De Sitter Lecture Series in Theoretical Physics 2009
University of Groningen

Quantum phase transitions: from antiferromagnets and superconductors to black holes

Talk online: sachdev.physics.harvard.edu

PHYSICS
VE RI TAS
HARVARD

Outline

1. Introduction to quantum phase transitions:
   quantum spin systems and relativistic field theories

2. Quantum phase transitions in *d*-wave
   superconductors and metals

3. The AdS/CFT correspondence:
   quantum criticality at strong coupling

4. The cuprate high temperature superconductors:
   competing orders and quantum criticality

Outline

1. Introduction to quantum phase transitions:
   quantum spin systems and relativistic field theories

2. Quantum phase transitions in *d*-wave
   superconductors and metals

3. The AdS/CFT correspondence:
   quantum criticality at strong coupling

4. The cuprate high temperature superconductors:
   competing orders and quantum criticality

Yejin Huh, Harvard

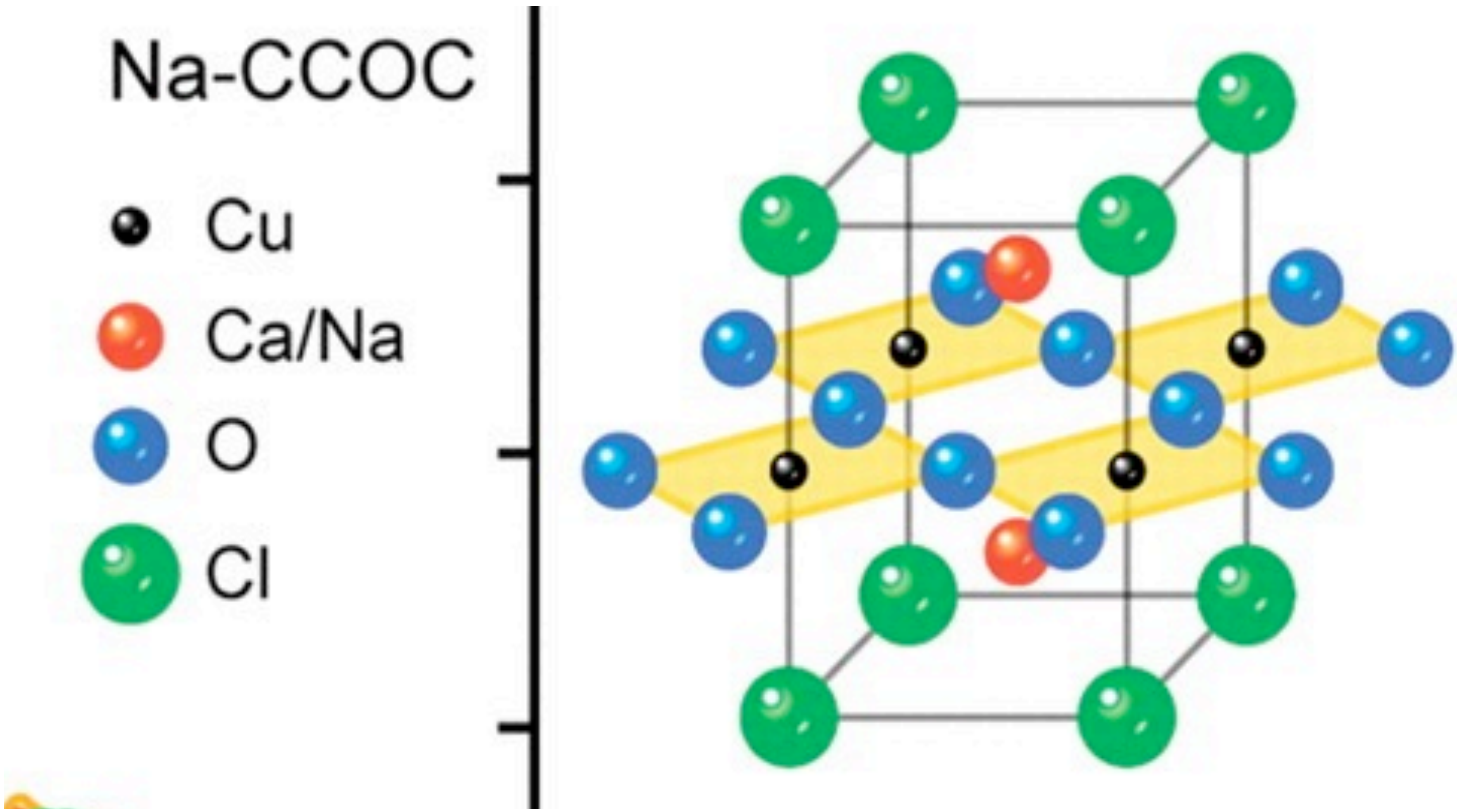
Max Metlitski, Harvard

PHYSICS

VE RI TAS

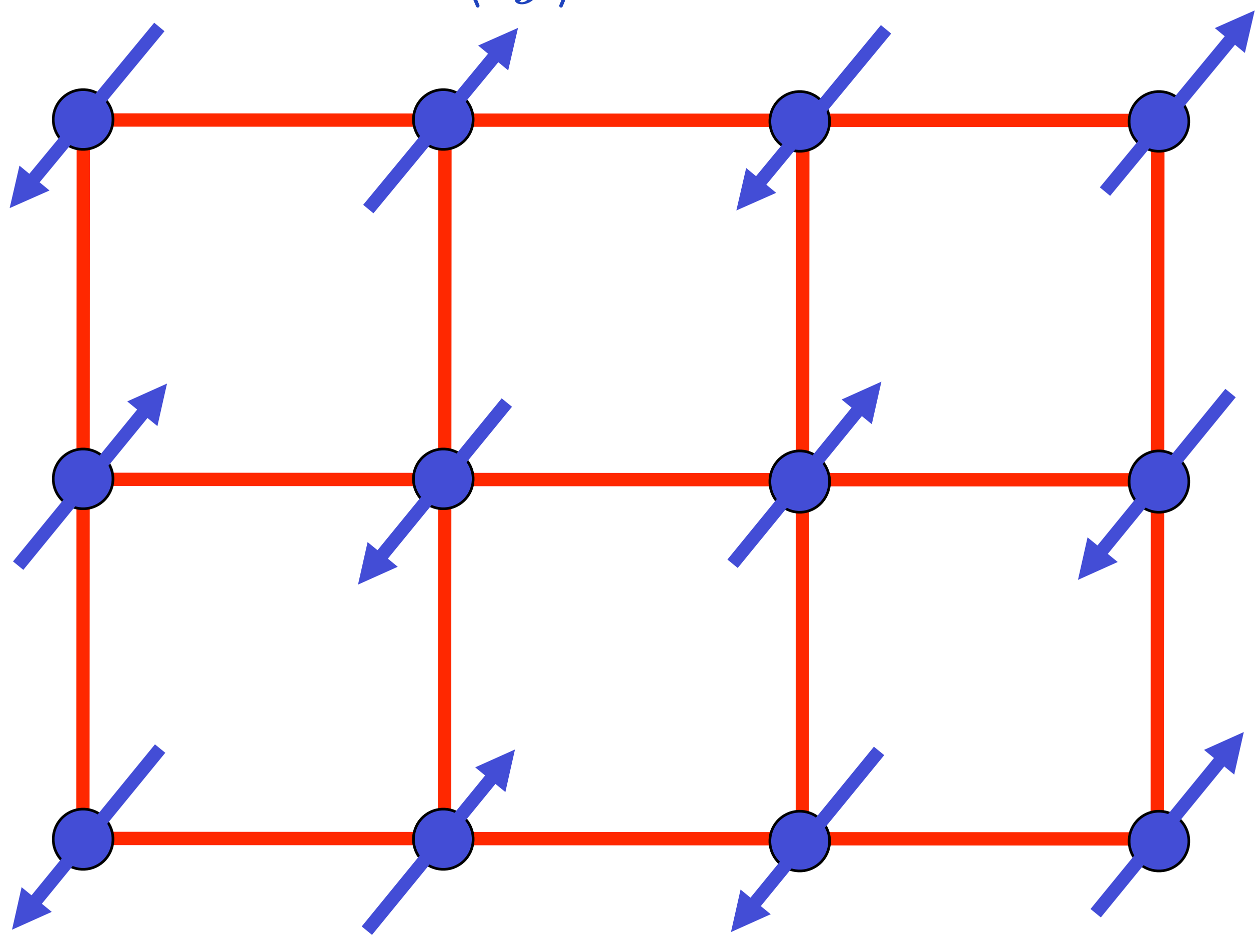
HARVARD

The cuprate superconductors



Square lattice antiferromagnet

$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$



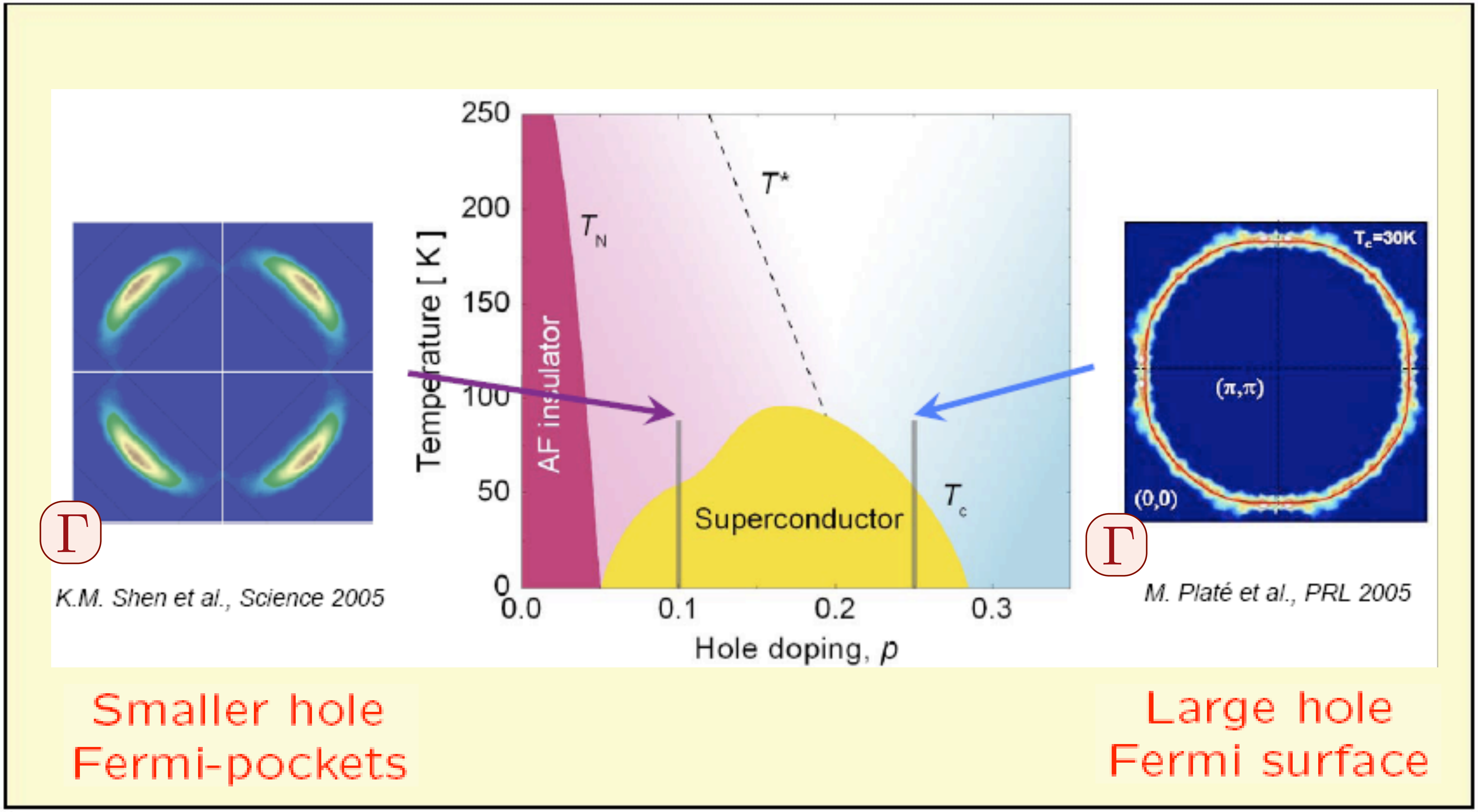
Ground state has long-range Néel order

Order parameter is a single vector field $\vec{\varphi} = \eta_i \vec{S}_i$

