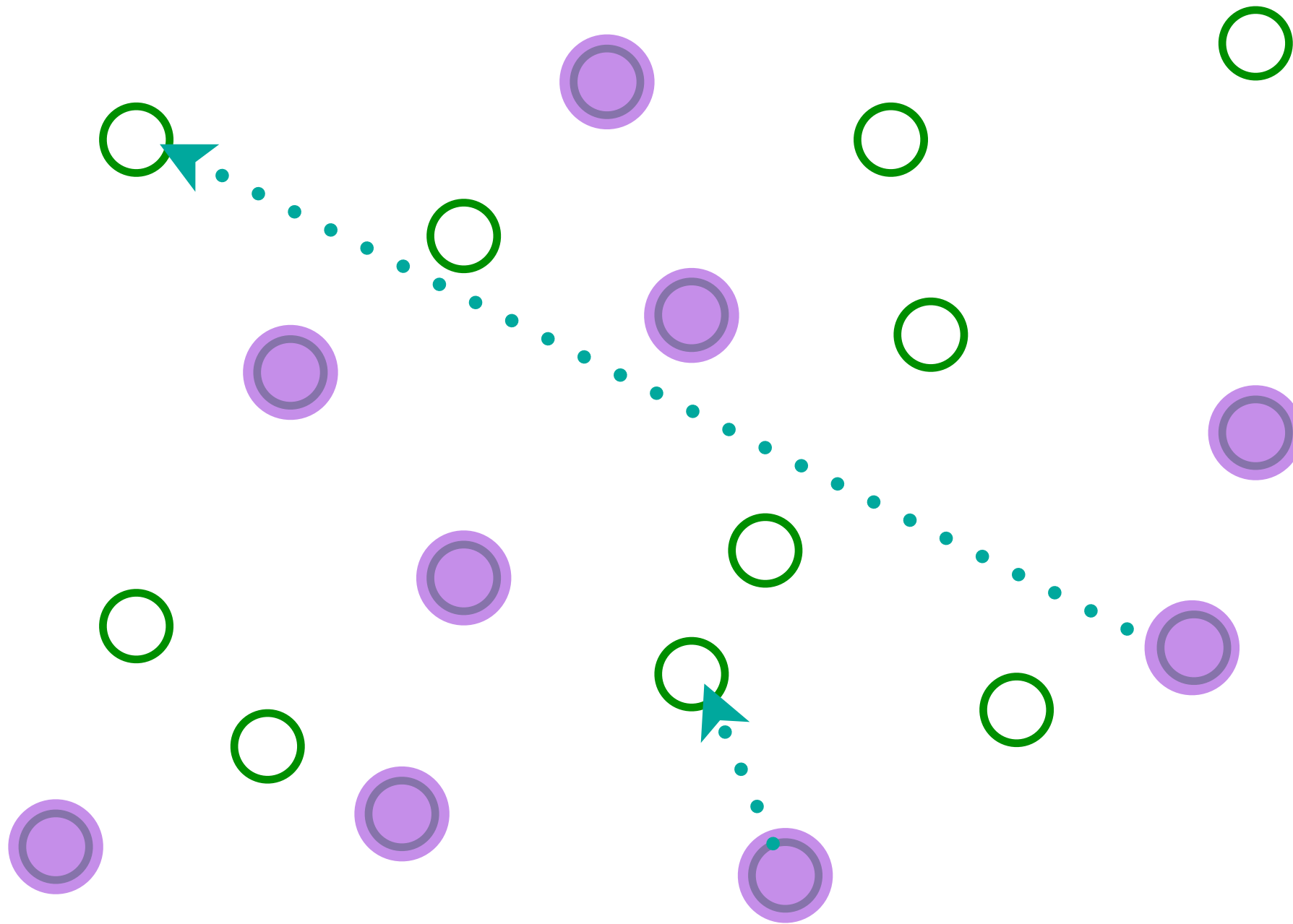
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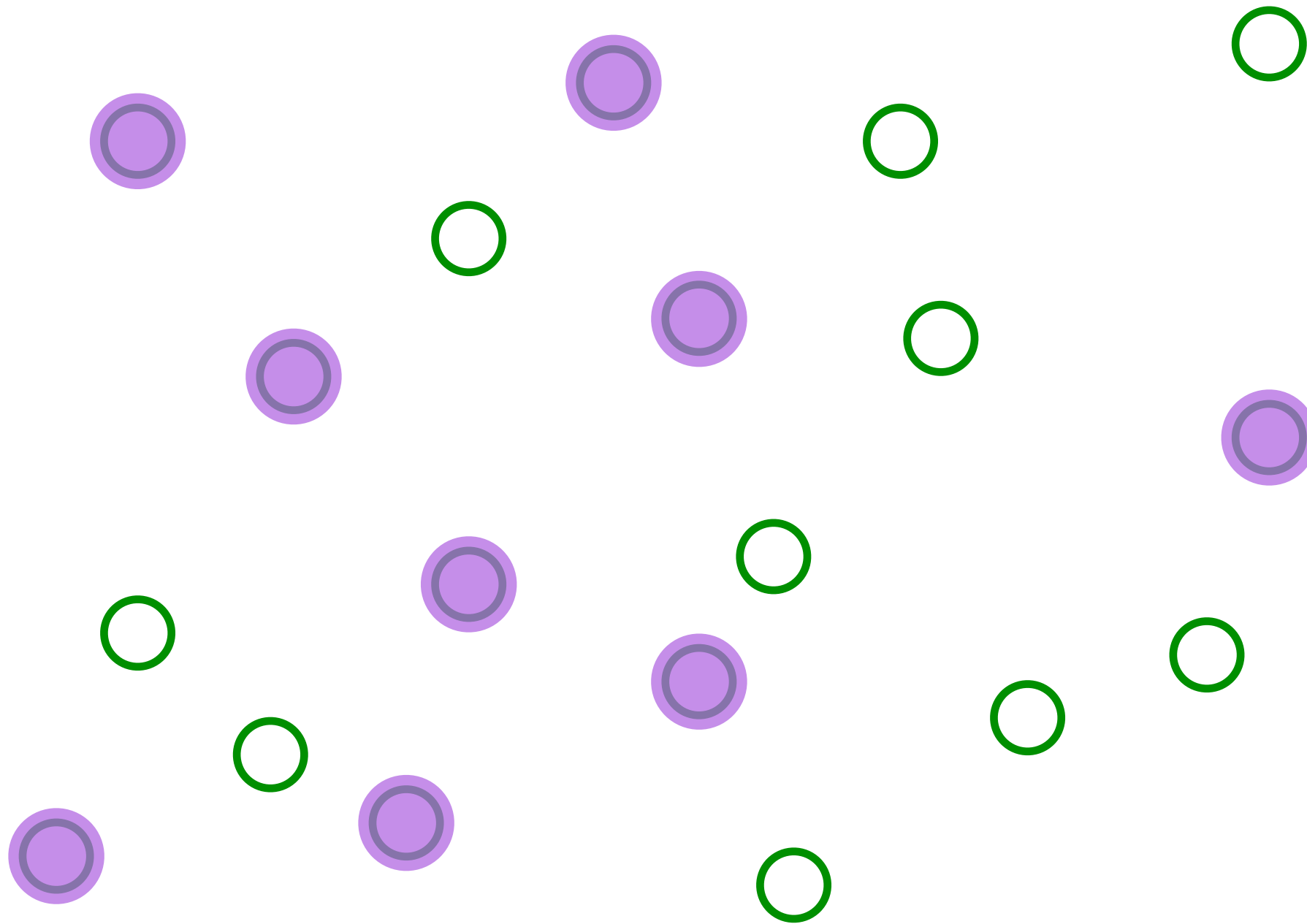
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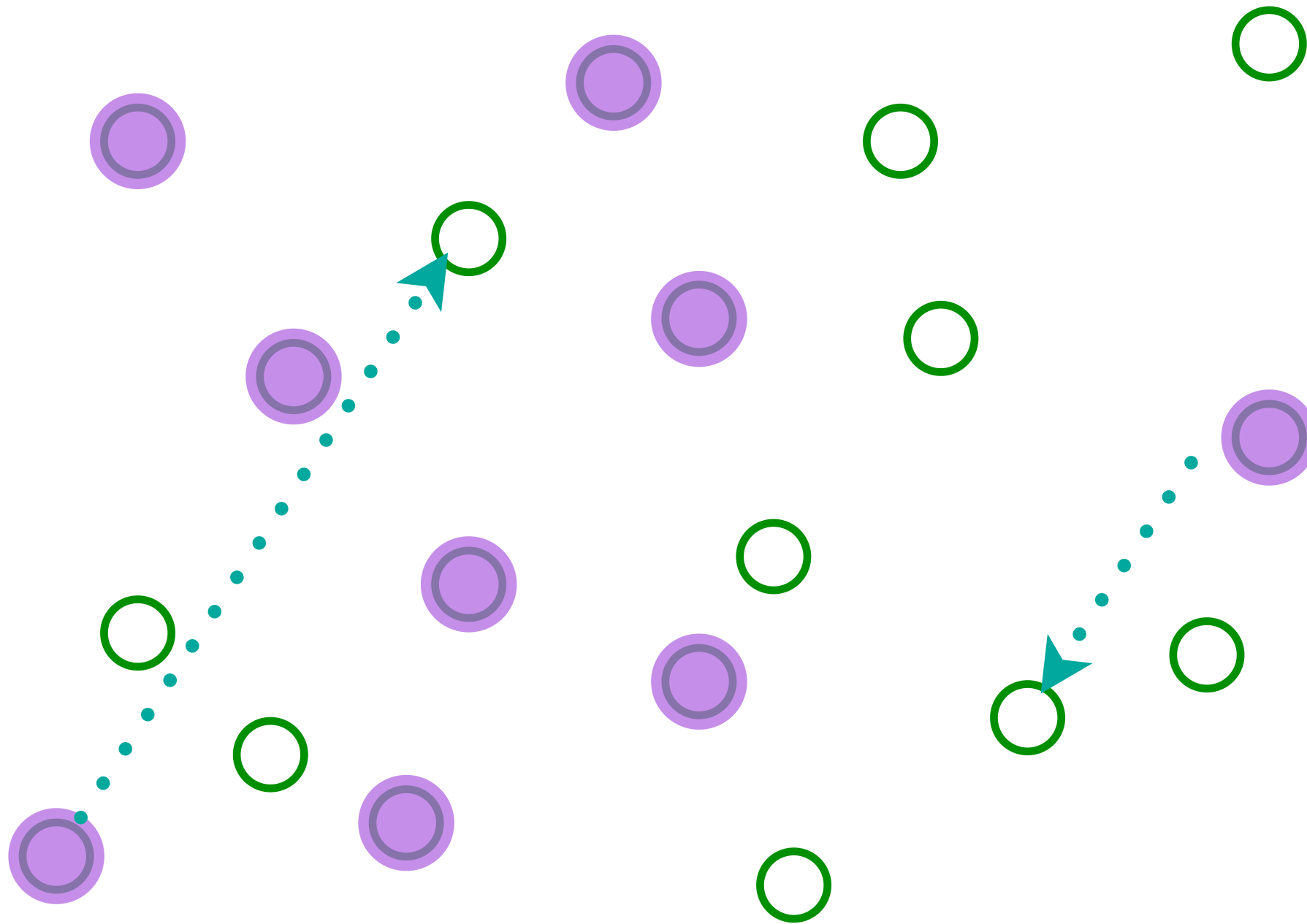
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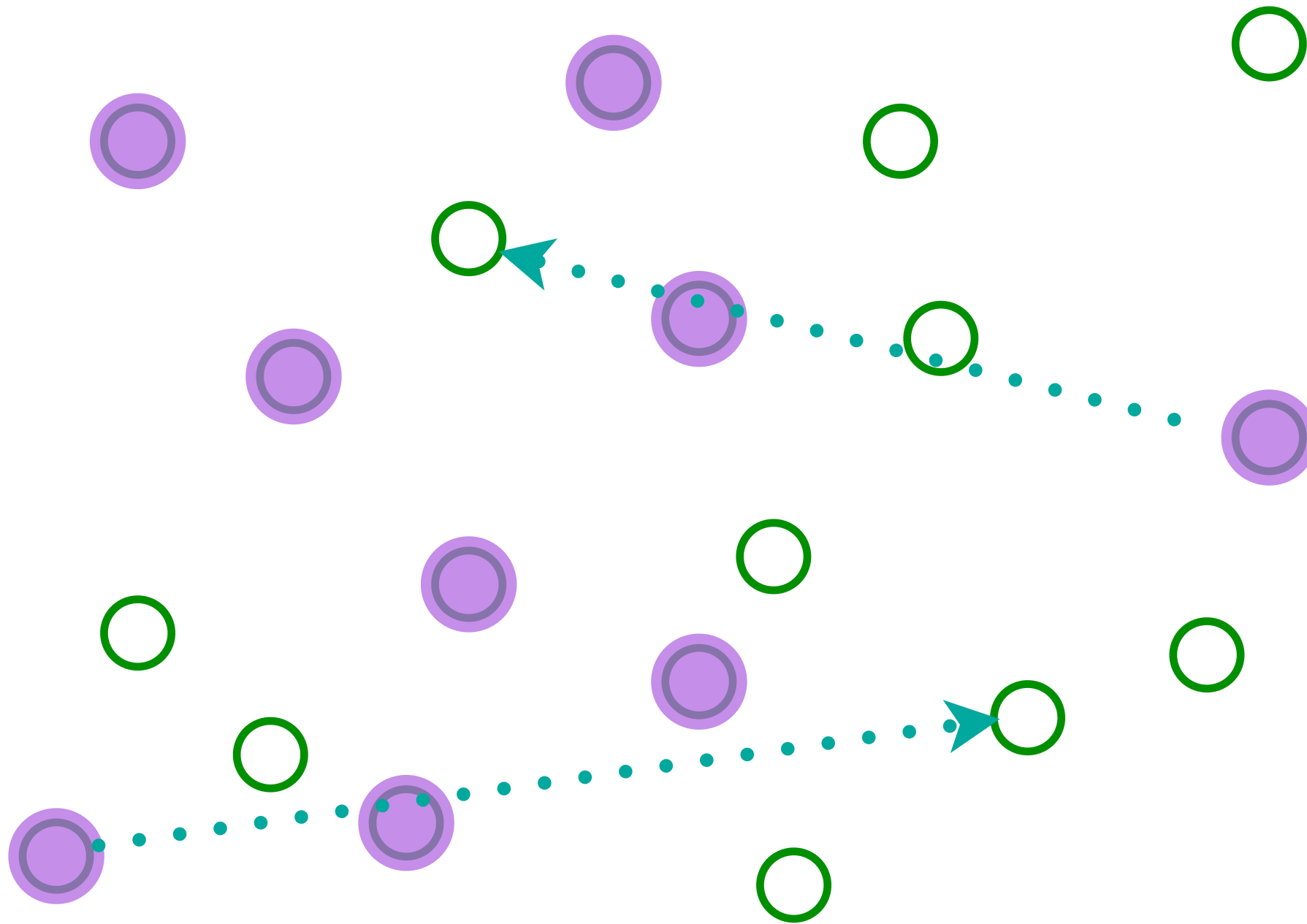
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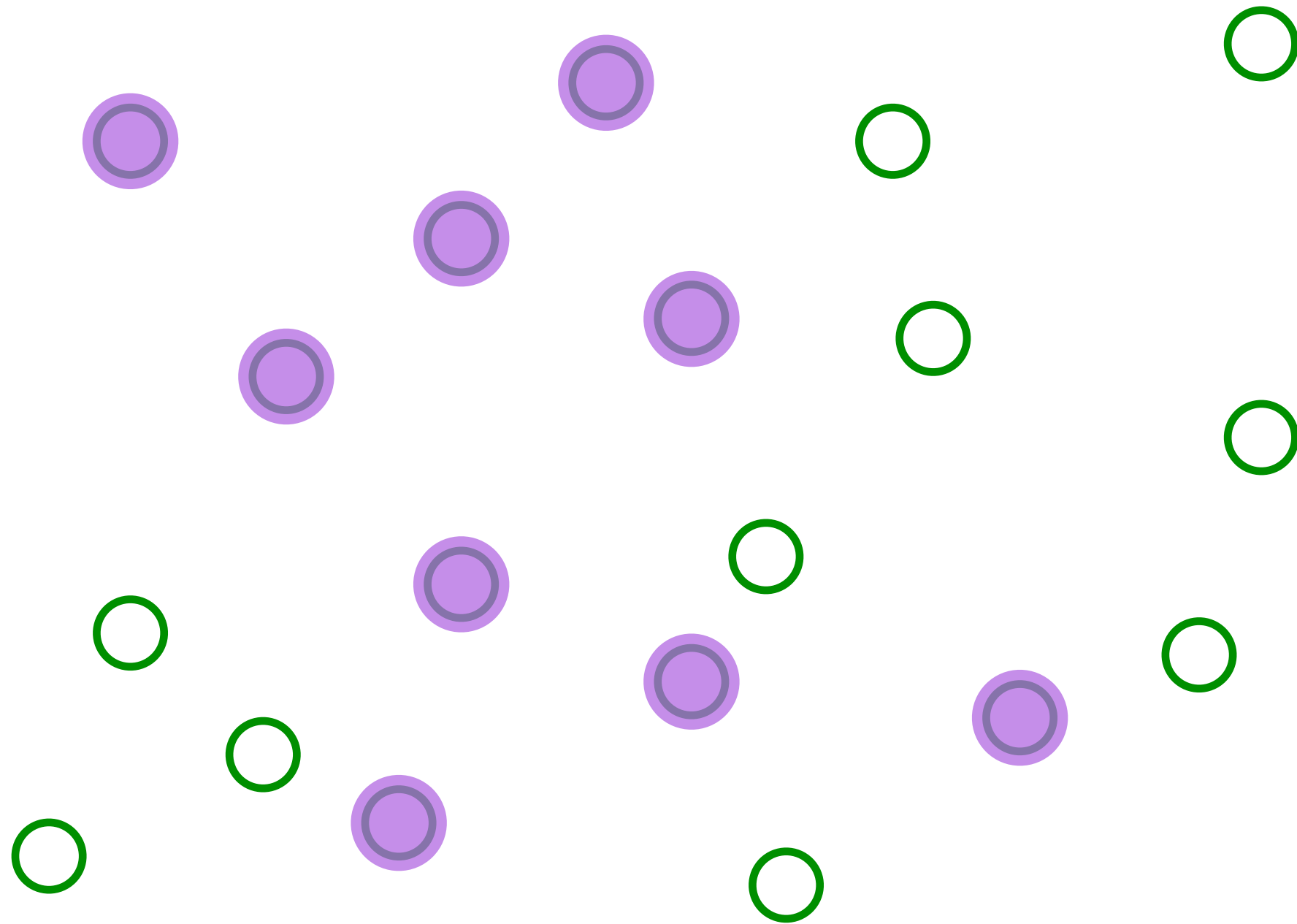