$\eta_i = \pm 1$ on two sublattices

$\langle \vec{\varphi} \rangle \neq 0$ in Néel state.

Central ingredients in cuprate phase diagram: antiferromagnetism, superconductivity, and change in Fermi surface



Antiferro-
magnetism

d-wave
supercon-
ductivity

Fermi
surface

Outline

2. Quantum phase transitions in *d*-wave superconductors and metals

A. d-wave superconductivity

B. Discrete symmetry breaking in a d-wave superconductor: reflection (nematic ordering) or time-reversal

C. Nematic ordering in a metal

Outline

2. Quantum phase transitions in *d*-wave superconductors and metals

A. d-wave superconductivity

B. Discrete symmetry breaking in a d-wave superconductor: reflection (nematic ordering) or time-reversal

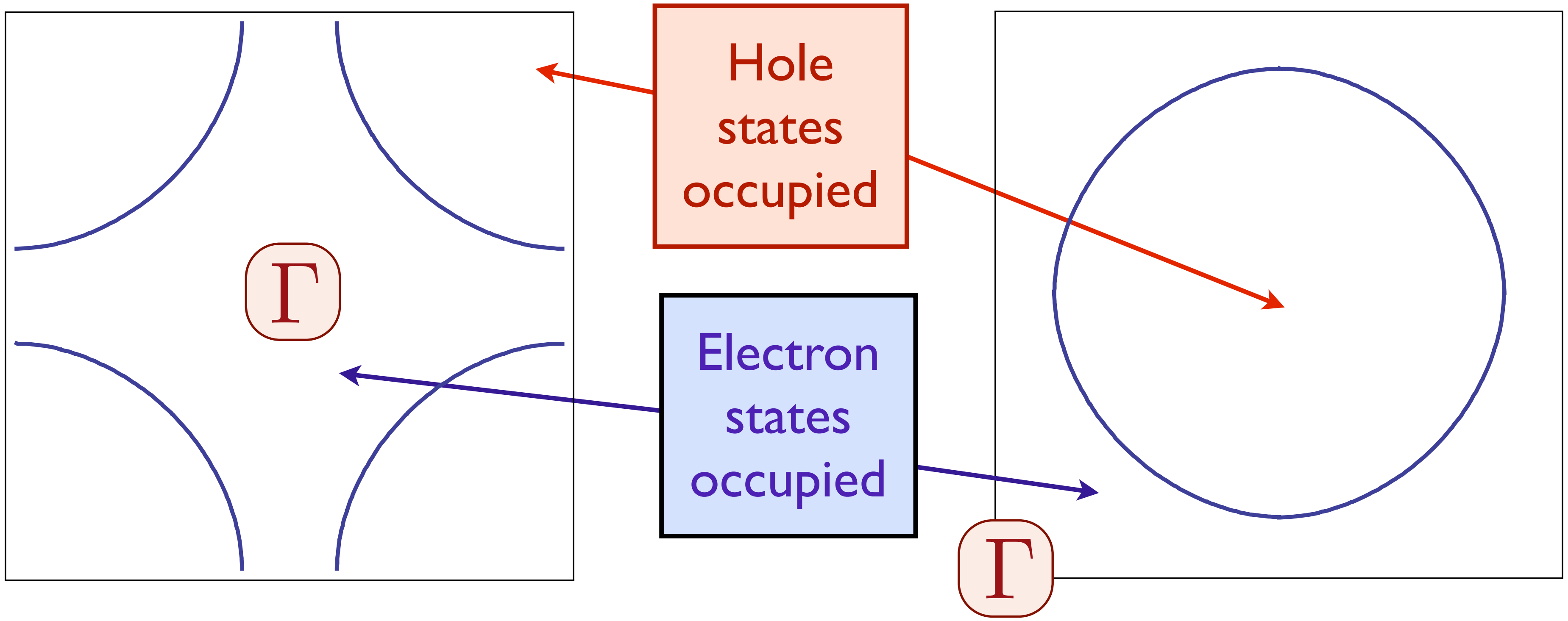
C. Nematic ordering in a metal

Antiferro-
magnetism

d-wave
supercon-
ductivity

Fermi
surface

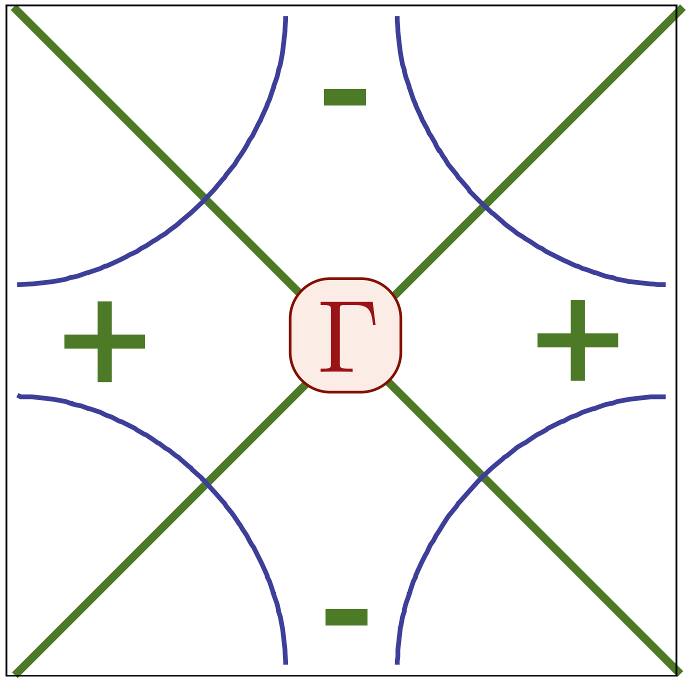
d-wave superconductivity in cuprates

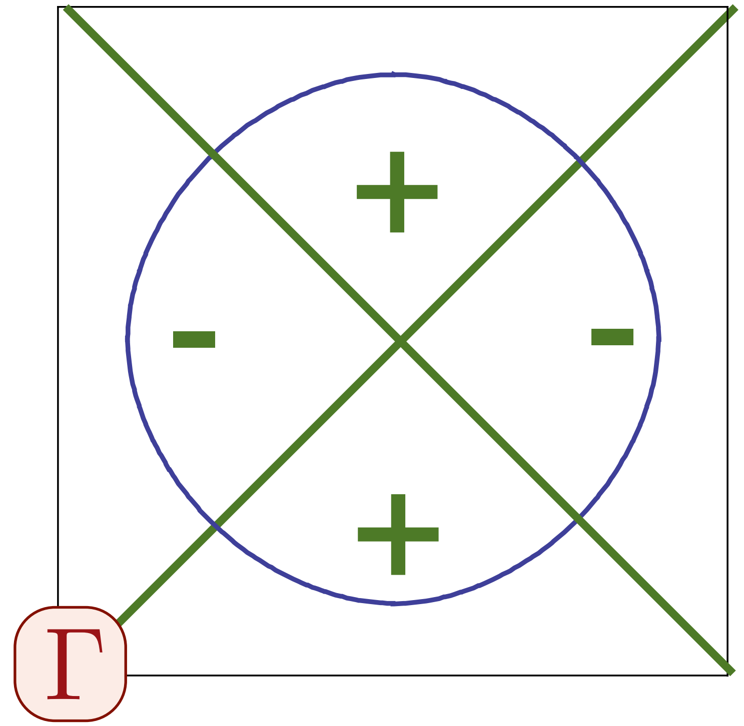


$$H_0 = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \equiv \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha}$$

- Begin with free electrons.

d-wave superconductivity in cuprates

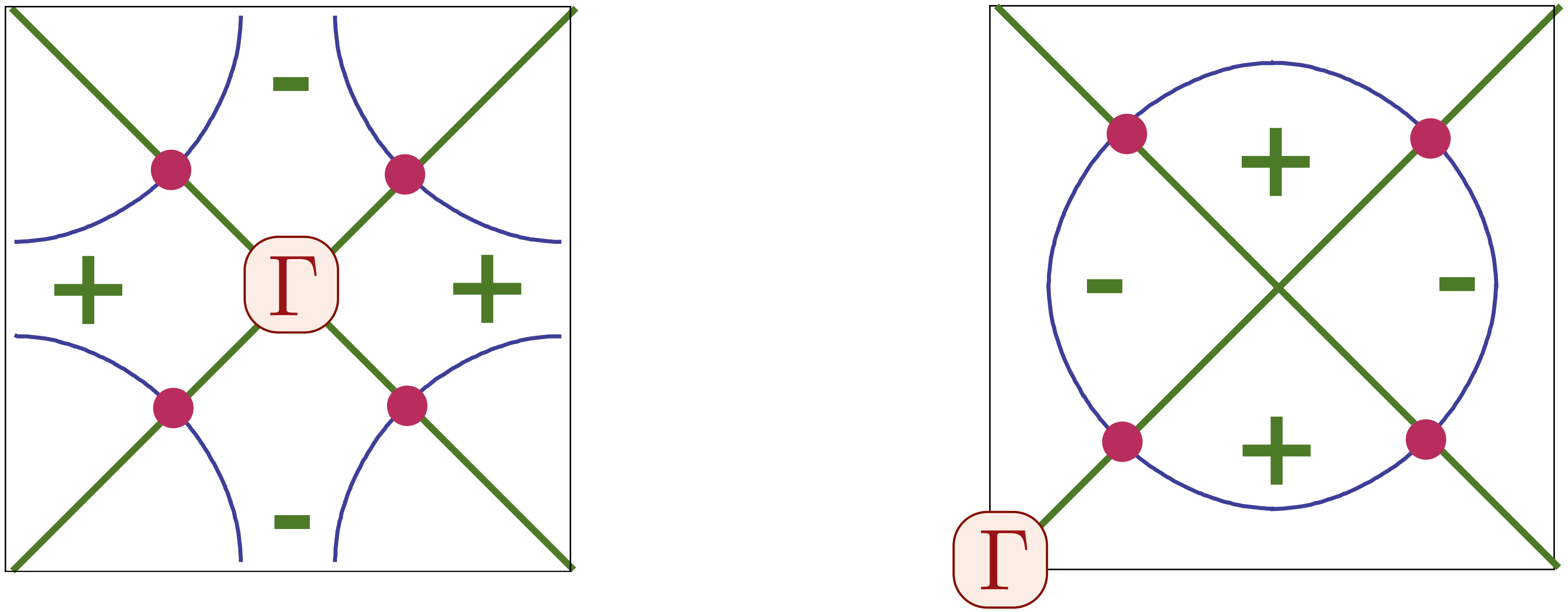




$$H = \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \text{c.c.} \right)$$

- Begin with free electrons.
- Add d-wave pairing interaction $\Delta_k \sim \cos k_x - \cos k_y$ which vanishes along diagonals

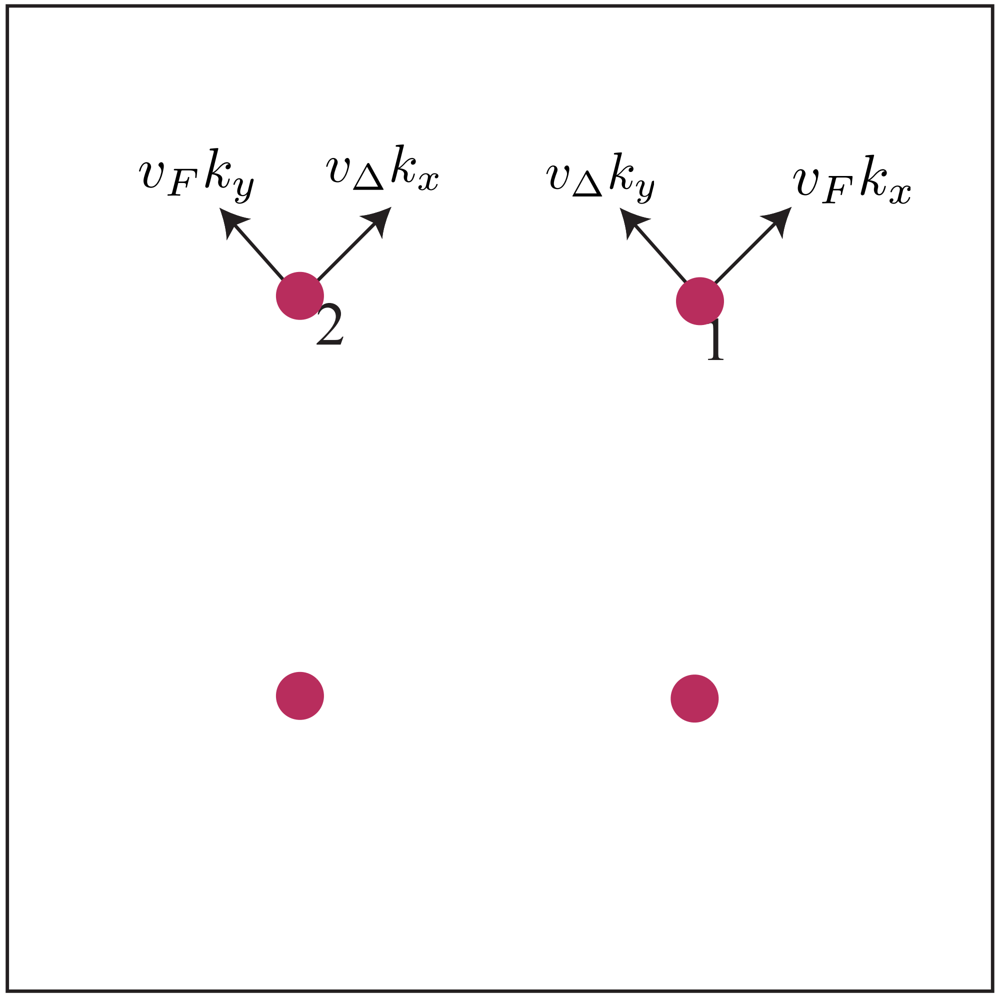
d-wave superconductivity in cuprates



$$H = \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + \text{c.c.} \right)$$

- Begin with free electrons.
- Add d-wave pairing interaction Δ_k which vanishes along diagonals
- Obtain Bogoliubov quasiparticles with dispersion $\sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$

d-wave superconductivity in cuprates



4 two-component Dirac fermions

$$S_\Psi = \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi^\dagger_{1a} \left(-i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x\right) \Psi_{1a}$$

$$+ \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi^\dagger_{2a} \left(-i\omega_n + v_F k_y \tau^z + v_\Delta k_x \tau^x\right) \Psi_{2a}.$$