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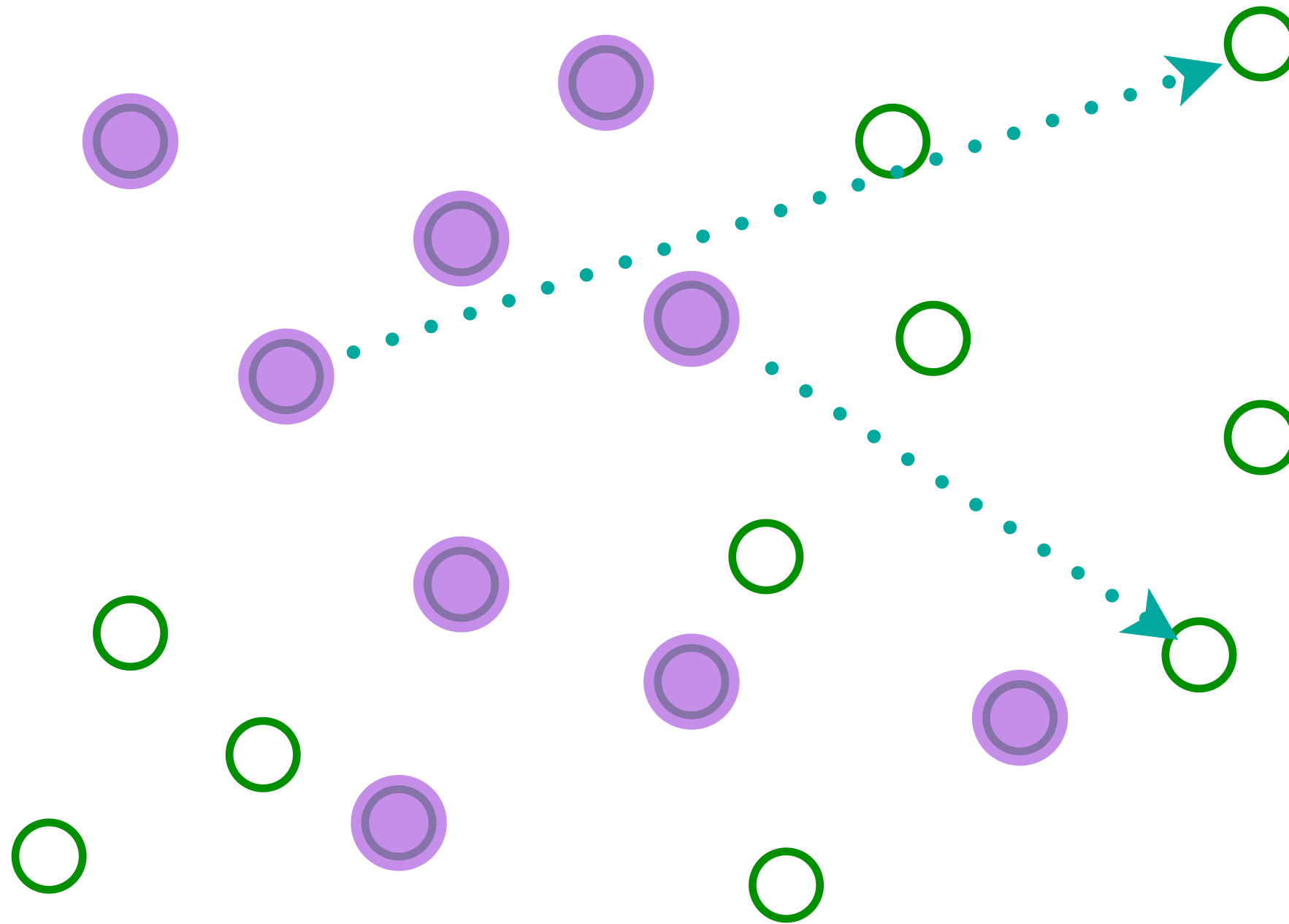
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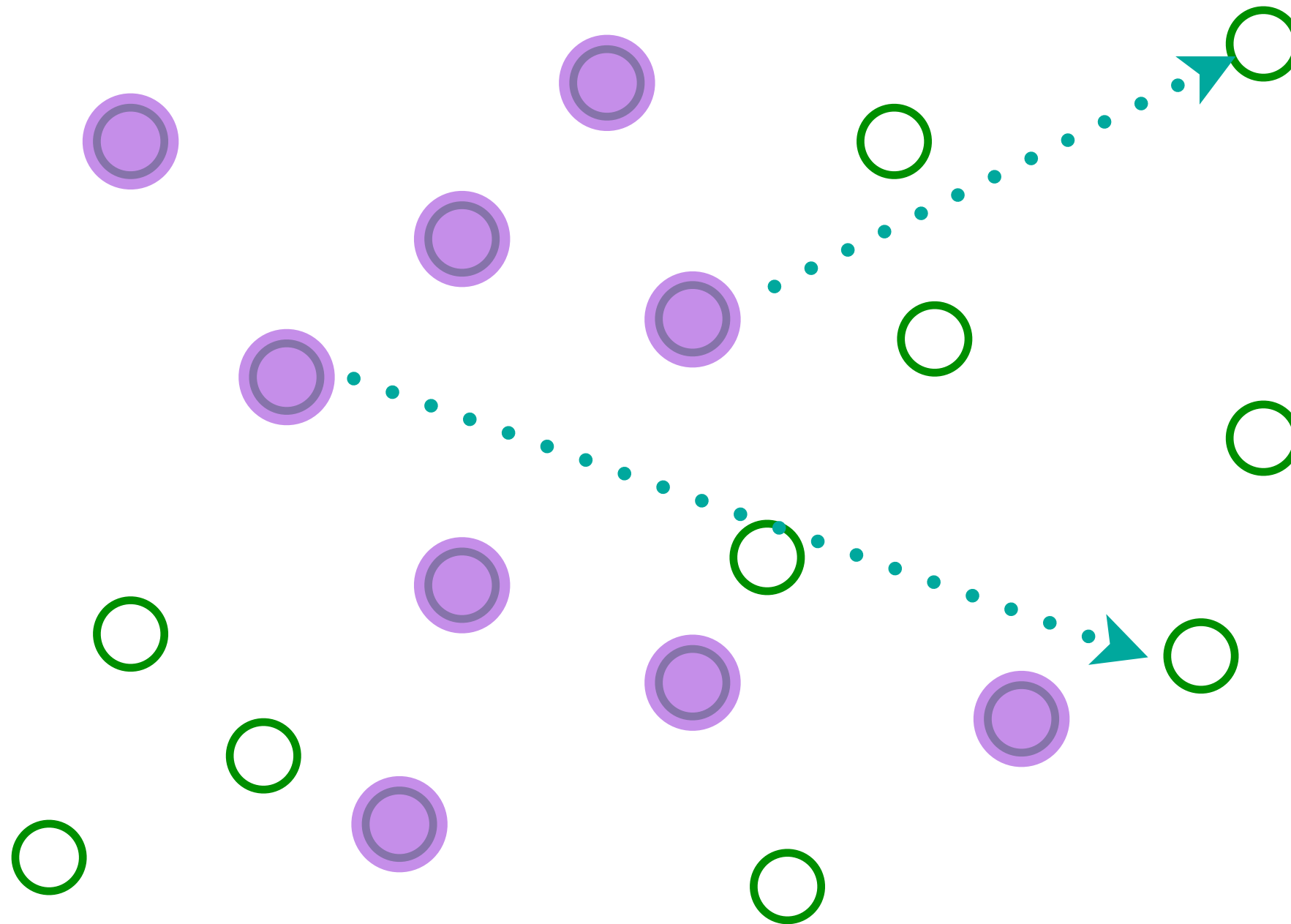
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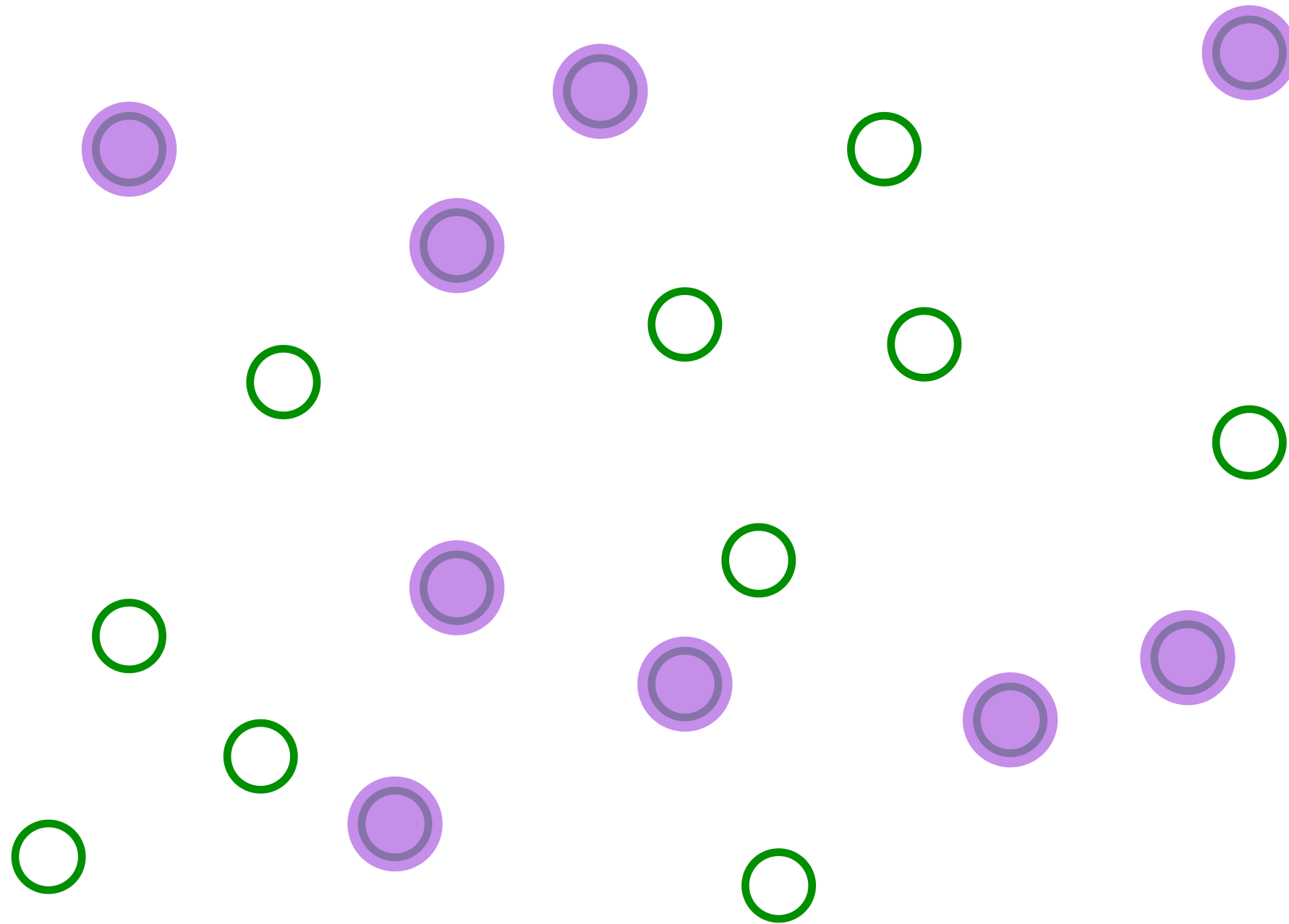
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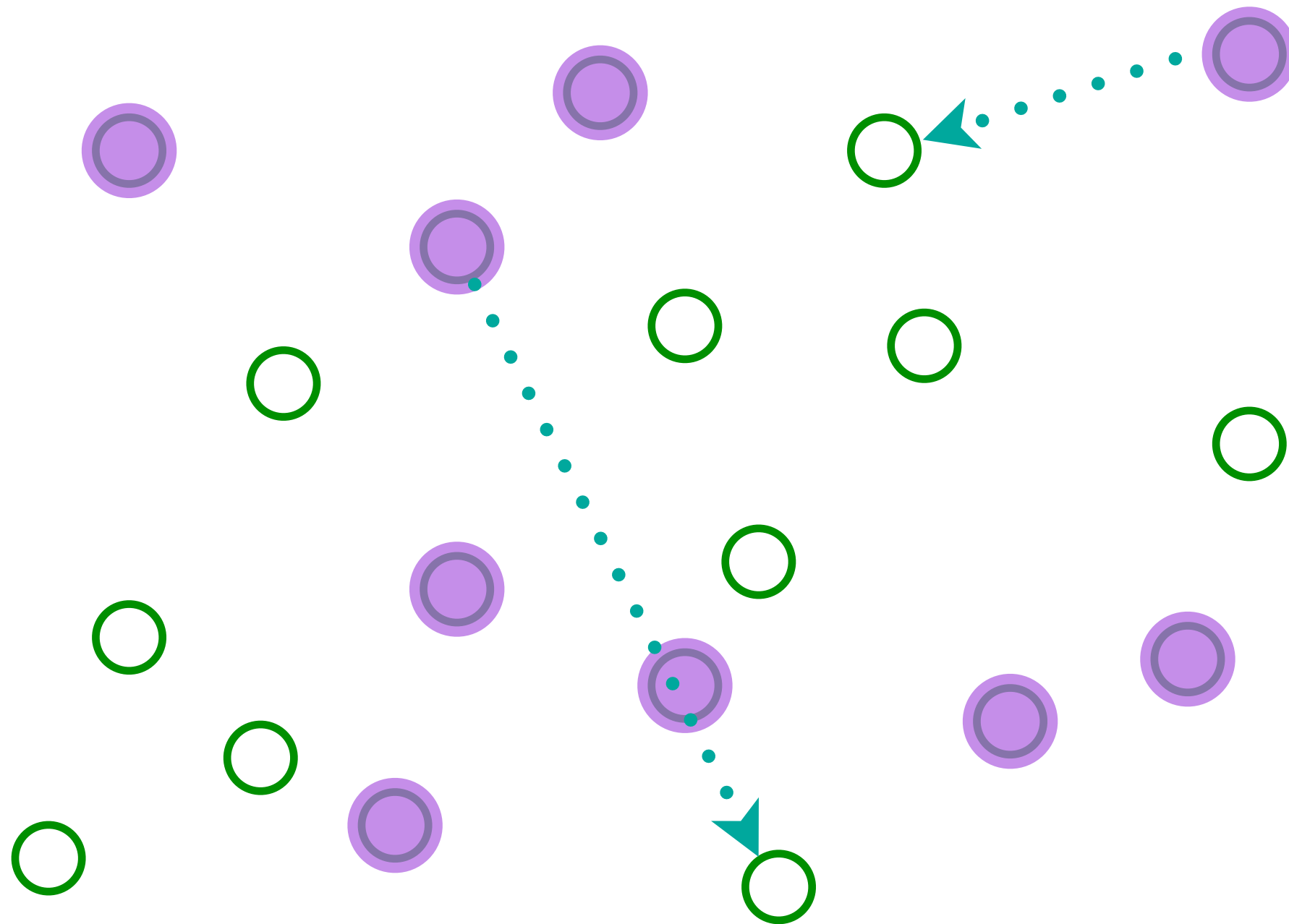
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The SYK model