Outline

2. Quantum phase transitions in *d*-wave superconductors and metals

A. d-wave superconductivity

B. Discrete symmetry breaking in a d-wave superconductor: reflection (nematic ordering) or time-reversal

C. Nematic ordering in a metal

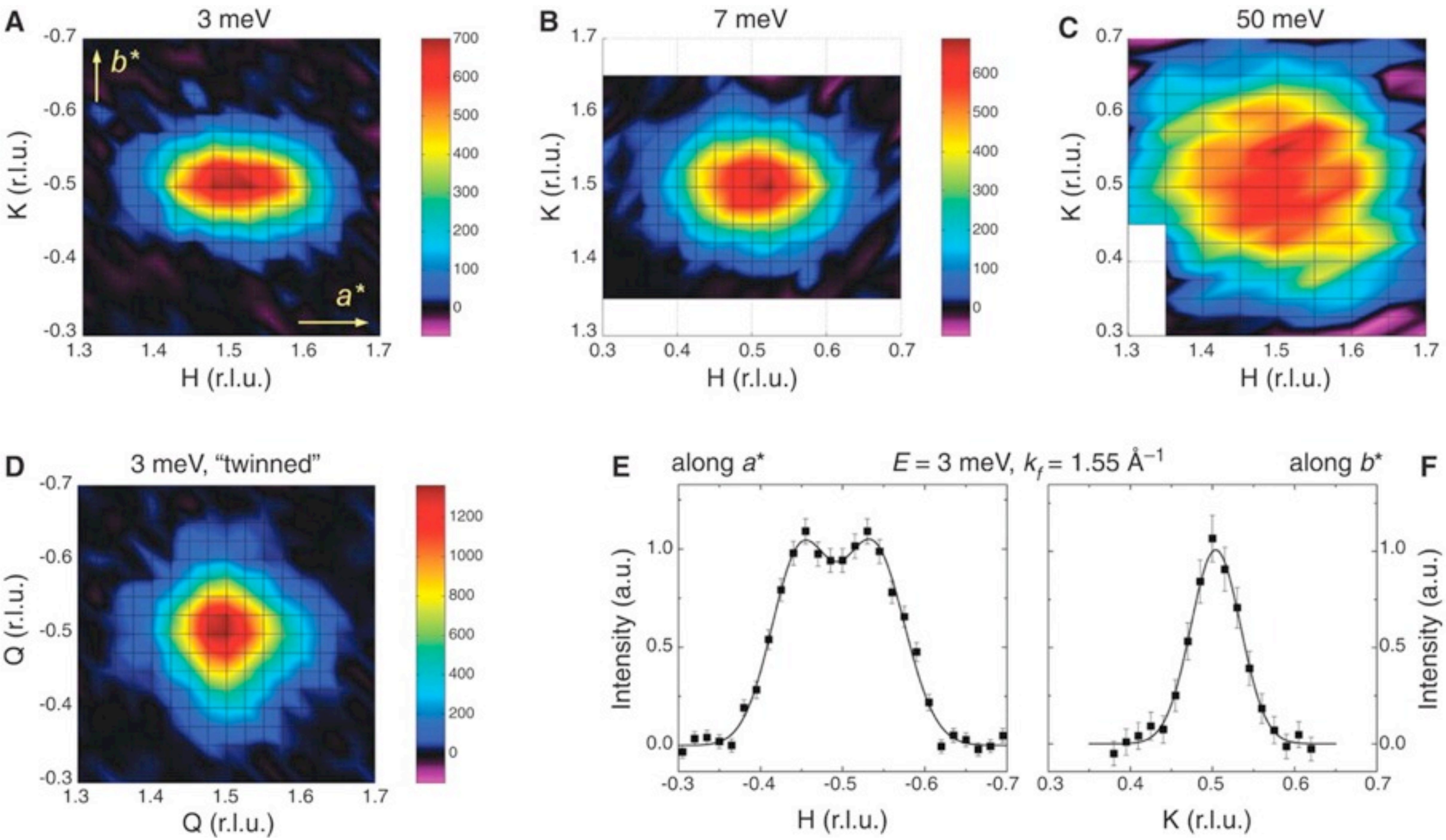
Outline

2. Quantum phase transitions in *d*-wave superconductors and metals

A. d-wave superconductivity

B. Discrete symmetry breaking in a d-wave superconductor: reflection (nematic ordering) or time-reversal

C. Nematic ordering in a metal

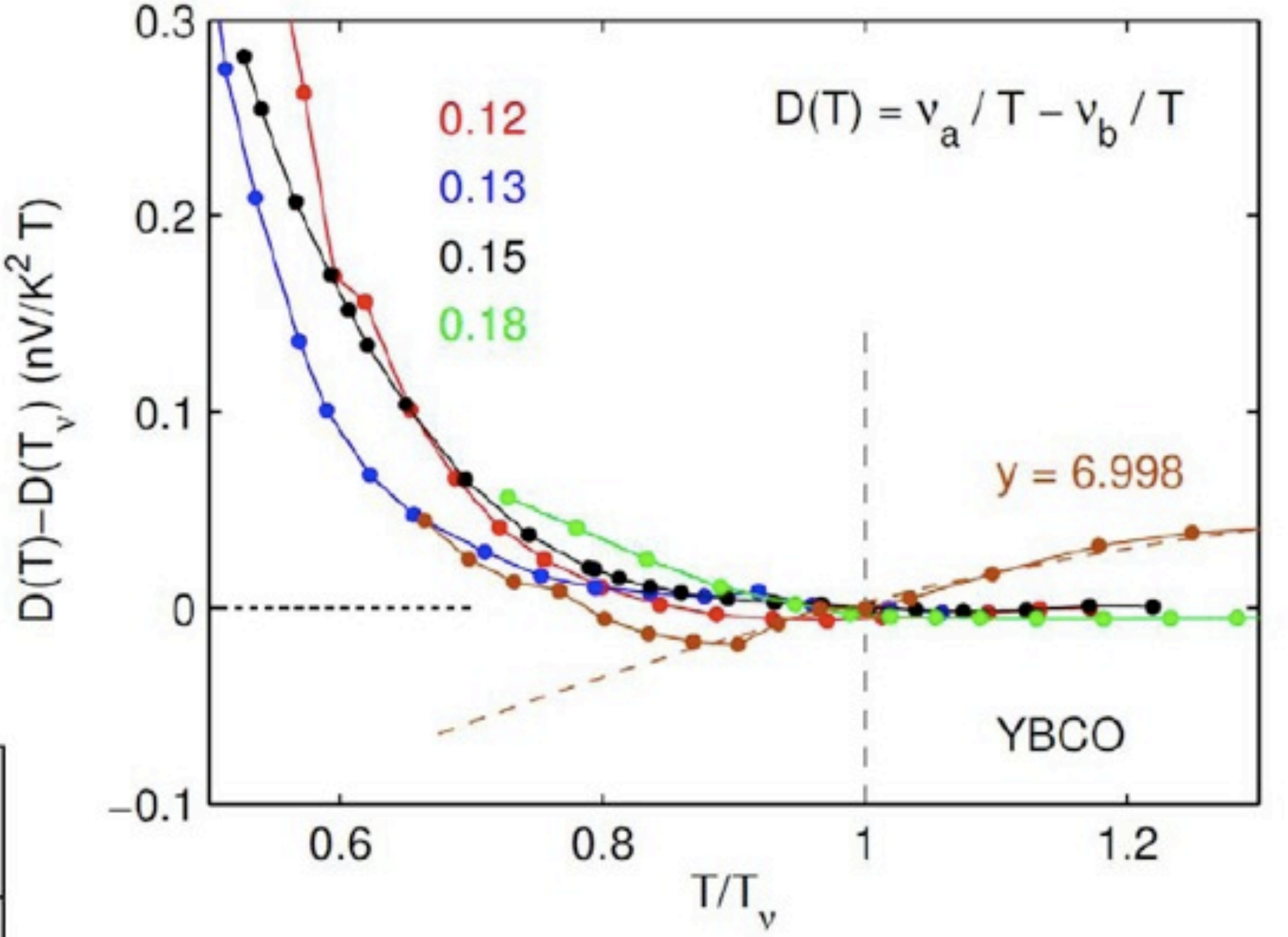


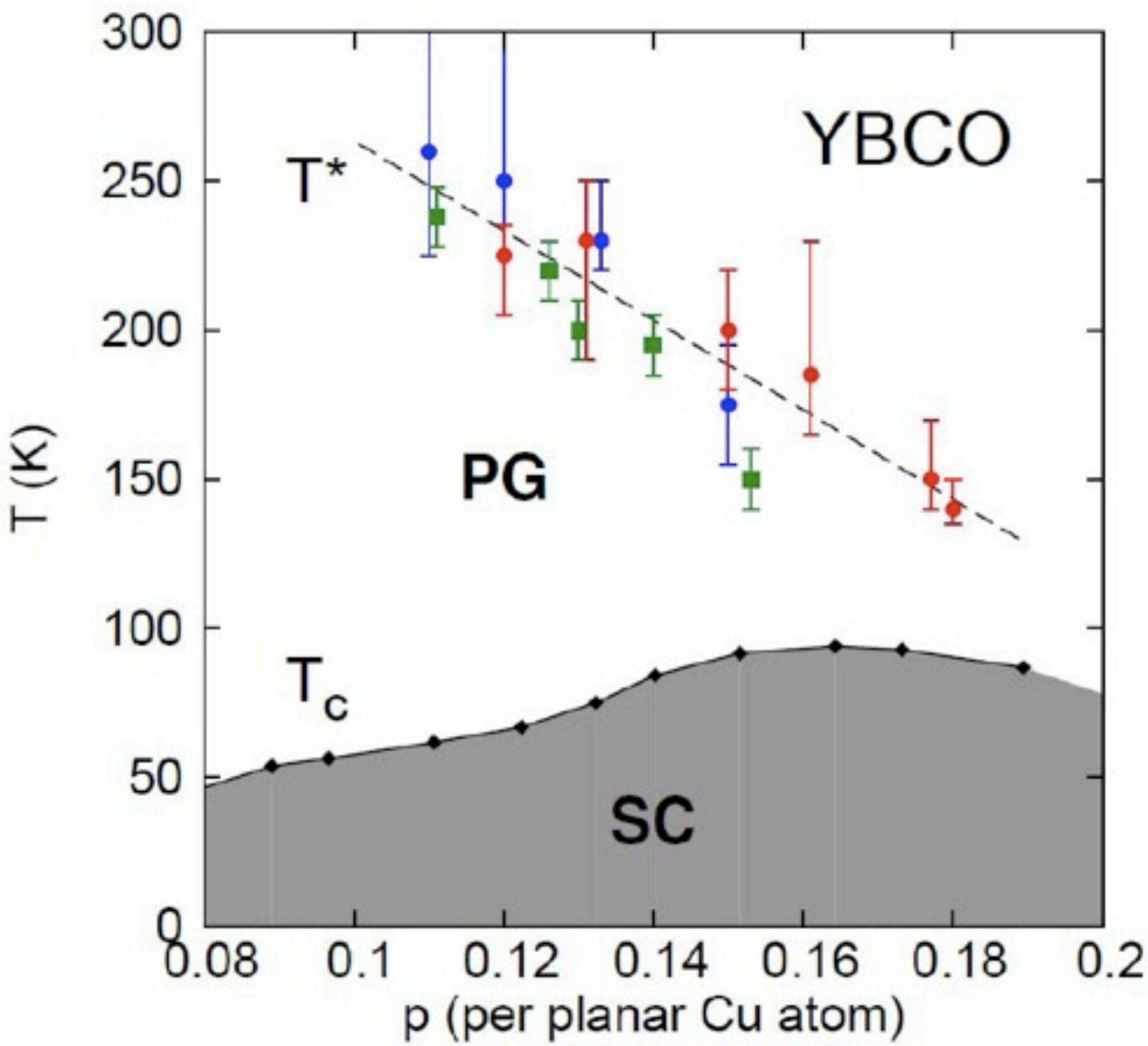
Nematic order in YBCO

V. Hinkov, D. Haug, B. Fauqué, P. Bourges, Y. Sidis, A. Ivanov, C. Bernhard, C. T. Lin, and B. Keimer , *Science* **319**, 597 (2008)

Broken rotational symmetry in the pseudogap phase of a high-Tc superconductor

R. Daou, J. Chang, David LeBoeuf, Olivier Cyr-Choiniere, Francis Laliberte, Nicolas Doiron-Leyraud, B. J. Ramshaw, Ruixing Liang, D. A. Bonn, W. N. Hardy, and Louis Taillefer
arXiv: 0909.4430





S. A. Kivelson, E. Fradkin, and V. J. Emery, *Nature* **393**, 550 (1998).

d-wave superconductivity in cuprates

Now consider a discrete spontaneous symmetry breaking, with Ising symmetry, described by a real scalar field ϕ.
Two cases of experimental interest are:

- Break 4-fold lattice rotation symmetry to 2-fold lattice rotations: leads to a superconductor with **nematic** order: equivalent to $d_{x^2-y^2} + s$ pairing.

$$H = H_\phi + \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \text{c.c.} \right)$$

$$H_\phi = \phi \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \text{c.c.}$$

d-wave superconductivity in cuprates

Now consider a discrete spontaneous symmetry breaking, with Ising symmetry, described by a real scalar field ϕ.
Two cases of experimental interest are:

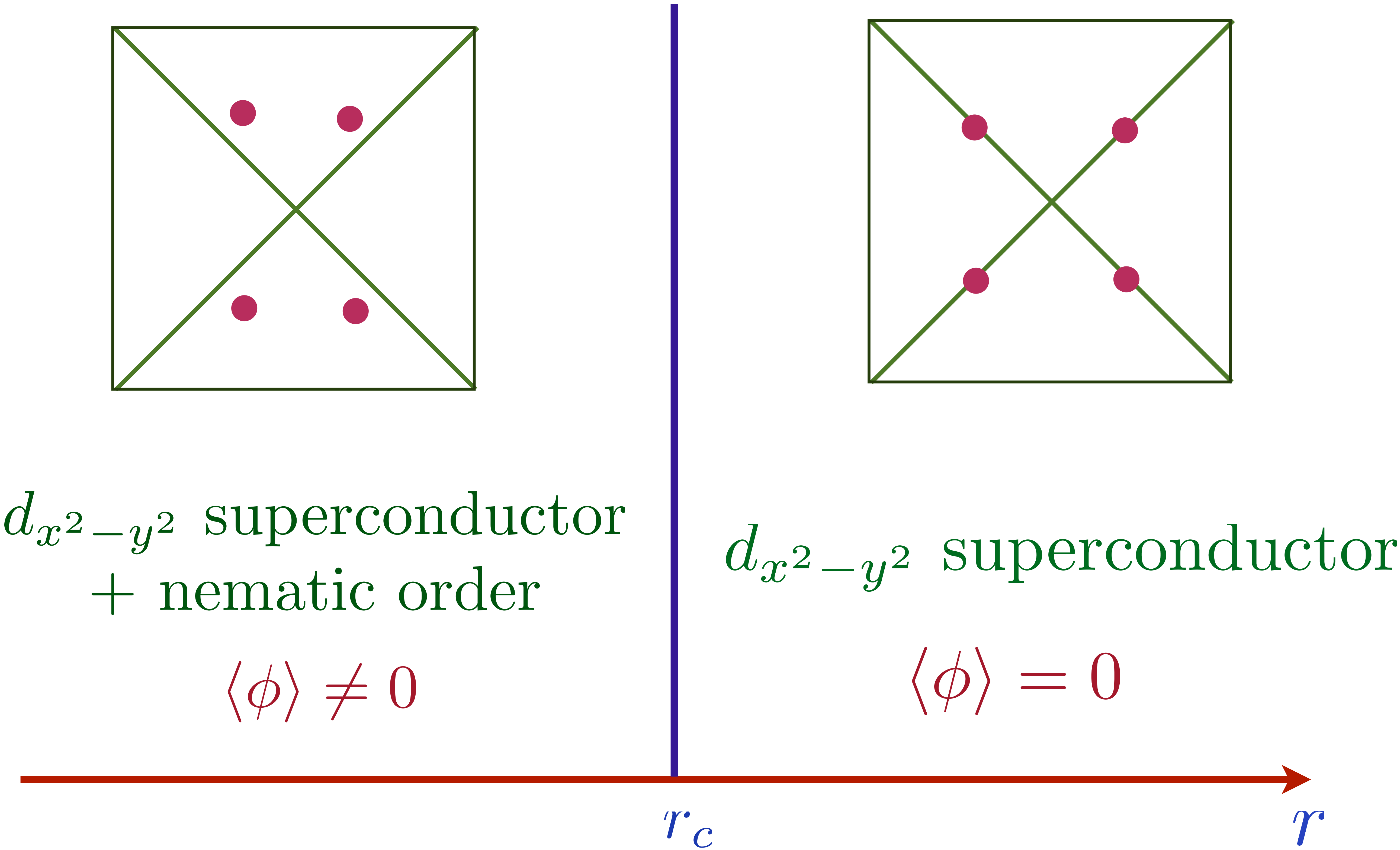
- Break 4-fold lattice rotation symmetry to 2-fold lattice rotations: leads to a superconductor with **nematic** order: equivalent to $d_{x^2-y^2} + s$ pairing.