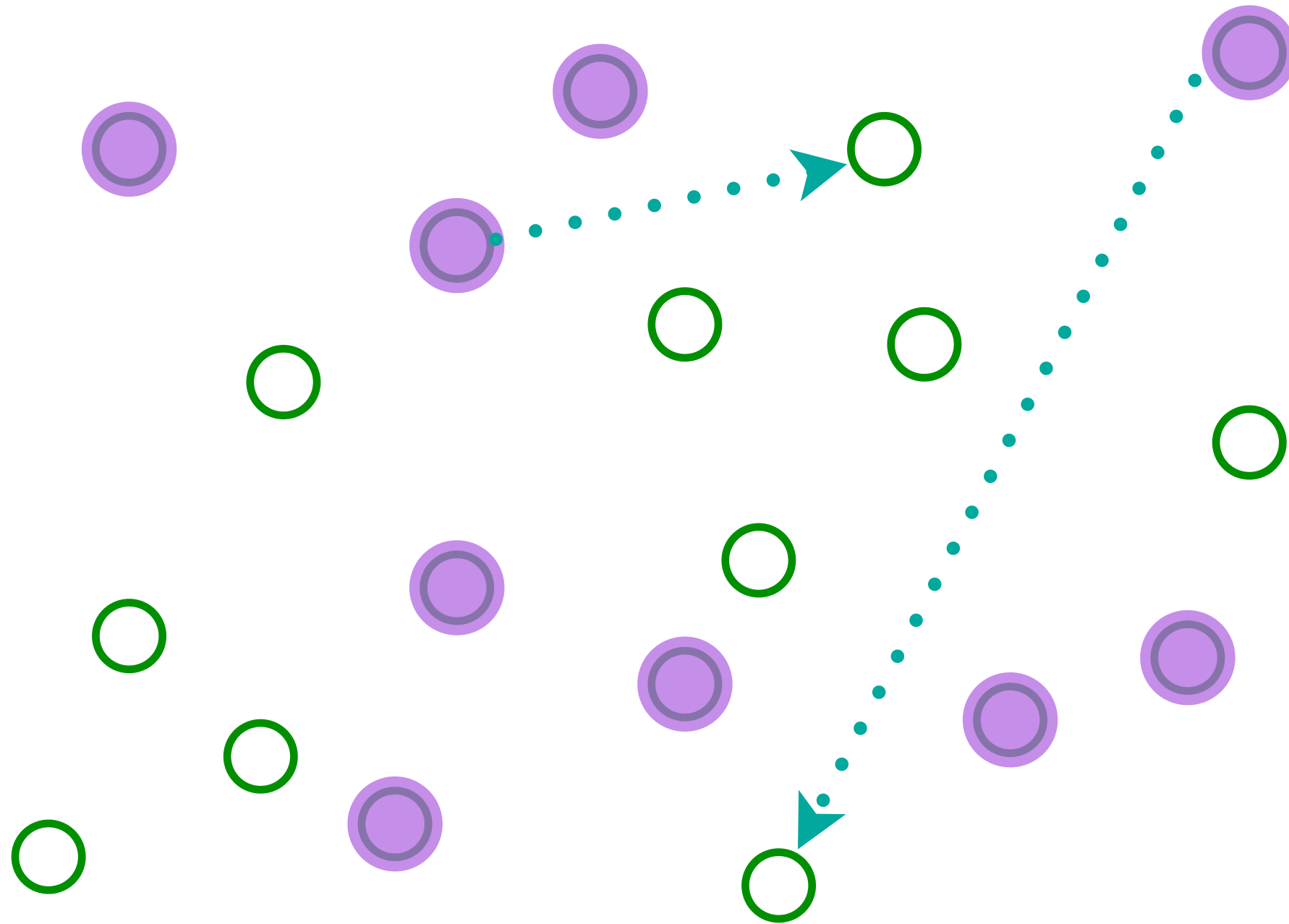
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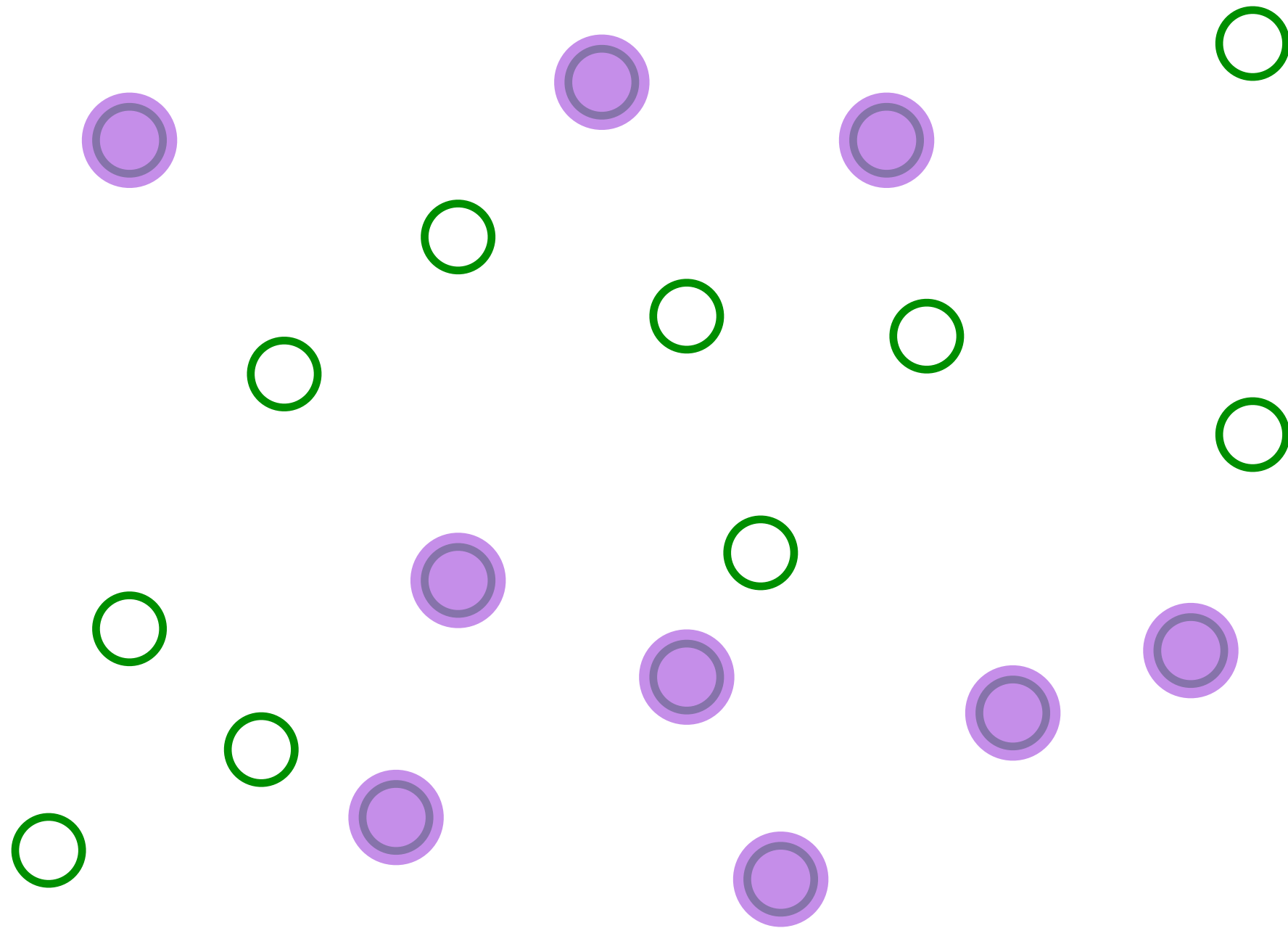
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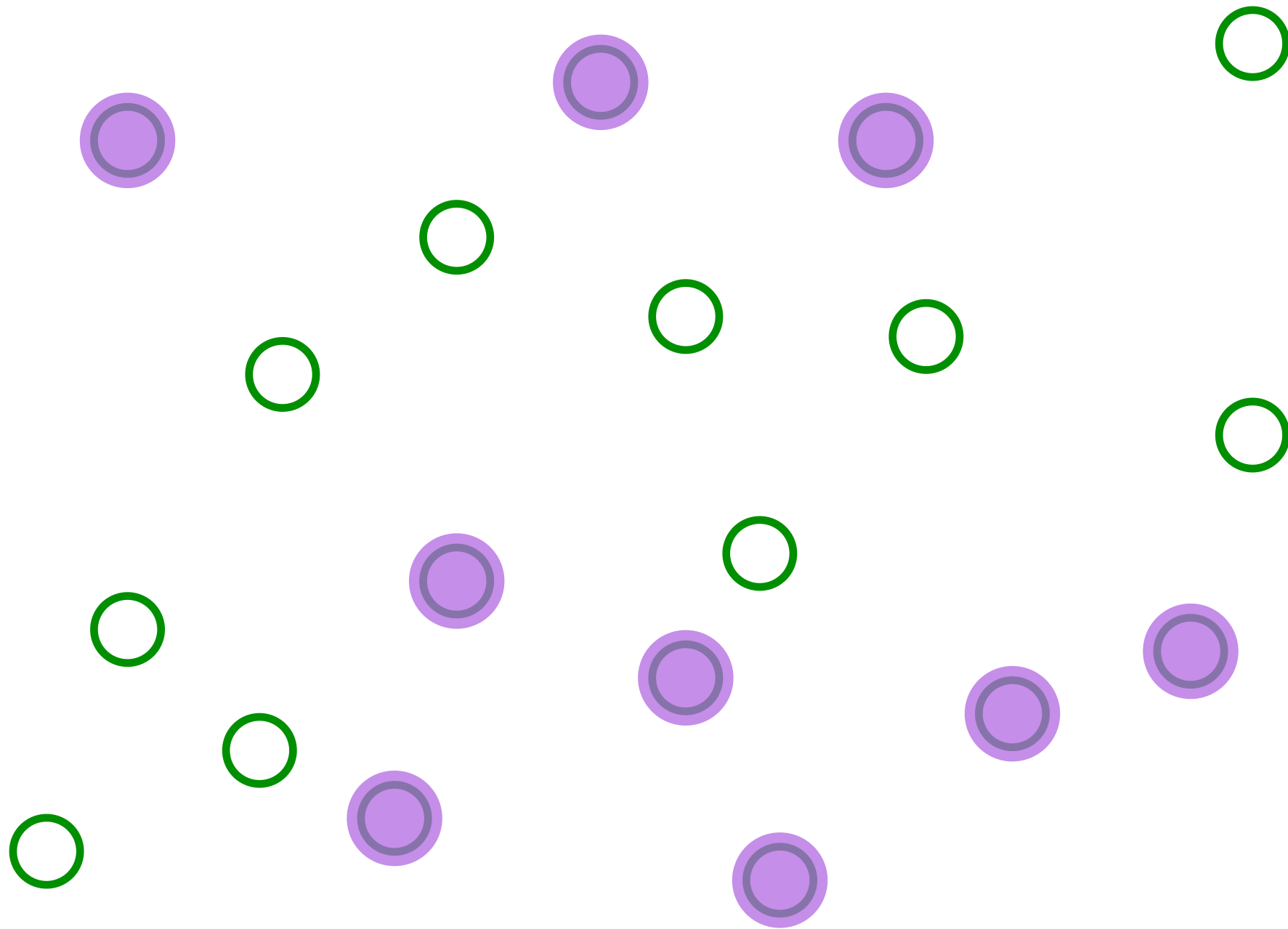
Entangle electrons pairwise randomly