- Time-reversal symmetry breaking: leads to a $d_{x^2-y^2} + id_{xy}$ superconductor, in which the Dirac fermions are massive

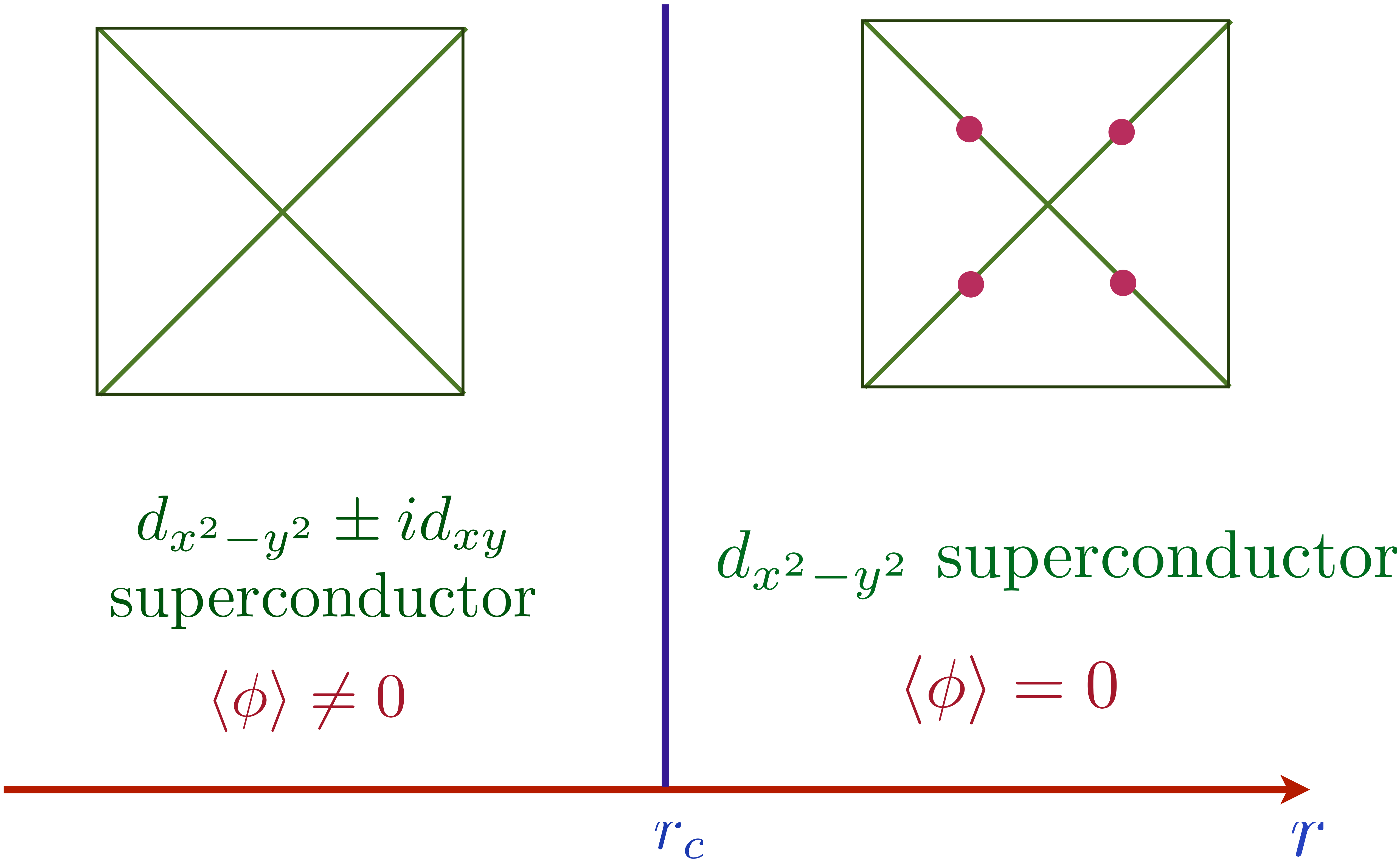
$$H = H_\phi + \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \text{c.c.} \right)$$

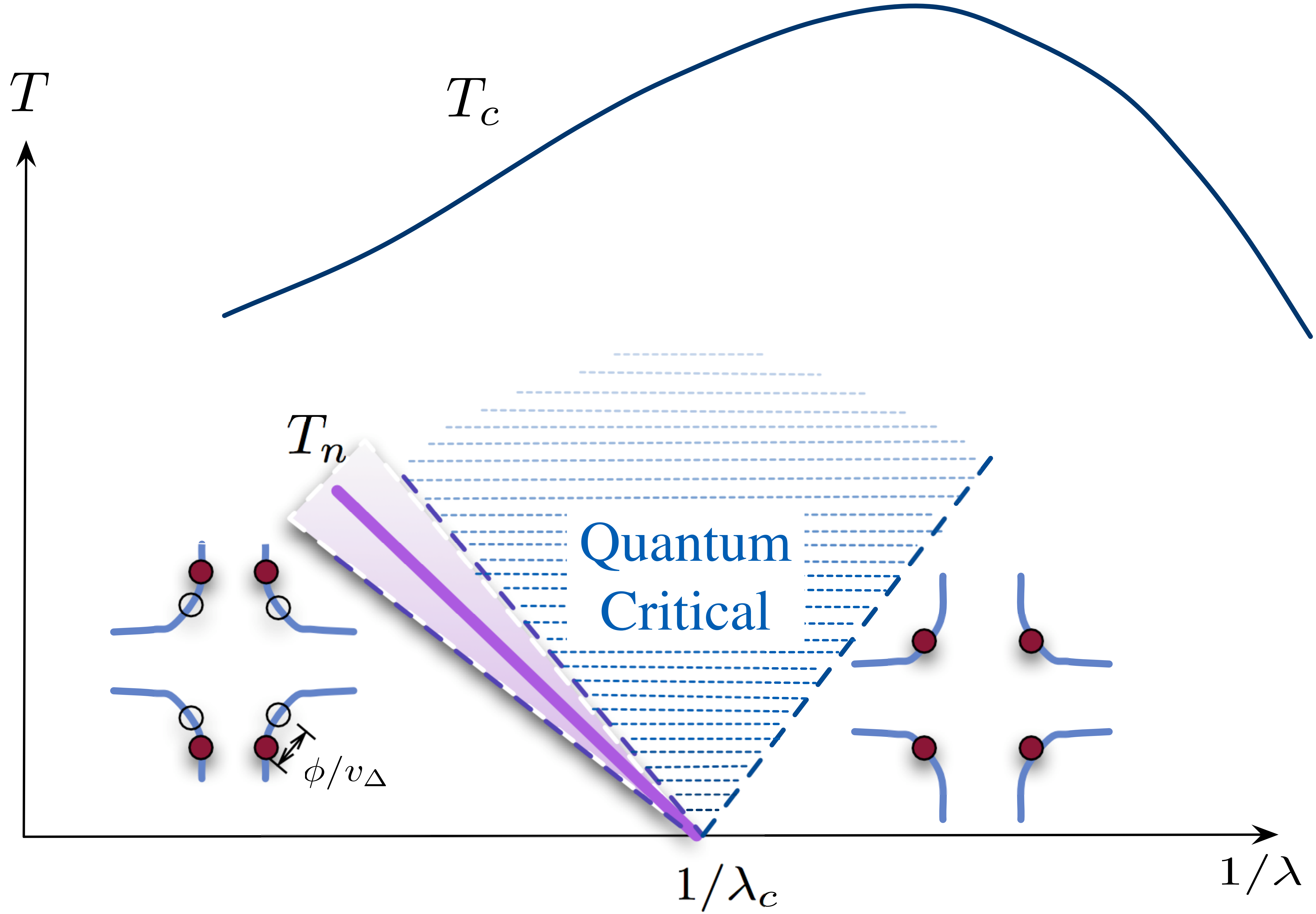
$$H_\phi = i\phi \sum_{\mathbf{k}} \sin k_x \sin k_y c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \text{c.c.}$$

Lattice rotation symmetry breaking



Time-reversal symmetry breaking





M. Vojta, Y. Zhang, and S. Sachdev, Phys. Rev. Lett. **85**, 4940 (2000)
E.-A. Kim, M. J. Lawler, P. Oreto, S. Sachdev, E. Fradkin, S.A. Kivelson, Phys. Rev. B **77**, 184514 (2008).

Discrete symmetry breaking in d-wave superconductors

Field theory for transition with Ising order described by a real scalar field ϕ:

$$\mathcal{S} = \mathcal{S}_\Psi + \mathcal{S}_\phi + \mathcal{S}_{\Psi\phi}$$

4 two-component Dirac fermions

$$\begin{aligned}\mathcal{S}_\Psi &= \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi^\dagger_{1a} \left(-i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x\right) \Psi_{1a} \\ &+ \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi^\dagger_{2a} \left(-i\omega_n + v_F k_y \tau^z + v_\Delta k_x \tau^x\right) \Psi_{2a}.\end{aligned}$$

Discrete symmetry breaking in d-wave superconductors

Field theory for transition with Ising order described by a real scalar field ϕ:

$$\mathcal{S} = \mathcal{S}_\Psi + \mathcal{S}_\phi + \mathcal{S}_{\Psi\phi}$$

4 two-component Dirac fermions

$$\begin{aligned}\mathcal{S}_\Psi &= \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{1a}^\dagger \left(-i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x\right) \Psi_{1a} \\ &+ \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{2a}^\dagger \left(-i\omega_n + v_F k_y \tau^z + v_\Delta k_x \tau^x\right) \Psi_{2a}.\end{aligned}$$

Ising field theory

$$\mathcal{S}_\phi = \int d^2x d\tau \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{c^2}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 + \frac{u_0}{24}\phi^4\right];$$

Ising order and Dirac fermions couple via a "Yukawa" term.

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \left(\Psi^\dagger_{1a} \tau^x \Psi_{1a} + \Psi^\dagger_{2a} \tau^x \Psi_{2a} \right) \right],$$

Nematic ordering

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \left(\Psi^\dagger_{1a} \tau^y \Psi_{1a} + \Psi^\dagger_{2a} \tau^y \Psi_{2a} \right) \right]$$

Time reversal symmetry breaking

M. Vojta, Y. Zhang, and S. Sachdev, Physical Review Letters **85**, 4940 (2000)

Ising order and Dirac fermions couple via a "Yukawa" term.