The SYK model



Entangle electrons pairwise randomly

The SYK model



This describes both a strange metal and a black hole!

The SYK model

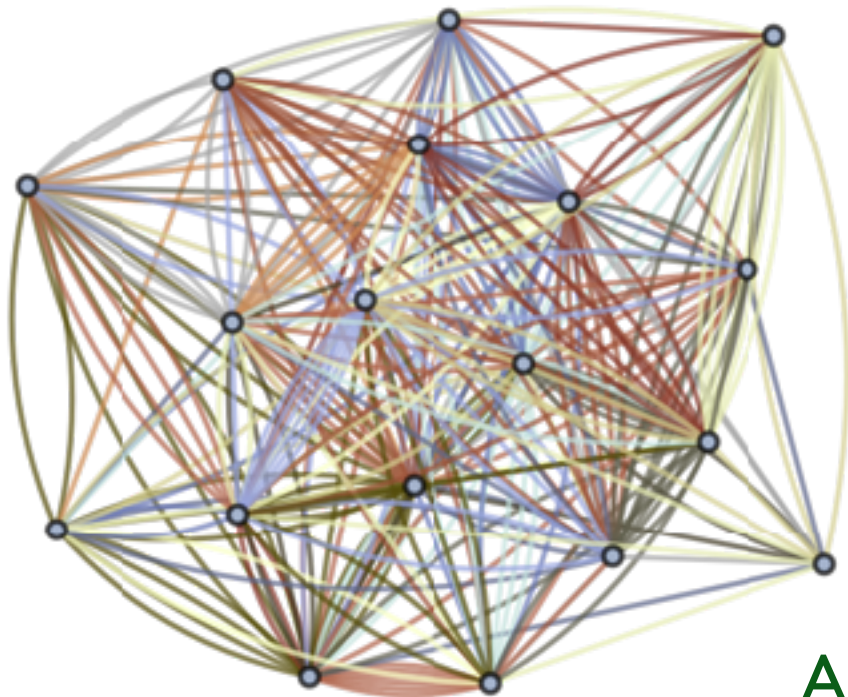
(See also: the “2-Body Random Ensemble” in nuclear physics; did not obtain the large N limit; T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^N U_{ij;k\ell} c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$Q = \frac{1}{N} \sum_i c_i^\dagger c_i$$

$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$
 $N \rightarrow \infty$ yields critical strange metal.



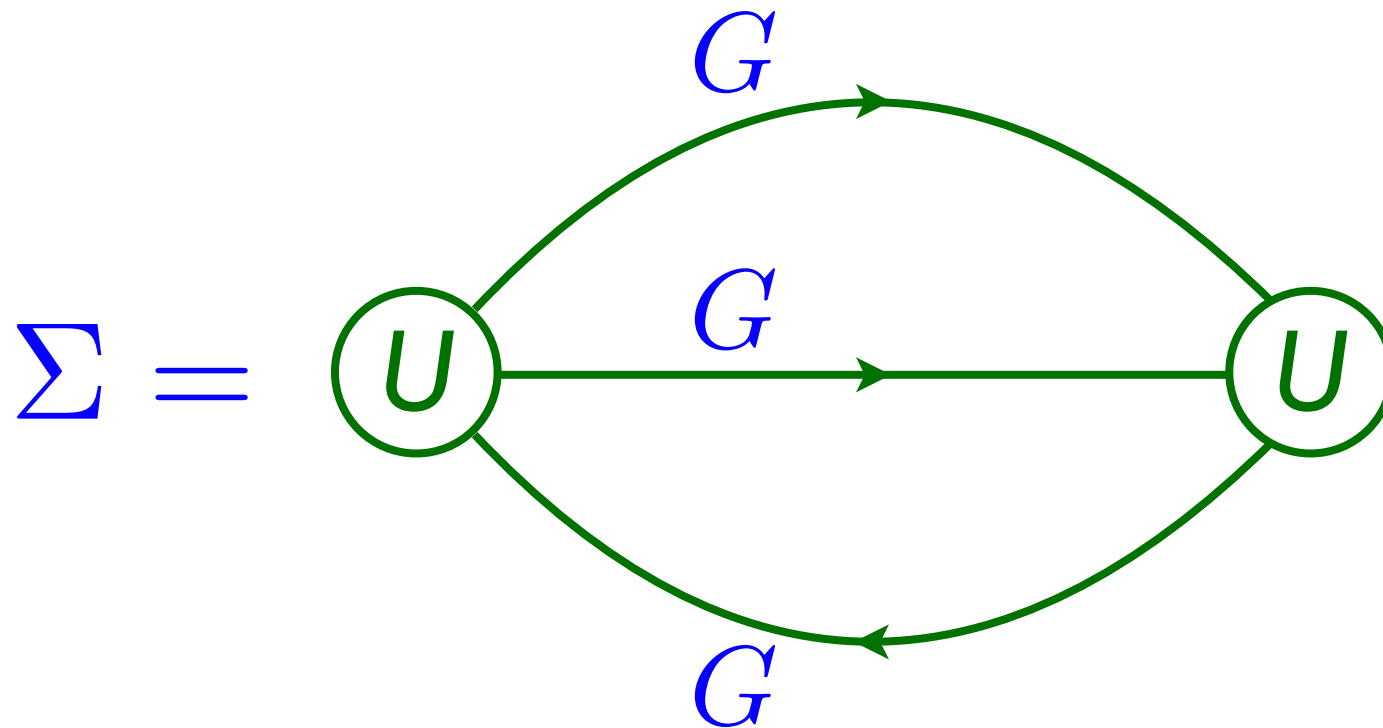
S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

The SYK model

Feynman graph expansion in U_{ijkl} , and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$



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$$G(\tau = 0^-) = \mathcal{Q}.$$

Low frequency analysis shows that the solutions must be gapless and obey

$$\Sigma(z) = \mu - \frac{e^{i(\pi/4+\theta)}}{A} \sqrt{z} + \dots \quad , \quad G(z) = \frac{A e^{-i(\pi/4+\theta)}}{\sqrt{z}}$$

where $A = (\pi/U^2 \cos(2\theta))^{1/4}$. The value of θ is universally related to \mathcal{Q} by a Luttinger-Ward functional analysis similar to that used to establish the Luttinger theorem of Fermi liquid theory:

$$\mathcal{Q} = \frac{1}{2} - \frac{\theta}{\pi} - \frac{\sin(2\theta)}{4}$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = Q.$$

At $T > 0$, we obtain a solution with a conformal structure

$$G(\tau) = -A \frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1 + e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T\tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \quad ,$$

where the ‘particle-hole asymmetry’ is determined by \mathcal{E}

$$e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)} \quad .$$

The SYK model

There are 2^N many body levels with energy E . Shown are all values of E for a single cluster of size $N = 12$. The $T \rightarrow 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\frac{ds_0}{dQ} = 2\pi\mathcal{E}.$$

At $Q = 1/2$,

$$s_0 = \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\dots$$

where G is Catalan's constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

Many-body level spacing $\sim 2^{-N} = e^{-N \ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-N s_0}$

The SYK model

$$\Omega(T) - E_0 = N \left[-s_0 T - \frac{1}{2}(\gamma + 4\pi^2 \mathcal{E}^2 K) T^2 + \mathcal{O}(T^3) \right] + 2T \ln \left(\frac{U}{T} \right) \dots$$

is the grand potential, where $K = d\mathcal{Q}/d\mu \sim 1/U$ is the compressibility/ N , $\gamma \sim 1/U$ will appear later in the co-efficient of the Schwarzian, and the N^0 term arises from fluctuations about the large N theory described by the Schwarzian.

The inversion from $\Omega(T)$ to the *many*-body density of states, $D(E)$, requires terms in $\Omega(T)$ which are exponentially small in N (not shown above) from the Schwarzian action, yielding terms which are not small in $D(E)$. We obtain

$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} d \left(E - \frac{p^2}{2NK} \right)$$

where $N\mathcal{Q} + p$ is the integer fermion number, $d(E) = 0$ for $E < E_0$, and

$$d(E) \sim \exp(Ns_0) \sinh \left(\sqrt{2N\gamma(E - E_0)} \right), \quad E > E_0, \quad e^{-cN} \ll \gamma(E - E_0) \ll N$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N .

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, arXiv:1611.04650;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv:1612.00849 ;
A.M. Garcia-Garcia and J.J.M. Verbaarschot, arXiv:1701.06593; D. Bagrets, A. Altland, and A. Kamenev, arXiv:1702.08902;
D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467; Yingfei Gu and S. Sachdev, unpublished.

A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N \left(-\frac{\pi^2}{6} \rho_0 T^2 + \mathcal{O}(T^4) \right) + \dots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.

We can also define the *many* body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to be performed with care (it does not commute with the $1/N$ expansion), and we obtain

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and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

The SYK model

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$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} d \left(E - \frac{p^2}{2NK} \right)$$

where $N\mathcal{Q} + p$ is the integer fermion number, $d(E) = 0$ for $E < E_0$, and

$$d(E) \sim \boxed{\exp(Ns_0)} \sinh \left(\sqrt{2N\gamma(E - E_0)} \right) , \quad E > E_0 , \quad e^{-cN} \ll \gamma(E - E_0) \ll N$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N .

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, arXiv:1611.04650;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv:1612.00849 ;
A.M. Garcia-Garcia and J.J.M. Verbaarschot, arXiv:1701.06593; D. Bagrets, A. Altland, and A. Kamenev, arXiv:1702.08902;
D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467; Yingfei Gu and S. Sachdev, unpublished.

The SYK model

No quasiparticles

- Rapid local thermal equilibration (of fermion correlators) in a ‘Planckian’ time

$$\tau_{\text{eq}} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \rightarrow 0.$$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

A. Eberlein, V. Kasper, S. Sachdev, and
J. Steinberg, PRB **96**, 205123 (2017)

Established by solution of Schwinger-Keldysh equations for a quench.

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Established by solution of Schwinger-Keldysh equations for a quench.

- Presence of quasiparticles should slow down thermalization, so *all* quantum systems obey

$$\tau_{\text{eq}} > C \frac{\hbar}{k_B T} \quad , \quad \text{as } T \rightarrow 0.$$

S. Sachdev, *Quantum Phase Transitions*,
Cambridge (1999)

Absence of quasiparticles \Leftrightarrow Fastest possible thermalization

Other quantum models without quasiparticles

- Rapid local thermal equilibration in a ‘Planckian’ time

$$\tau_{\text{eq}} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \rightarrow 0.$$

- Ising model in a transverse field in 2 dimensions at its quantum critical point, $g = g_c$. Described by the Wilson-Fisher fixed point of ϕ^4 quantum field theory in 2+1 dimensions

$$H = - \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^x$$

$\sigma_i^{x,z}$ are the Pauli operators on site i .

- Other strongly-coupled conformal field theories.

Absence of quasiparticles \Leftrightarrow Fastest possible thermalization

1. Random matrix quasiparticle model

$q=2$, complex SYK

2. Matter without quasiparticles

$q=4$, complex SYK

3. The Schwarzian theory

4. Connections to black holes
with AdS_2 horizons

5. Connections to strange metals

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \mu - \frac{1}{A} \sqrt{z} + \dots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

At frequencies $\ll U$, the $i\omega + \mu$ can be dropped, and without it equations are invariant under the reparametrization and gauge transformations.

The singular part of the self-energy and the Green's function obey

$$\int_0^\beta d\tau_2 \Sigma_{\text{sing}}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma_{\text{sing}}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

$$\int_0^\beta d\tau_2 \Sigma(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

These equations are invariant under

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

By using $f(\sigma) = \tan(\pi T \sigma) / (\pi T)$ we can

now obtain the $T > 0$ solution from the $T = 0$ solution.

Let us write the large N saddle point solutions of S as

$$\begin{aligned} G_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-1/2} \\ \Sigma_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-3/2}. \end{aligned}$$

The saddle point will be invariant under a reparamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\tilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for Σ). It turns out this is true only for the $\text{SL}(2, \mathbb{R})$ transformations under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to $\text{SL}(2, \mathbb{R})$ by the saddle point.

The Schwarzian theory of the SYK model

After introducing replicas $a = 1 \dots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp \left[- \sum_{ia} \int_0^\beta d\tau c_{ia}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{ia} - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \right].$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \right) \right].$$

The Schwarzian theory of the SYK model

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$
$$S = \ln \det [\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)]$$
$$+ \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) [G(\tau_2, \tau_1) + (U^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \Sigma(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

A. Kitaev, 2015

S. Sachdev, PRX **5**, 041025 (2015)

The Schwarzian theory of the SYK model

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_1) \mathcal{D}\Sigma(\tau_1, \tau_2) e^{-NS[G, \Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s , and only focus on the “Nambu-Goldstone” modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau) \mathcal{D}\phi(\tau) e^{-NS_{\text{eff}}[f, \phi]}.$$

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv:1612.00849;
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746;
Yingfei Gu and S. Sachdev, unpublished

The Schwarzian theory of the SYK model

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\text{eff}}[f, \phi] = \frac{K}{2} \int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi\mathcal{E}T)\partial_\tau f)^2 - \frac{\gamma}{4\pi^2} \int_0^{1/T} d\tau \{ \tan(\pi T f(\tau)), \tau \},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, the couplings K , γ , and \mathcal{E} can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g, \tau\} \equiv \frac{g'''}{g'} - \frac{3}{2} \left(\frac{g''}{g'} \right)^2.$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\text{eff}} \left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0 \right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv:1612.00849;
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Yingfei Gu and S. Sachdev, unpublished

1. Random matrix quasiparticle model

$q=2$, complex SYK

2. Matter without quasiparticles

$q=4$, complex SYK

3. The Schwarzian theory

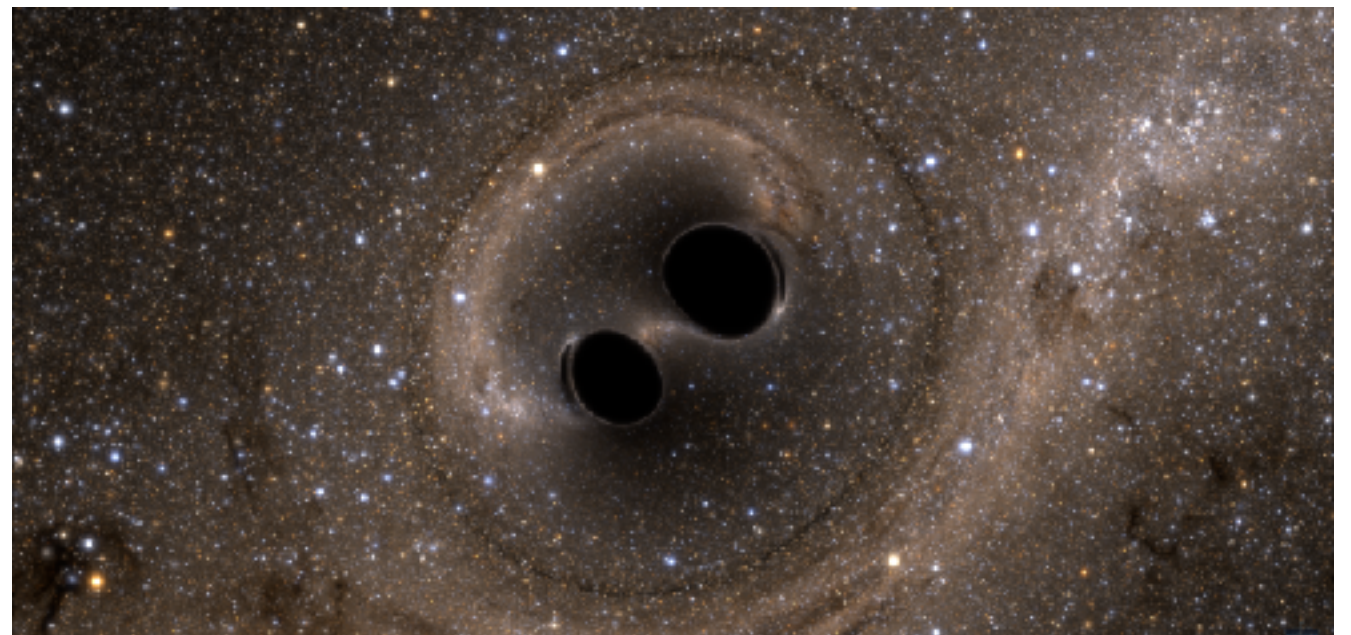
4. Connections to black holes
with AdS_2 horizons

5. Connections to strange metals

SYK models and black holes

- Black holes have an entropy proportional to their surface area, and a temperature, $T_H = \hbar c^3 / (8\pi G M k_B)$.
- Black holes relax to thermal equilibrium in a ‘Planckian’ time $\sim \hbar / (k_B T_H) = 8\pi G M / c^3$.
- Black holes in $d + 1$ spatial dimensions are similar to a quantum system without quasiparticles in d spatial dimensions.