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \left(\Psi_{1a}^\dagger \tau^x \Psi_{1a} + \Psi_{2a}^\dagger \tau^x \Psi_{2a} \right) \right],$$

Nematic ordering

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \left(\Psi_{1a}^\dagger \tau^y \Psi_{1a} + \Psi_{2a}^\dagger \tau^y \Psi_{2a} \right) \right]$$

Time reversal symmetry breaking

For the latter case *only*, with $v_F = v_\Delta = c$, theory reduces to relativistic Gross-Neveu model

M. Vojta, Y. Zhang, and S. Sachdev, Physical Review Letters **85**, 4940 (2000)

Expansion in number of fermion spin components N_f

Integrating out the fermions yields an effective action for the scalar order parameter

$$S_\phi = \frac{N_f}{v_\Delta v_F}\Gamma\left[\lambda_0\phi(x,\tau);\frac{v_\Delta}{v_F}\right] + \frac{N_f}{2}\int d^2x d\tau\Big(r\phi^2(x,\tau)\Big) + \text{irrelevant terms}$$

where Γ is a non-local and non-analytic functional of ϕ.

The theory has only 2 couplings constants: r and v_Δ/v_F.

Y. Huh and S. Sachdev, Physical Review B **78**, 064512 (2008).

Expansion in number of fermion spin components N_f

Integrating out the fermions yields an effective action for the nematic order parameter

$$S_\phi = \frac{N_f}{2}\int_{k,\omega} |\phi(k,\omega)|^2 \Bigg[r + \frac{\lambda_0^2}{8 v_F v_\Delta}\left(\frac{\omega^2 + v_F^2 k_x^2}{\sqrt{\omega^2 + v_F^2 k_x^2 + v_\Delta^2 k_y^2}} + (x \leftrightarrow y)\right)\Bigg]$$

$$+\text{higher order terms which cannot be neglected}$$

E.-A. Kim, M. J. Lawler, P. Oreto, S. Sachdev, E. Fradkin, S.A. Kivelson, arXiv:0705.4099

Expansion in number of fermion spin components N_f

Integrating out the fermions yields an effective action for the T-breaking order parameter

$$S_\phi = \frac{N_f}{2}\int_{k,\omega} |\phi(k,\omega)|^2 \Bigg[r + \frac{\lambda_0^2}{8 v_F v_\Delta}\left(\sqrt{\omega^2 + v_F^2 k_x^2 + v_\Delta^2 k_y^2} + (x \leftrightarrow y)\right)\Bigg]$$

$$+\text{higher order terms which cannot be neglected}$$

E.-A. Kim, M. J. Lawler, P. Oreto, S. Sachdev, E. Fradkin, S.A. Kivelson, arXiv:0705.4099

Expansion in number of fermion spin components N_f

Integrating out the fermions yields an effective action for the scalar order parameter

$$S_\phi = \frac{N_f}{v_\Delta v_F} \Gamma\left[\lambda_0 \phi(x,\tau); \frac{v_\Delta}{v_F}\right] + \frac{N_f}{2} \int d^2x d\tau \Big(r\phi^2(x,\tau) \Big) + \text{irrelevant terms}$$

where Γ is a non-local and non-analytic functional of ϕ.

The theory has only 2 couplings constants: r and v_Δ / v_F.

Y. Huh and S. Sachdev, Physical Review B **78**, 064512 (2008).

Expansion in number of fermion spin components N_f

Integrating out the fermions yields an effective action for the scalar order parameter

$$S_\phi = \frac{N_f}{v_\Delta v_F} \Gamma\left[\lambda_0 \phi(x,\tau); \frac{v_\Delta}{v_F}\right] + \frac{N_f}{2}\int d^2x d\tau \Big(r\phi^2(x,\tau) \Big) + \text{irrelevant terms}$$

where Γ is a non-local and non-analytic functional of ϕ.

There is a systematic expansion in powers of $1/N_f$ for renormalization group equations and all critical properties.

Y. Huh and S. Sachdev, Physical Review B **78**, 064512 (2008).

Outline

2. Quantum phase transitions in *d*-wave superconductors and metals

A. d-wave superconductivity

B. Discrete symmetry breaking in a d-wave superconductor: reflection (nematic ordering) or time-reversal

C. Nematic ordering in a metal

Outline

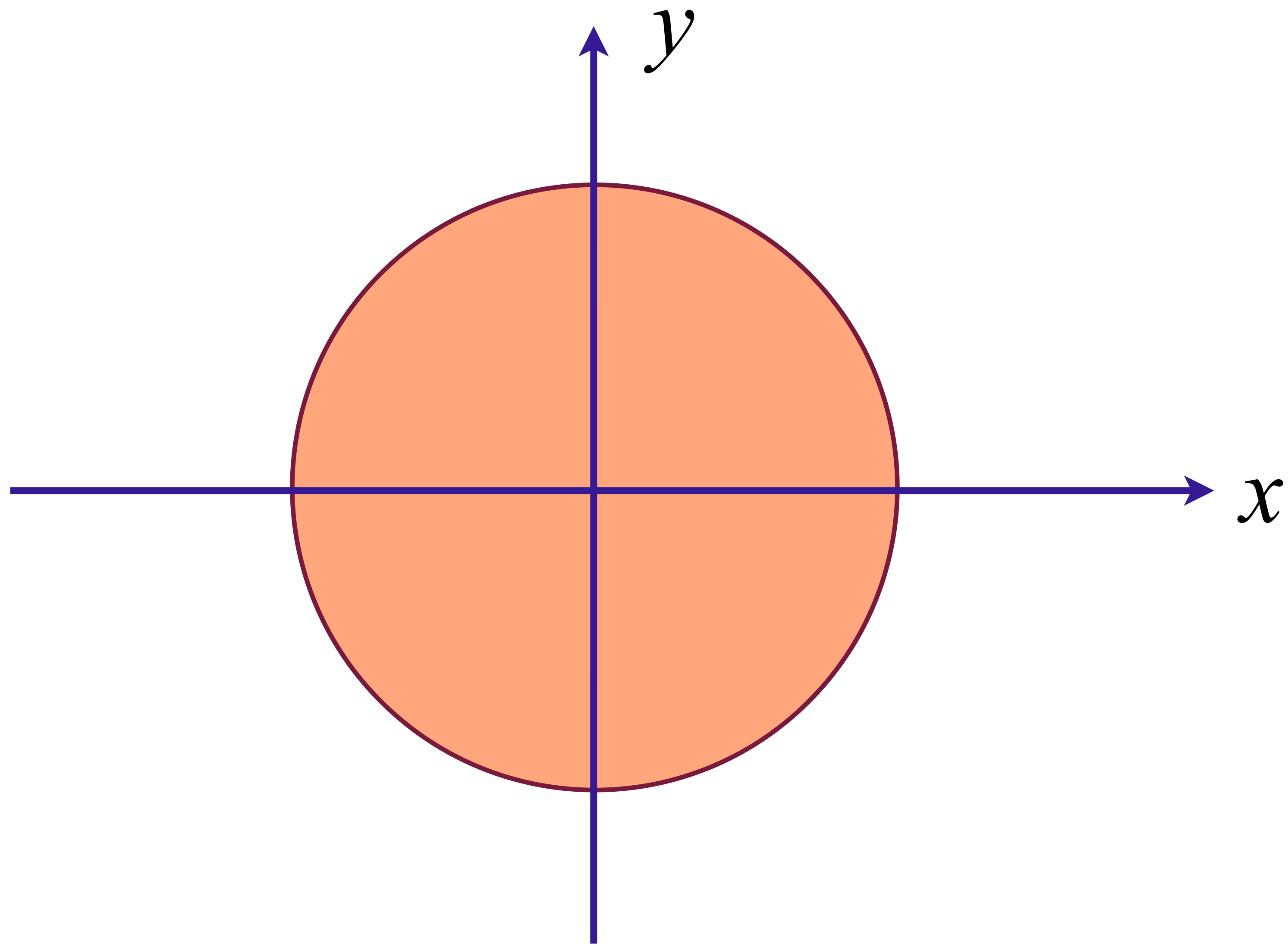
2. Quantum phase transitions in *d*-wave superconductors and metals

A. d-wave superconductivity

B. Discrete symmetry breaking in a d-wave superconductor: reflection (nematic ordering) or time-reversal

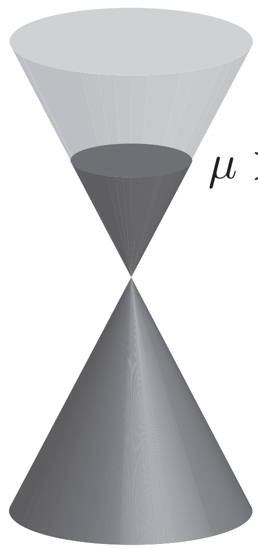
C. Nematic ordering in a metal

Quantum criticality of Pomeranchuk instability



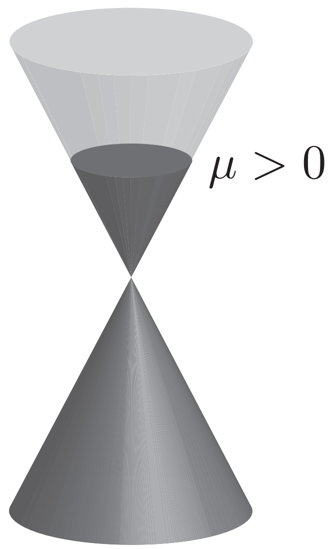
Fermi surface with full square lattice symmetry

Electron Green's function in Fermi liquid (T=0)



$$G(k,\omega) = \frac{Z}{\omega - v_F(k-k_F) - i\omega^2 \mathcal{F}\left(\frac{k-k_F}{\omega}\right)} + \ldots$$

Electron Green's function in Fermi liquid (T=0)