**Black
holes**



SYK models and black holes

PHYSICAL REVIEW LETTERS **105**, 151602 (2010)



Holographic Metals and the Fractionalized Fermi Liquid

Subir Sachdev

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(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti-de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a “small” Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, $\text{AdS}_2 \times \mathbb{R}^2$ physics of Reissner-Nordström black holes.

Bekenstein-Hawking entropy of AdS_2 horizon
at $T = 0 \Leftrightarrow N s_0$ entropy of SYK model

Black hole
horizon

$$\text{AdS}_2 \times S^2$$
$$ds^2 = (d\zeta^2 - dt^2)/\zeta^2 + d\vec{x}^2$$
$$\text{Gauge field: } A = (\mathcal{E}/\zeta)dt$$

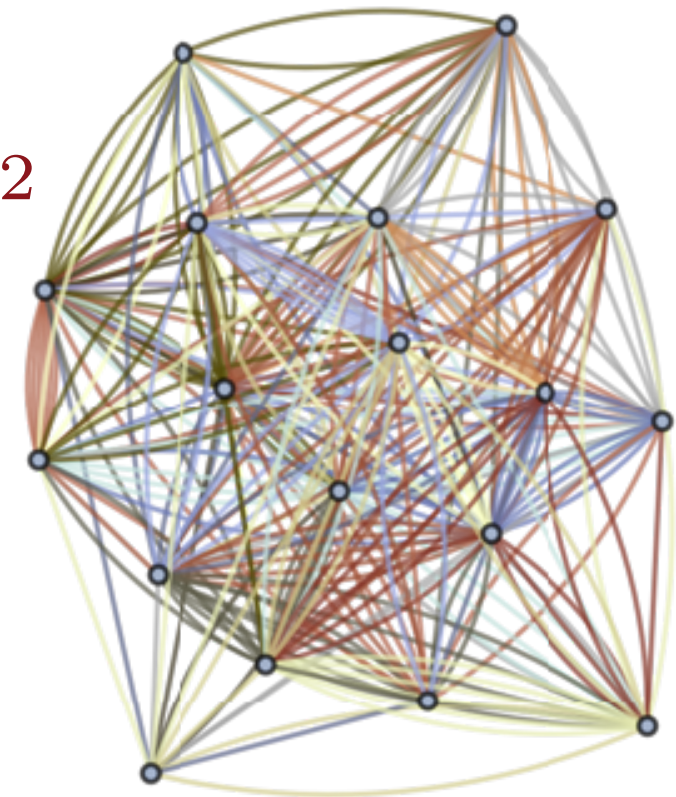
$\zeta = \infty$

ζ

charge
density \mathcal{Q}

S^2

\vec{x}



SYK models and black holes

Connections of SYK to gravity and AdS₂ horizons

- Reparameterization and gauge invariance are the ‘symmetries’ of the Einstein-Maxwell theory of gravity and electromagnetism
- SL(2,R) is the isometry group of AdS₂.

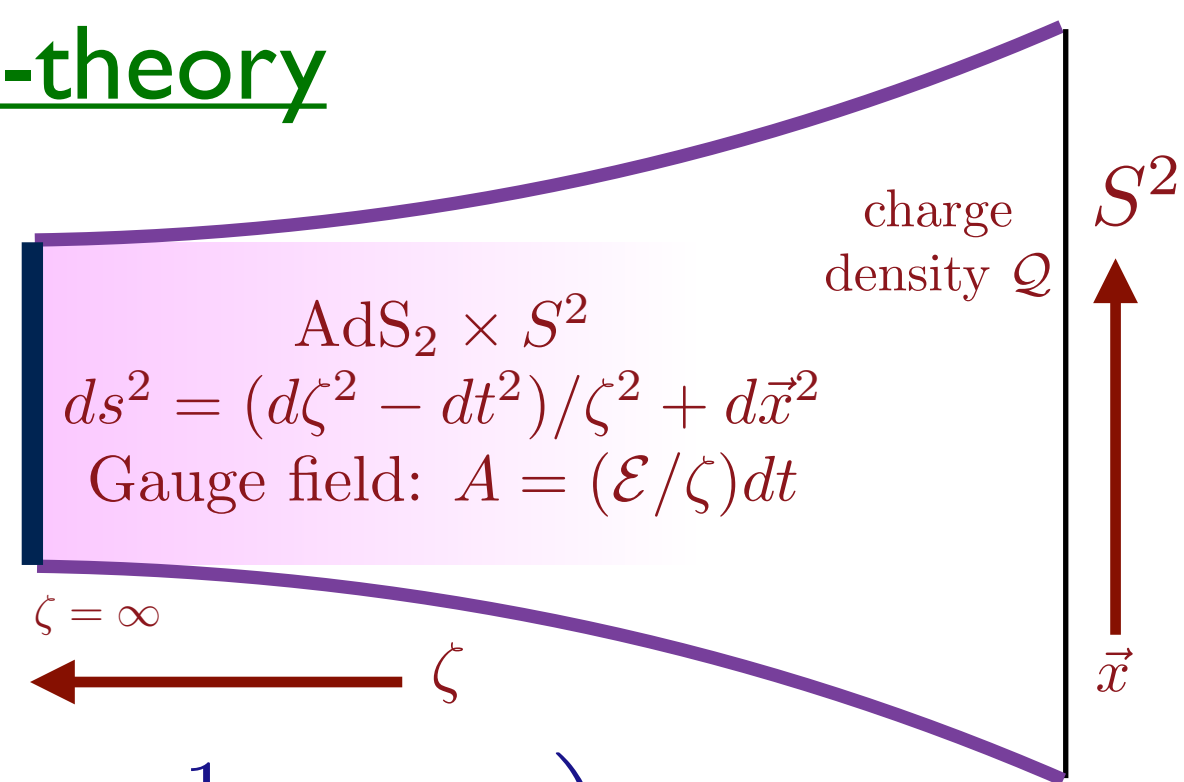
$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d}$$

with $ad - bc = 1$.



Einstein-Maxwell-theory



$$S_{4D} = \int d^4x \sqrt{-\hat{g}} \left(\hat{\mathcal{R}} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right),$$

- Has Reissner-Nördstrom-AdS charged black hole solution, with charge density Q , a near-horizon $\text{AdS}_2 \times S^2$ geometry, and surface electric field \mathcal{E} .
- The Bekenstein-Hawking black hole entropy S_{4D} obeys the same relation as the SYK model

$$\frac{\partial S_{4D}}{\partial Q} = 2\pi\mathcal{E},$$

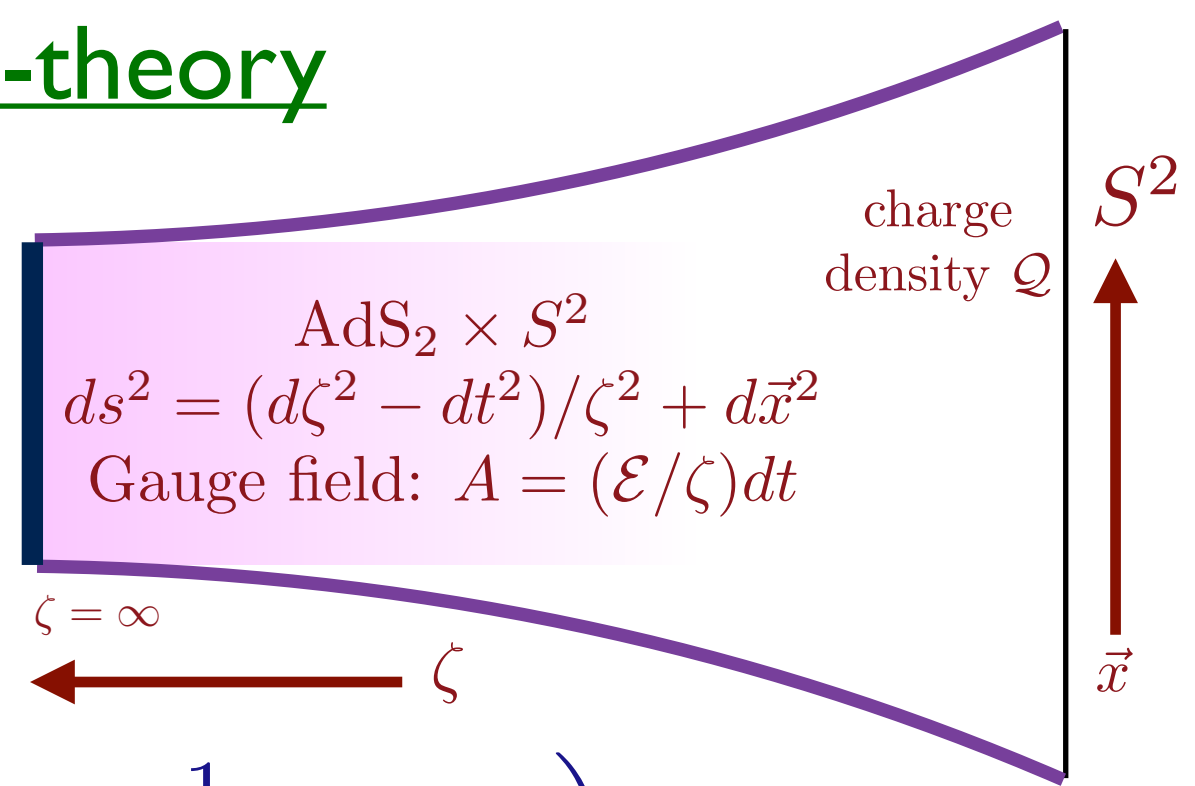
where \mathcal{E} is identified from the spectral asymmetry of probe particle Green's functions in both cases.



Einstein-Maxwell-theory

P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal,
arXiv:1802.09547;

A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia,
arXiv:1802.07746



$$S_{4D} = \int d^4x \sqrt{-\hat{g}} \left(\hat{\mathcal{R}} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right),$$

In the small black hole size limit, $T \ll 1/R$, where R is the radius of the black hole, the theory dimensionally reduces to an Einstein-Maxwell-dilaton theory in two dimensions (the Jackiw-Teitelbaum model), along with Maxwell term

$$S_{2D} = N s_0 + \int d^2x \sqrt{-g} \left(\Phi (\mathcal{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab} F^{ab} \right).$$

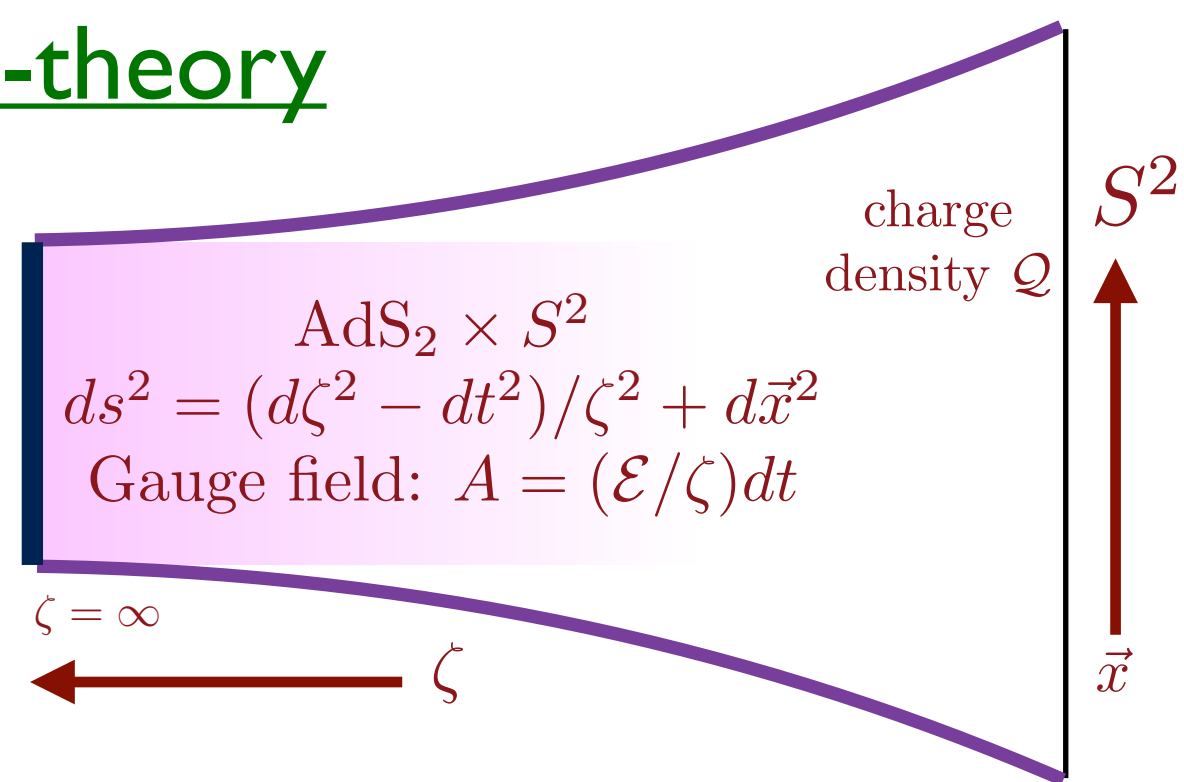
The dilaton Φ represents the radial oscillations of the small black hole.



Einstein-Maxwell-theory

P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal,
arXiv:1802.09547;

A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia,
arXiv:1802.07746



$$S_{2D} = N s_0 + \int d^2x \sqrt{-g} \left(\Phi (\mathcal{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab} F^{ab} \right).$$

There are no bulk quantum fluctuations of the metric in two-dimensional gravity, and there a further dimensional reduction to a 0+1 dimensional theory representing fluctuations of the AdS₂ boundary: this 0+1 dimensional turns out to be *precisely the Schwarzian theory obtained for the SYK model.*

J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;

K. Jensen, arXiv:1605.06098;

J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

1. Random matrix quasiparticle model

$q=2$, complex SYK

2. Matter without quasiparticles

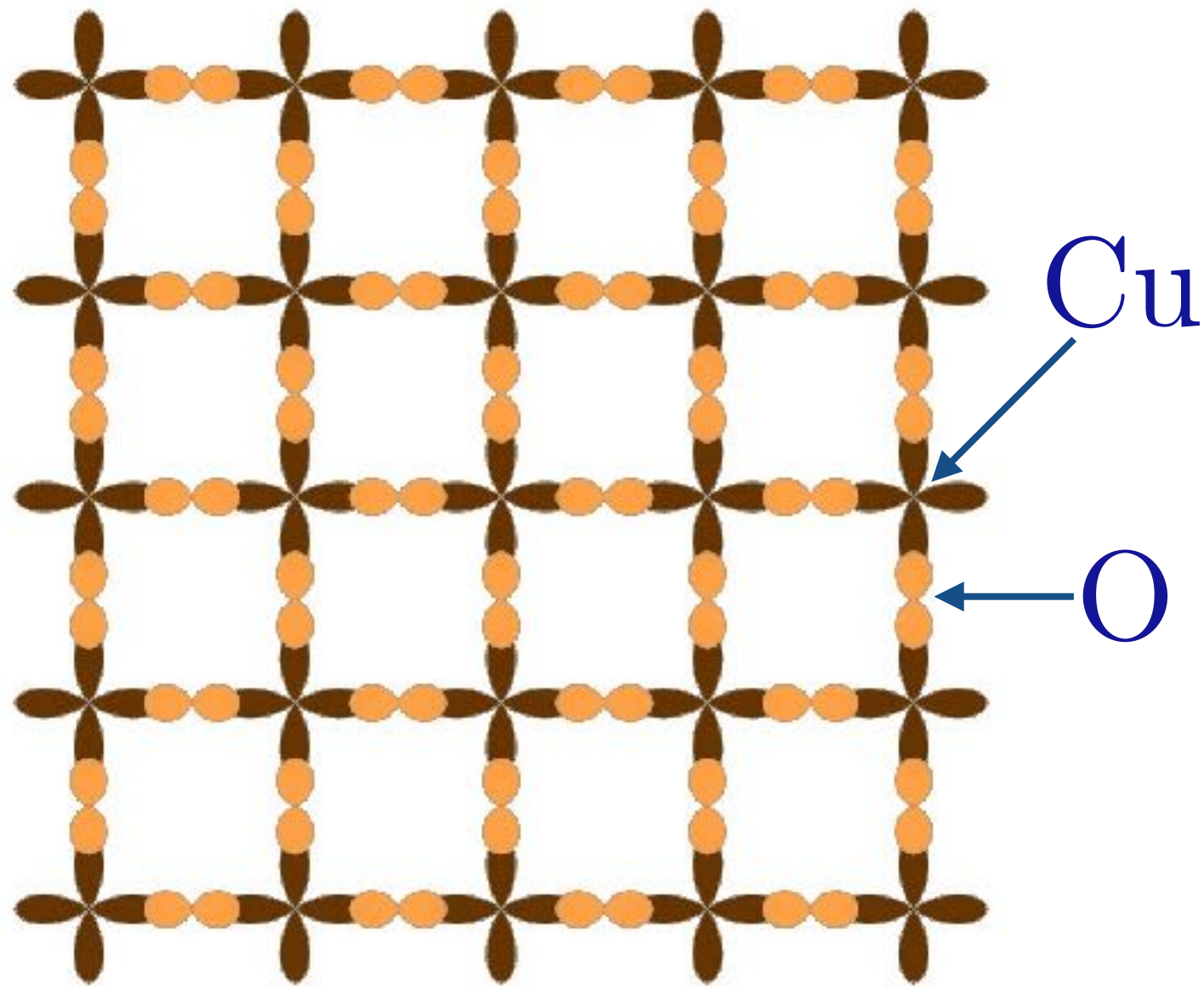
$q=4$, complex SYK

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with AdS_2 horizons

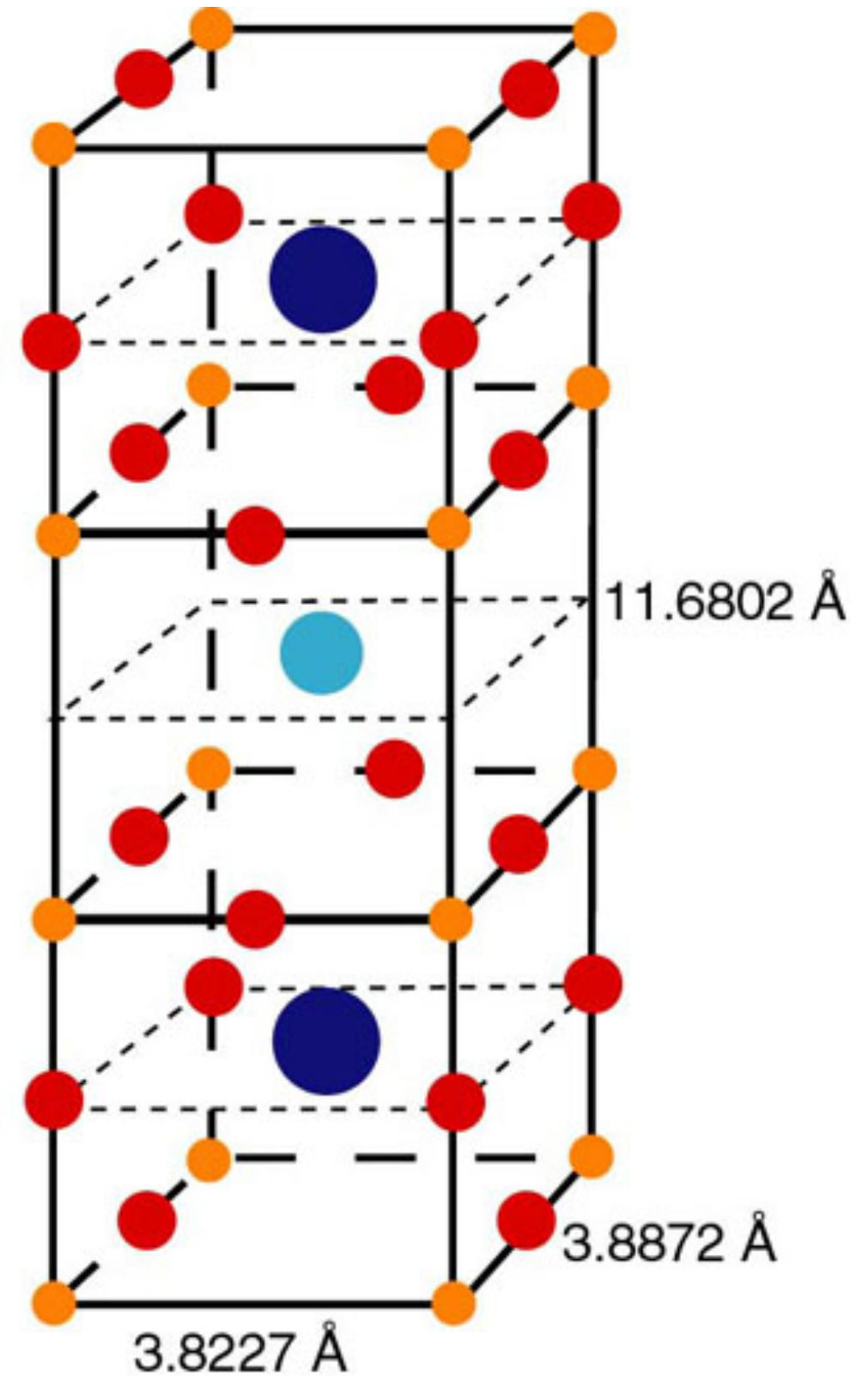
5. Connections to strange metals

High temperature superconductors



CuO_2 plane

Described by a Hubbard model
on the Cu sites



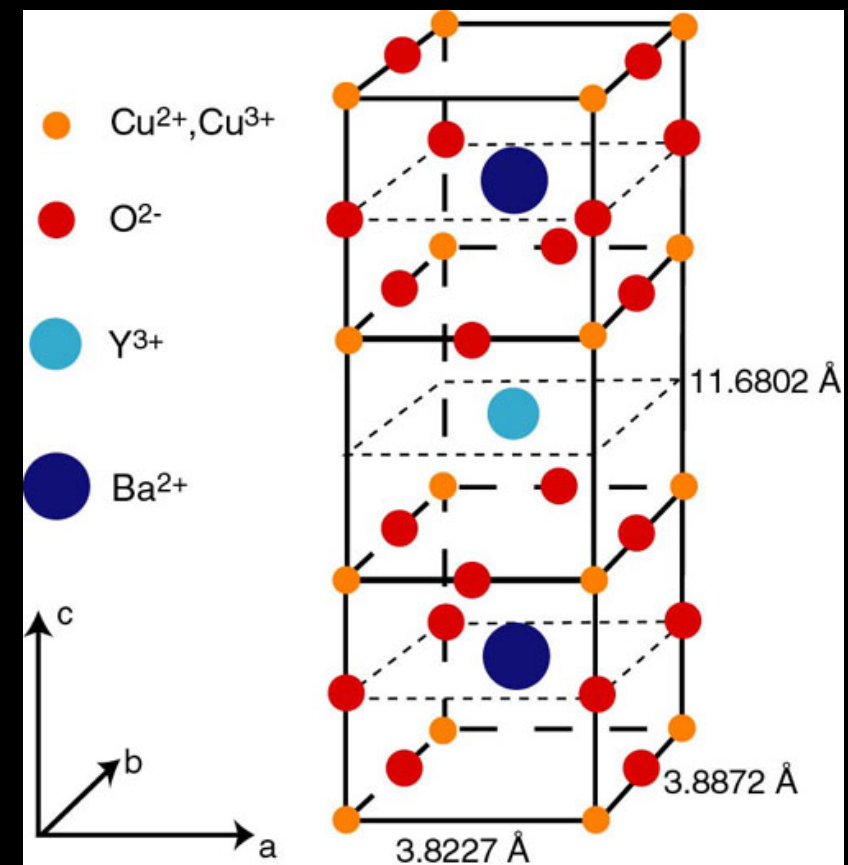
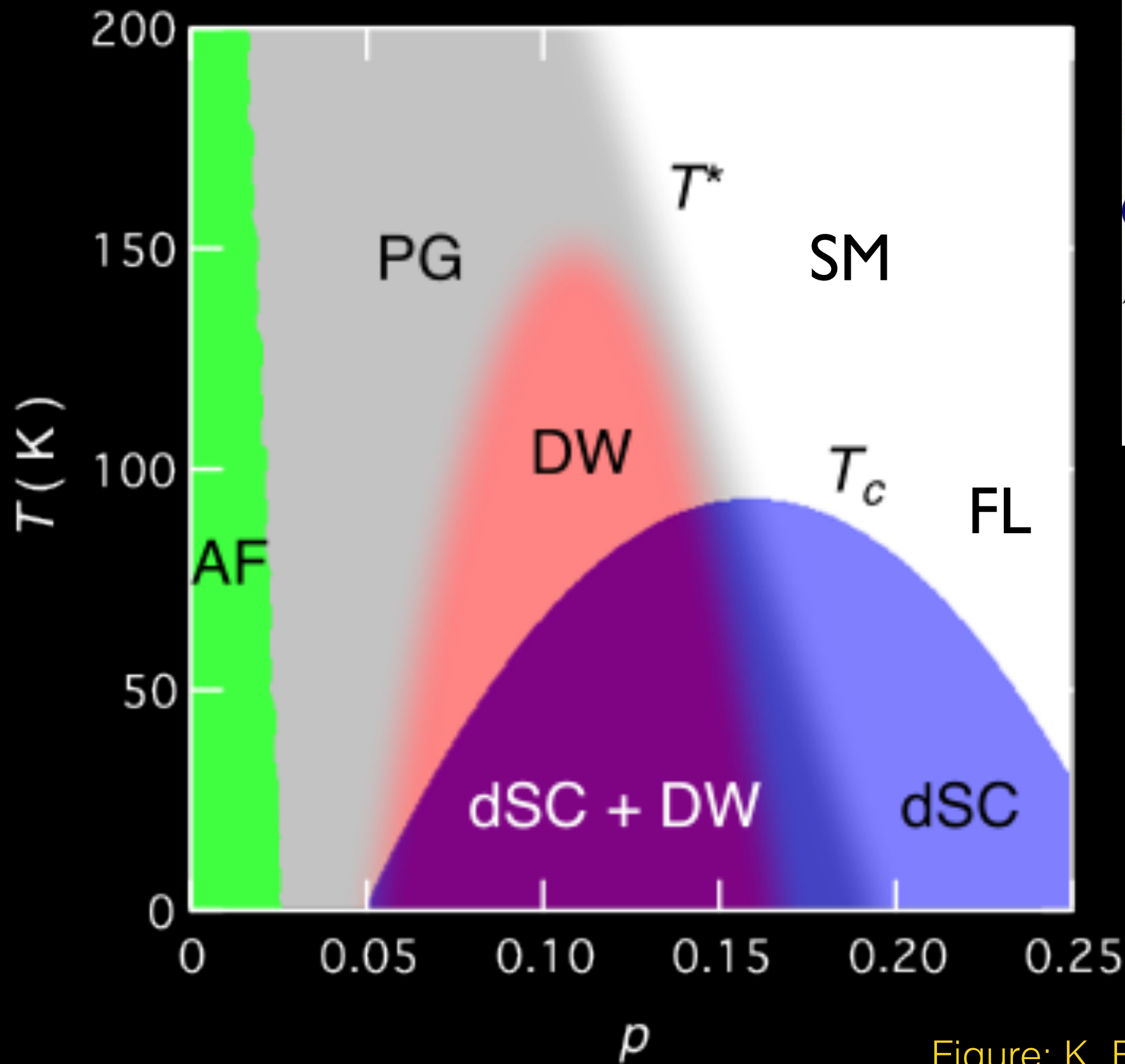