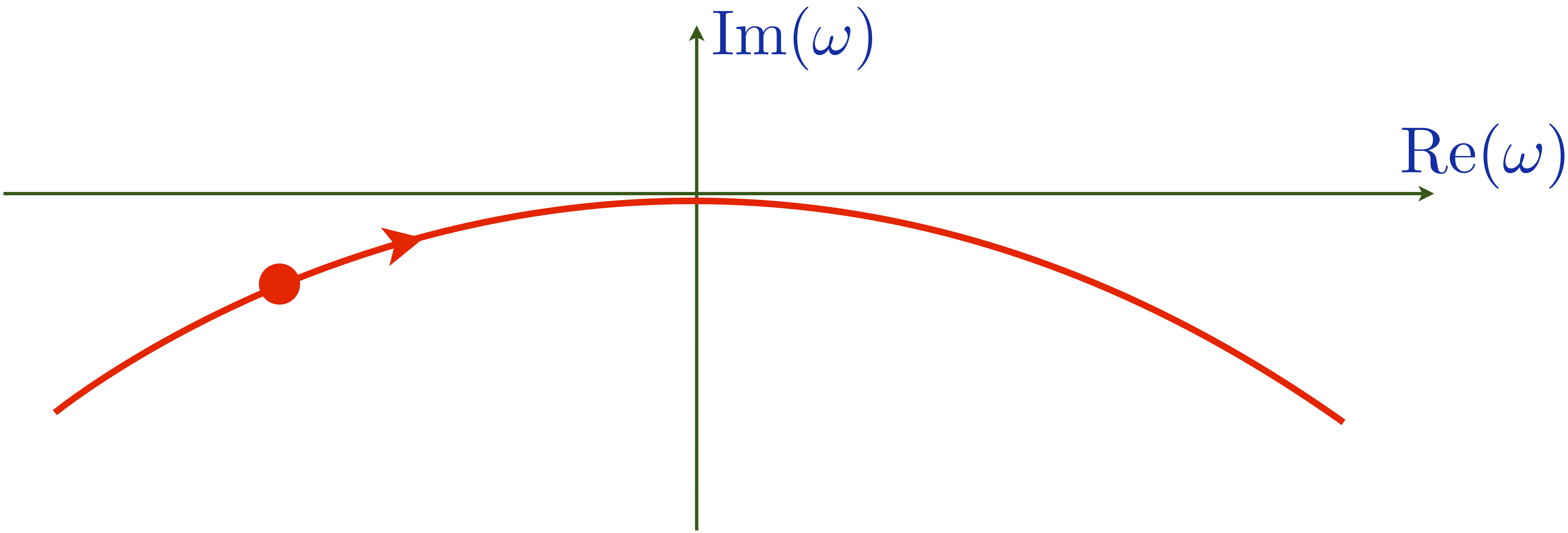


$$G(k,\omega) = \frac{Z}{\omega - v_F(k-k_F) - i\omega^2 \mathcal{F}\left(\frac{k-k_F}{\omega}\right)} + \ldots$$

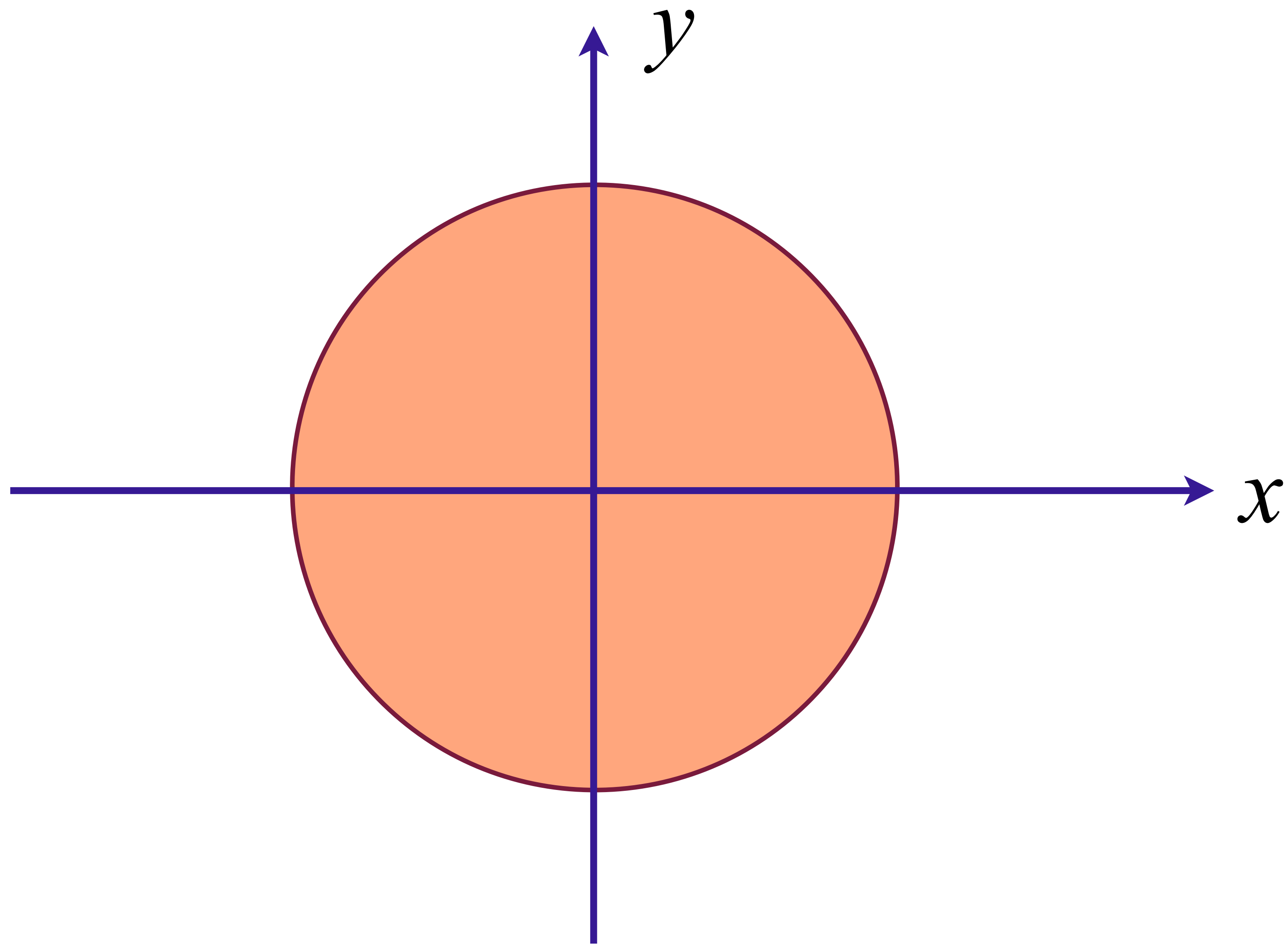
Green's function has a pole in the LHP at

$$\omega = v_F(k-k_F) - i\alpha(k-k_F)^2 + \ldots$$

Pole is at $\omega = 0$ precisely at $k = k_F$ *i.e.* on a sphere of radius k_F in momentum space. This is the *Fermi surface.*