Figure: K. Fujita and J. C. Seamus Davis

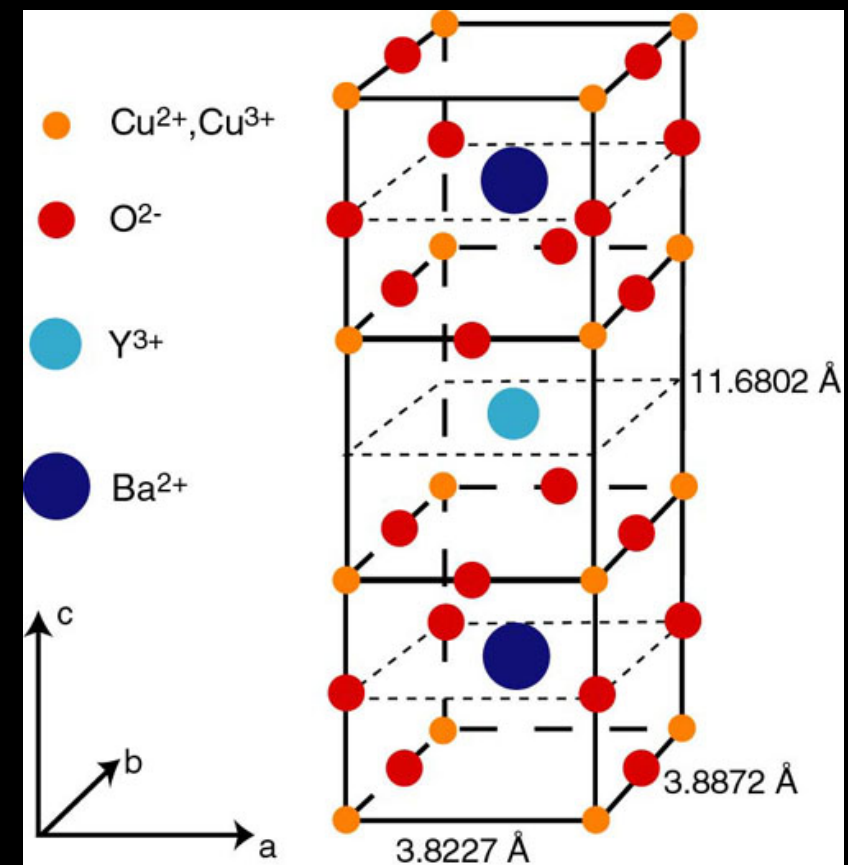
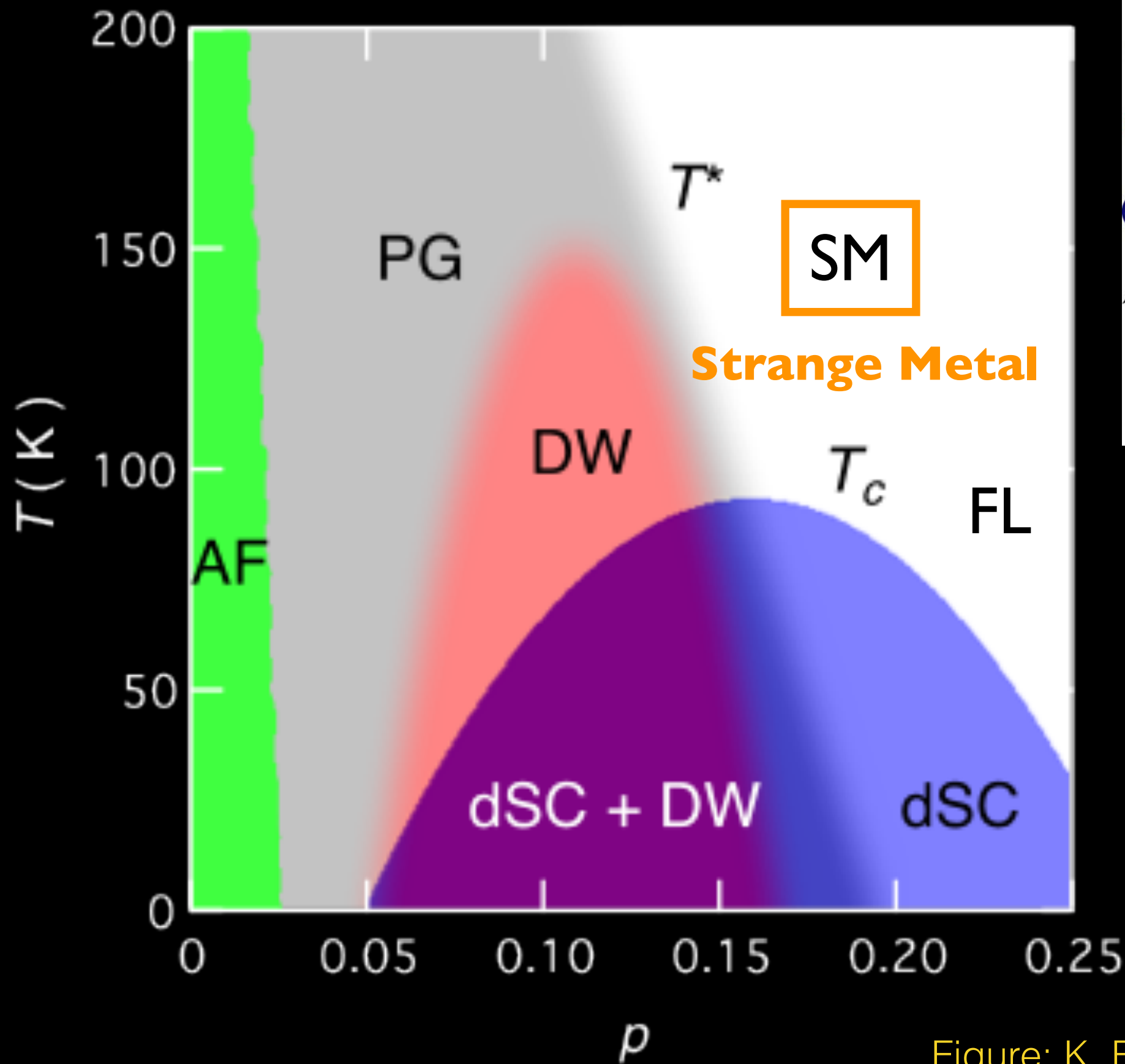
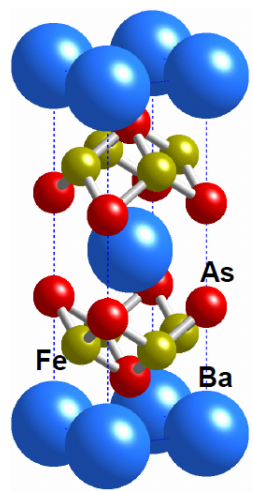
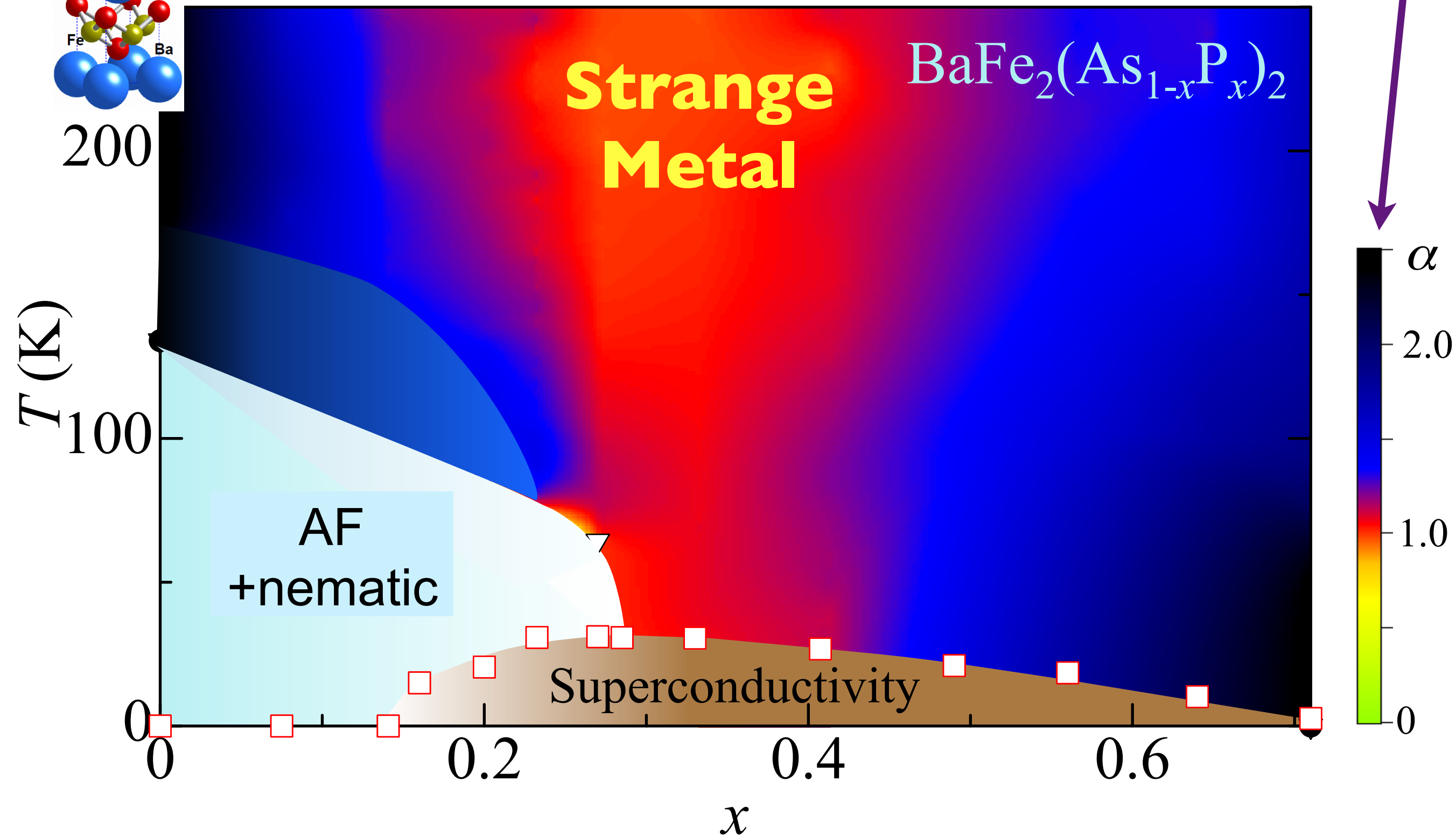


Figure: K. Fujita and J. C. Seamus Davis



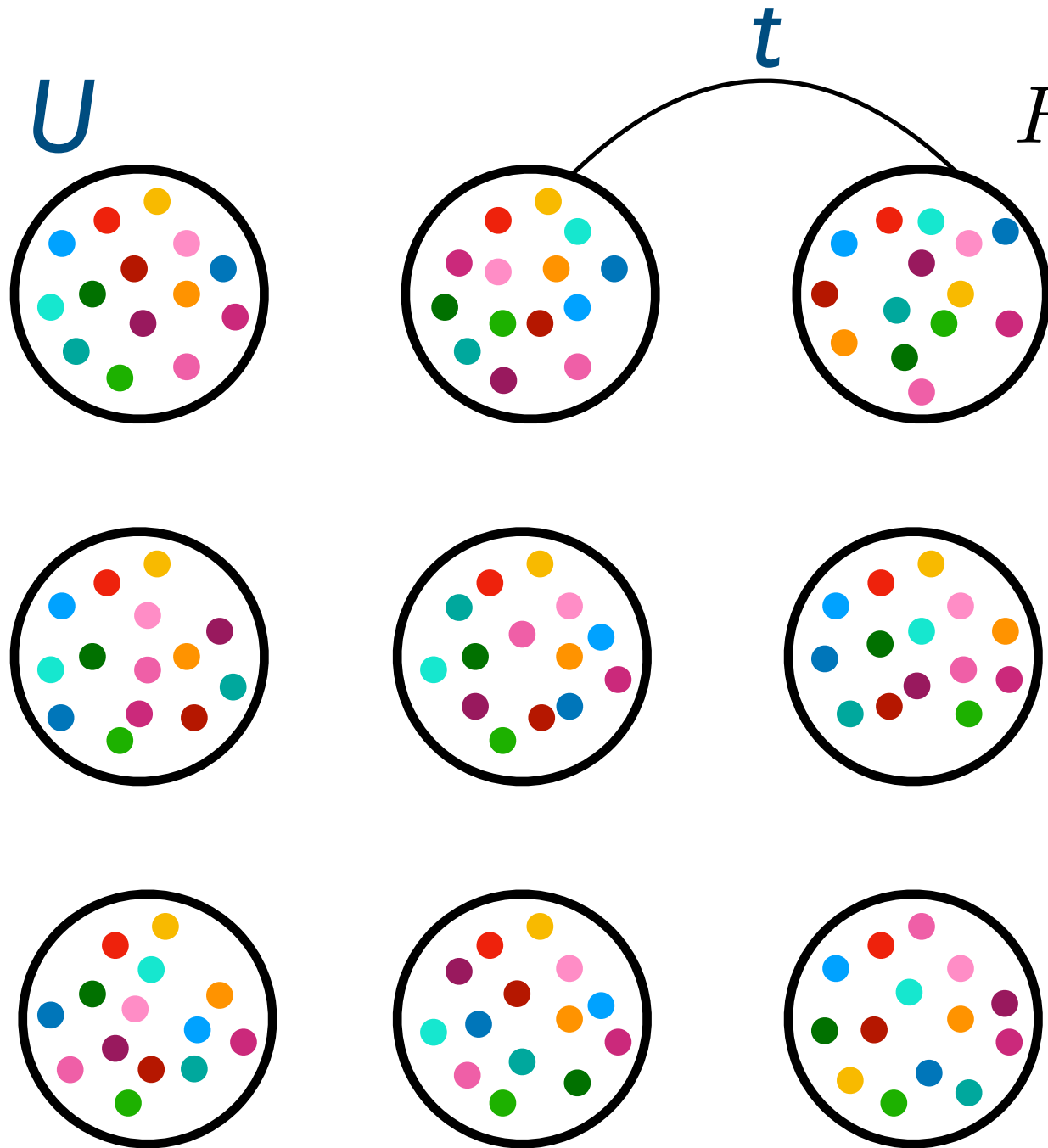
Resistivity
 $\sim \rho_0 + AT^\alpha$



S. Kasahara, T. Shibauchi, K. Hashimoto, K. Ikada, S. Tonegawa, R. Okazaki, H. Shishido, H. Ikeda, H. Takeya, K. Hirata, T. Terashima, and Y. Matsuda, *Physical Review B* **81**, 184519 (2010)

Coupled SYK Islands

SYK quantum islands of electrons with random hopping between them.



$$H = \sum_x \sum_{i < j, k < l} U_{ijkl,x} c_{ix}^\dagger c_{jx}^\dagger c_{kx} c_{lx} + \sum_{\langle xx' \rangle} \sum_{i,j} t_{ij,xx'} c_{i,x}^\dagger c_{j,x'}$$

$$\overline{|U_{ijkl}|^2} = \frac{2U^2}{N^3}$$

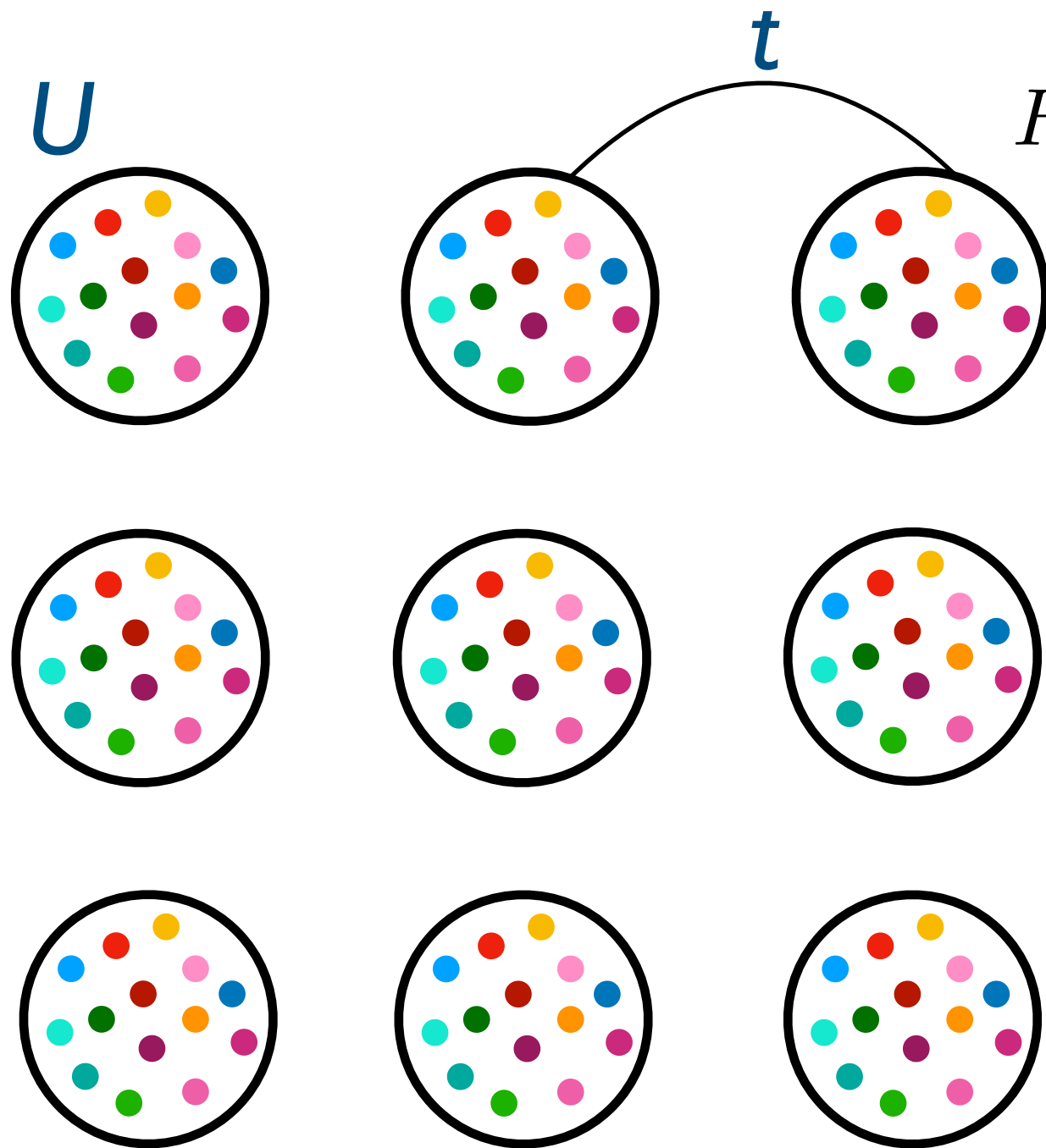
$$\overline{|t_{ij,xx'}|^2} = t_0^2/N$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Can also use non-random t , and the same U on all “islands”.



$$H = \sum_x \sum_{i < j, k < l} U_{ijkl} c_{ix}^\dagger c_{jx}^\dagger c_{kx} c_{lx} + \sum_{\langle xx' \rangle} \sum_{i, j} t_{ij} c_{i,x}^\dagger c_{j,x'}$$

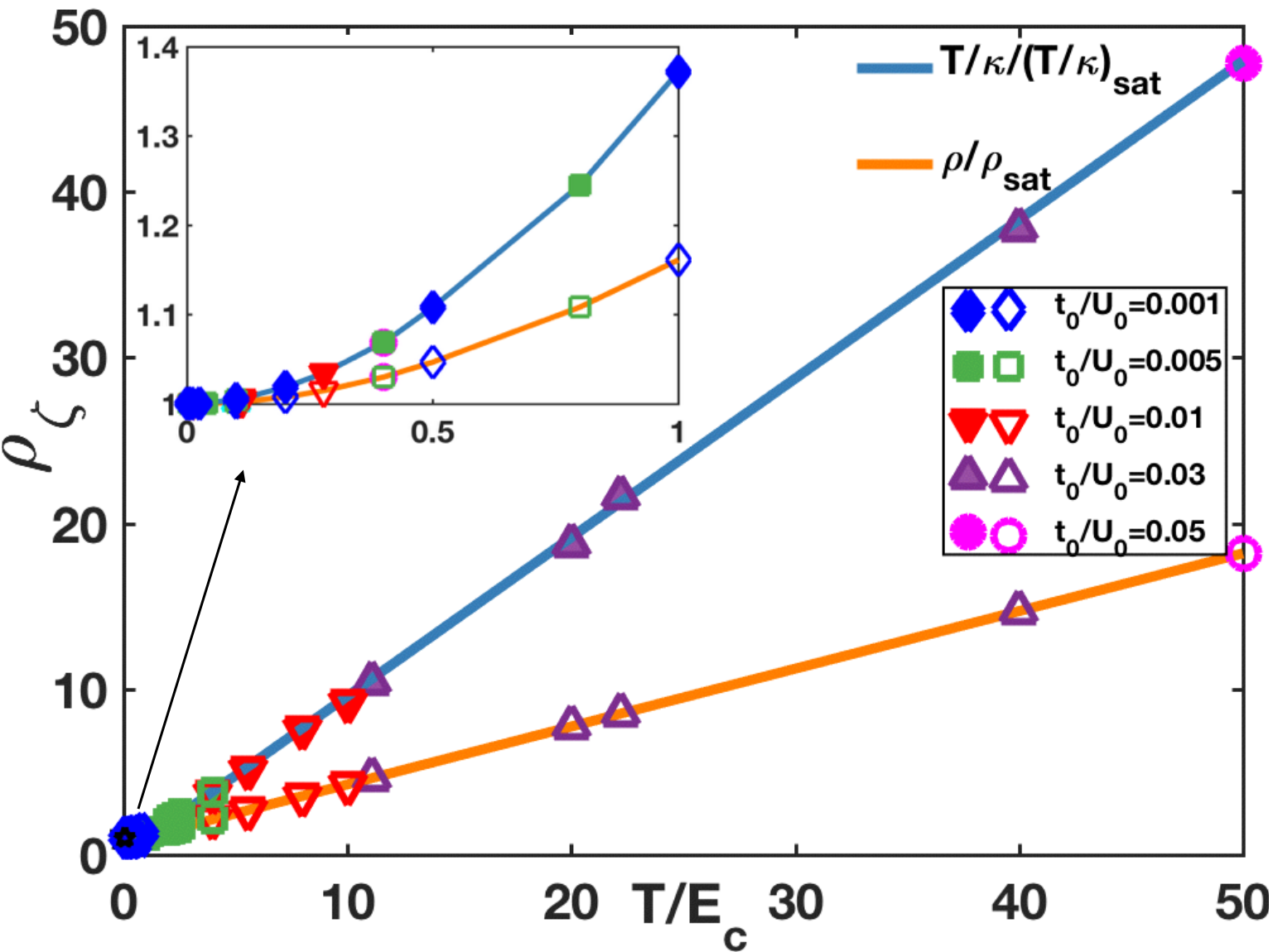
Pengfei Zhang, PRB **96**, 205138 (2017)

Debanjan Chowdhury, Yochai Werman, Erez Berg, T. Senthil, arXiv:1801.06178

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Low 'coherence' scale



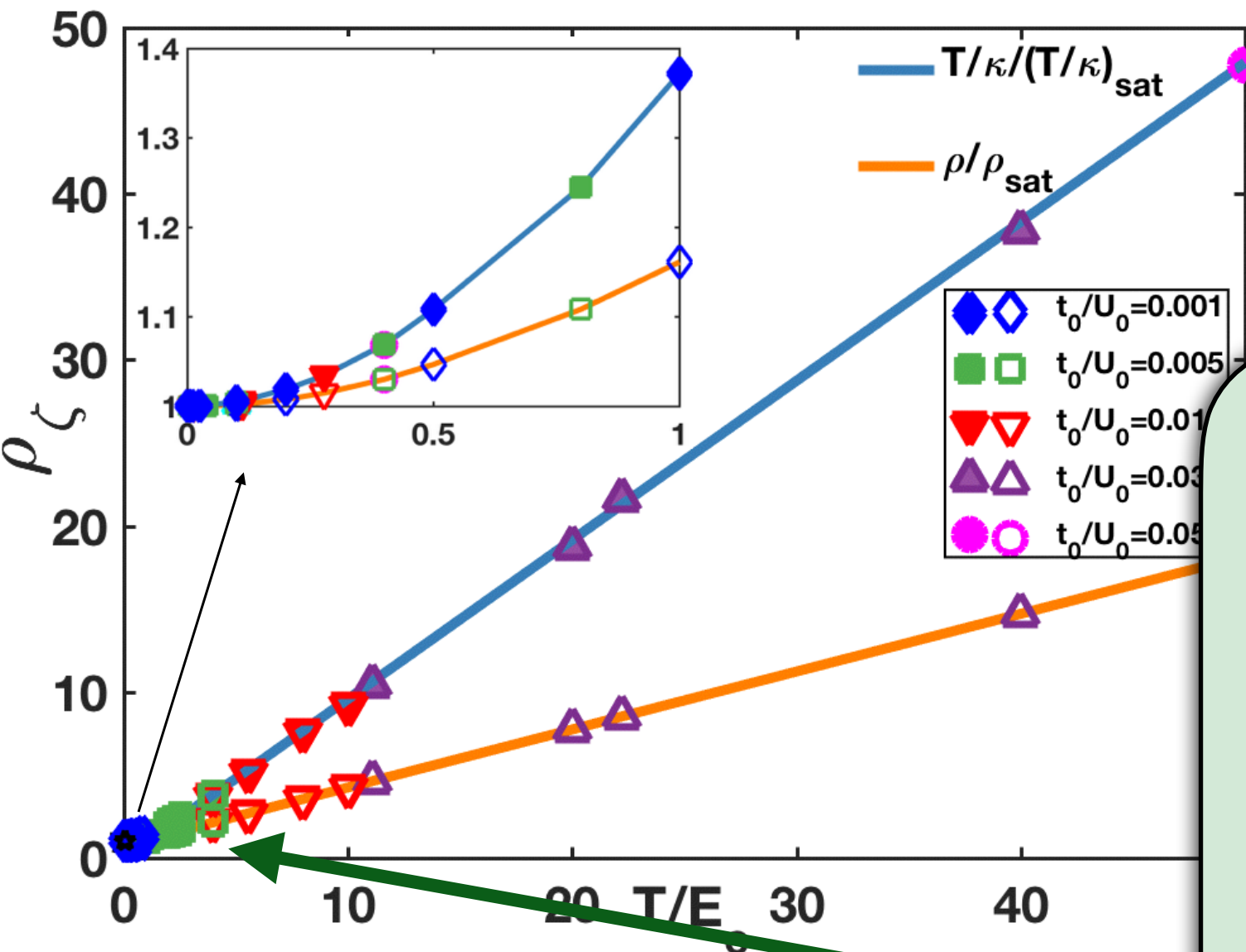
$$E_c \sim \frac{t_0^2}{U}$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Low 'coherence' scale



$$E_c \sim \frac{t_0^2}{U}$$

For $T < E_c$, the resistivity, ρ , and entropy density, s , are

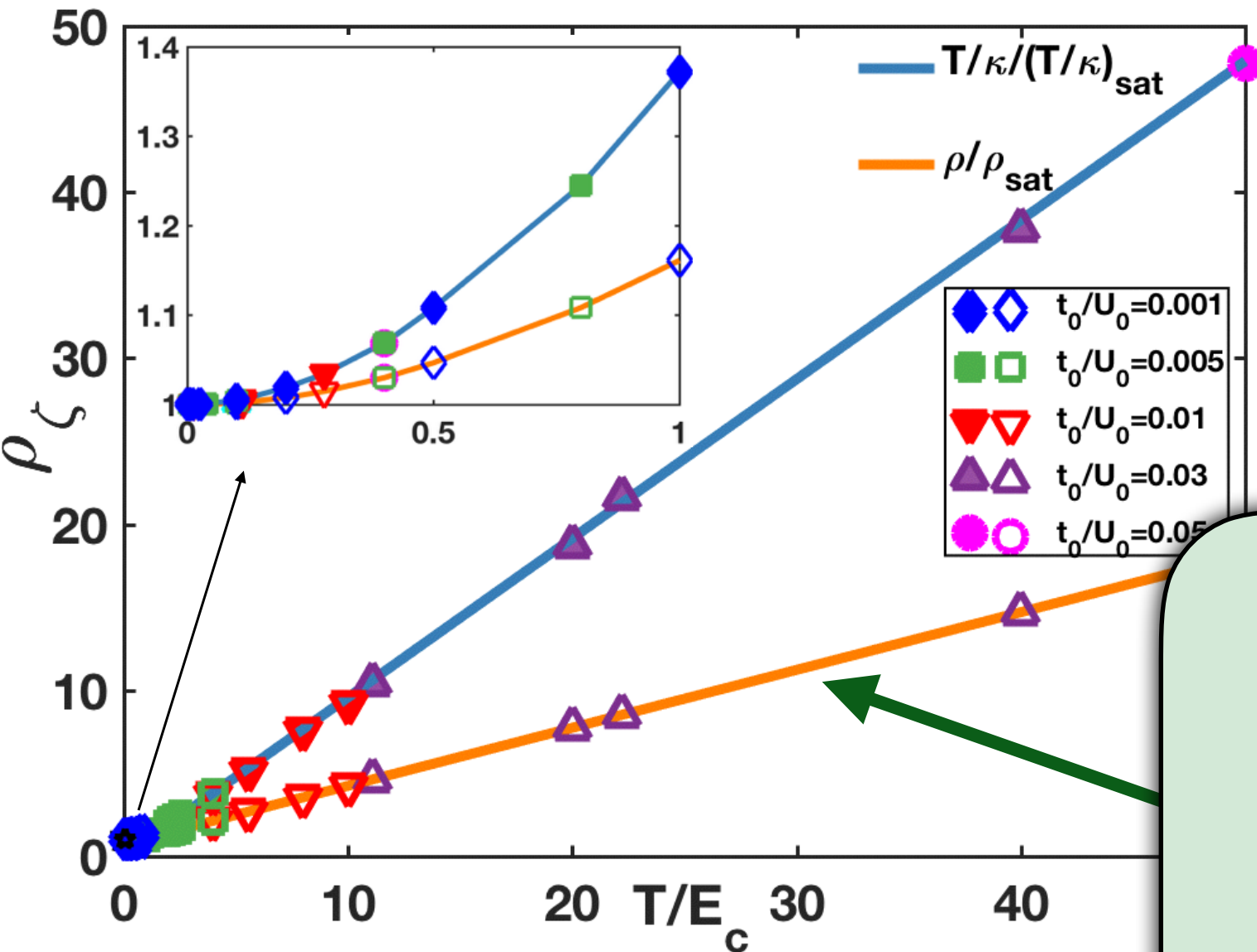
$$\rho = \frac{h}{e^2} \left[c_1 + c_2 \left(\frac{T}{E_c} \right)^2 \right]$$

$$s \sim s_0 \left(\frac{T}{E_c} \right)$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)
 See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Coupled SYK Islands

Low 'coherence' scale



$$E_c \sim \frac{t_0^2}{U}$$

For $E_c < T < U$, the resistivity, ρ , and entropy density, s , are

$$\rho \sim \frac{h}{e^2} \left(\frac{T}{E_c} \right), \quad s = s_0$$

Xue-Yang Song, Chao-Ming Jian, and L. Balents, PRL **119**, 216601 (2017)

See also A. Georges and O. Parcollet PRB **59**, 5341 (1999)

Quantum matter without quasiparticles

- Rapid local thermal equilibration (of fermion correlators) in a ‘Planckian’ time

$$\tau_{\text{eq}} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \rightarrow 0.$$

- Presence of quasiparticles should slow down thermalization, so *all* quantum systems obey

$$\tau_{\text{eq}} > C \frac{\hbar}{k_B T} \quad , \quad \text{as } T \rightarrow 0.$$

S. Sachdev, *Quantum Phase Transitions*,
Cambridge (1999)

Absence of quasiparticles \Leftrightarrow Fastest possible thermalization

Quantum matter without quasiparticles

- Planckian dynamics is realized in the ‘solvable’ SYK models
- Black holes thermalize in a time $\sim \hbar/(k_B T_H)$, where T_H is the Hawking temperature.
- A Schwarzian theory of a time reparameterization mode, with $SL(2, \mathbb{R})$ symmetry, describes the quantum dynamics of
 - the SYK models
 - black holes with near-extremal AdS_2 horizons