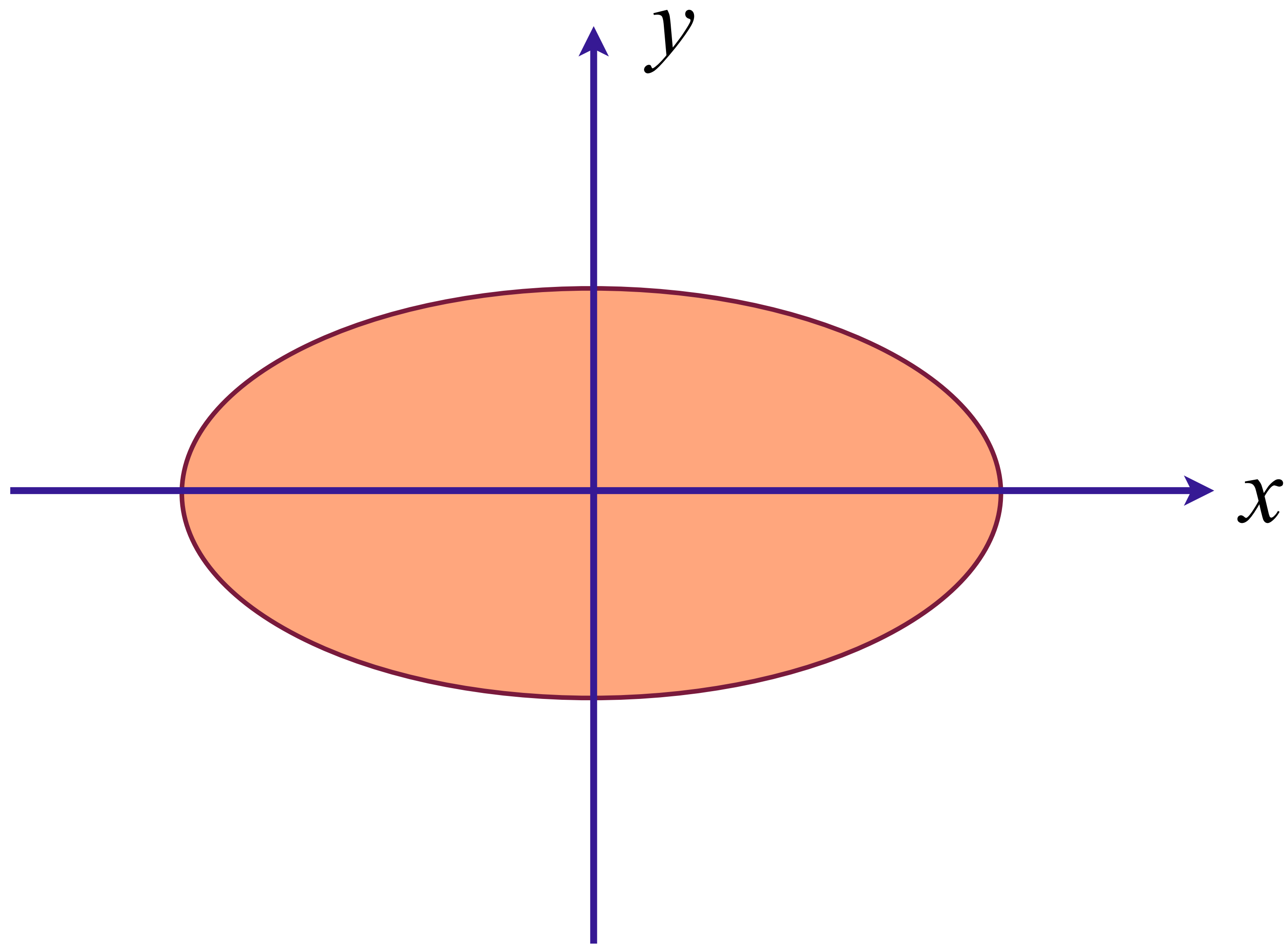


Quantum criticality of Pomeranchuk instability



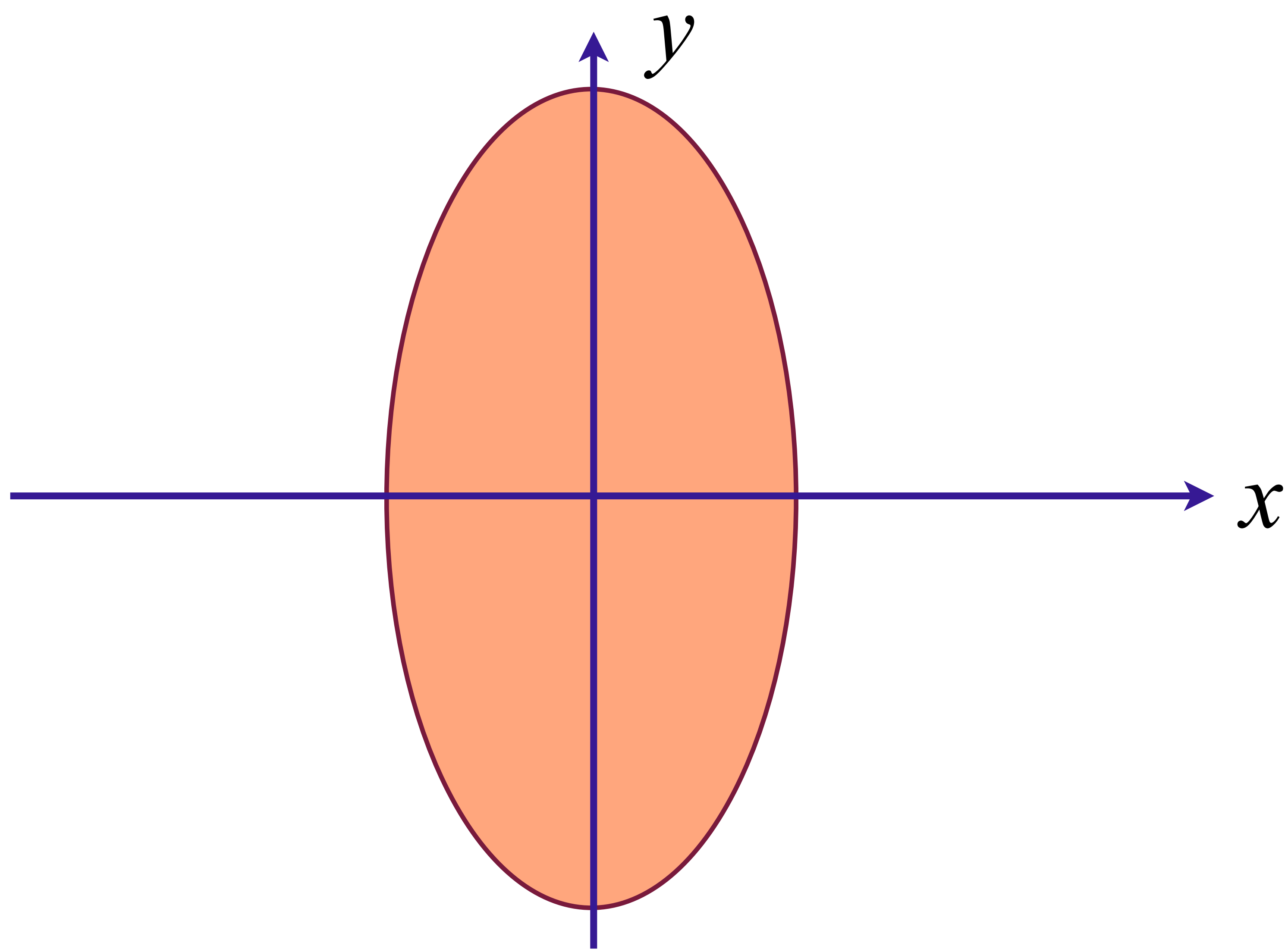
Fermi surface with full square lattice symmetry

Quantum criticality of Pomeranchuk instability



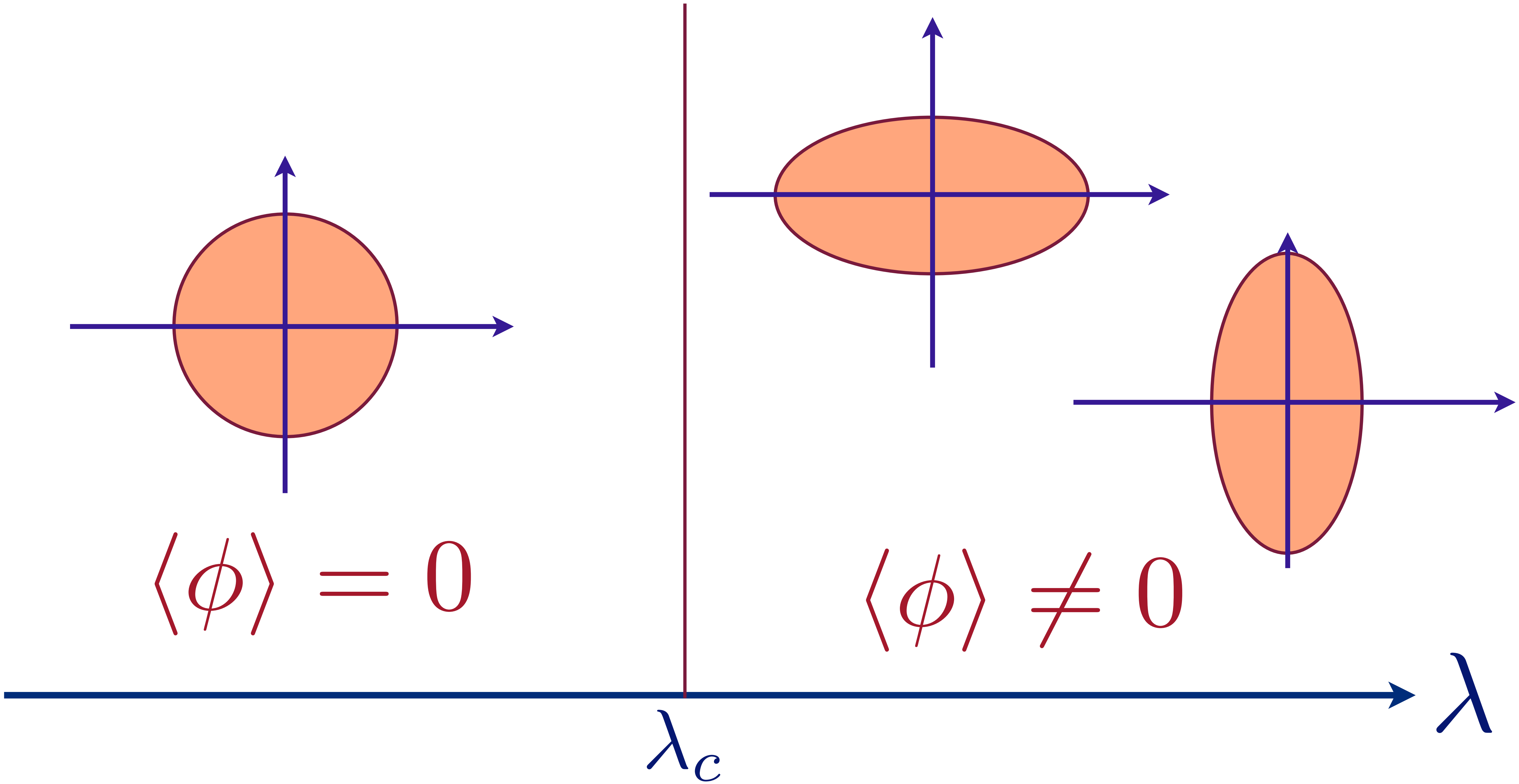
Spontaneous elongation along x direction:
Ising order parameter $\phi > 0$.

Quantum criticality of Pomeranchuk instability



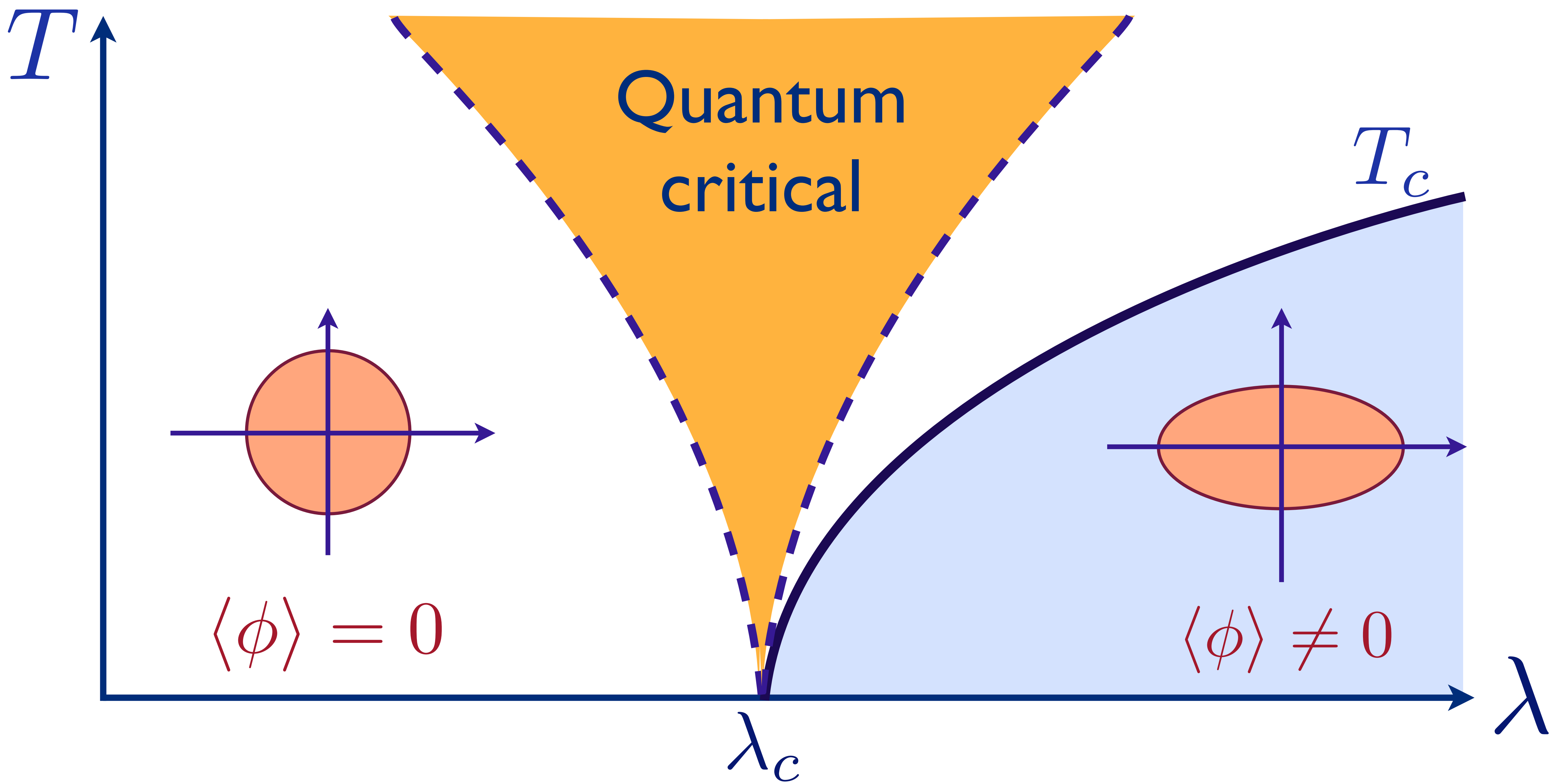
Spontaneous elongation along y direction:
Ising order parameter $\phi < 0$.

Quantum criticality of Pomeranchuk instability



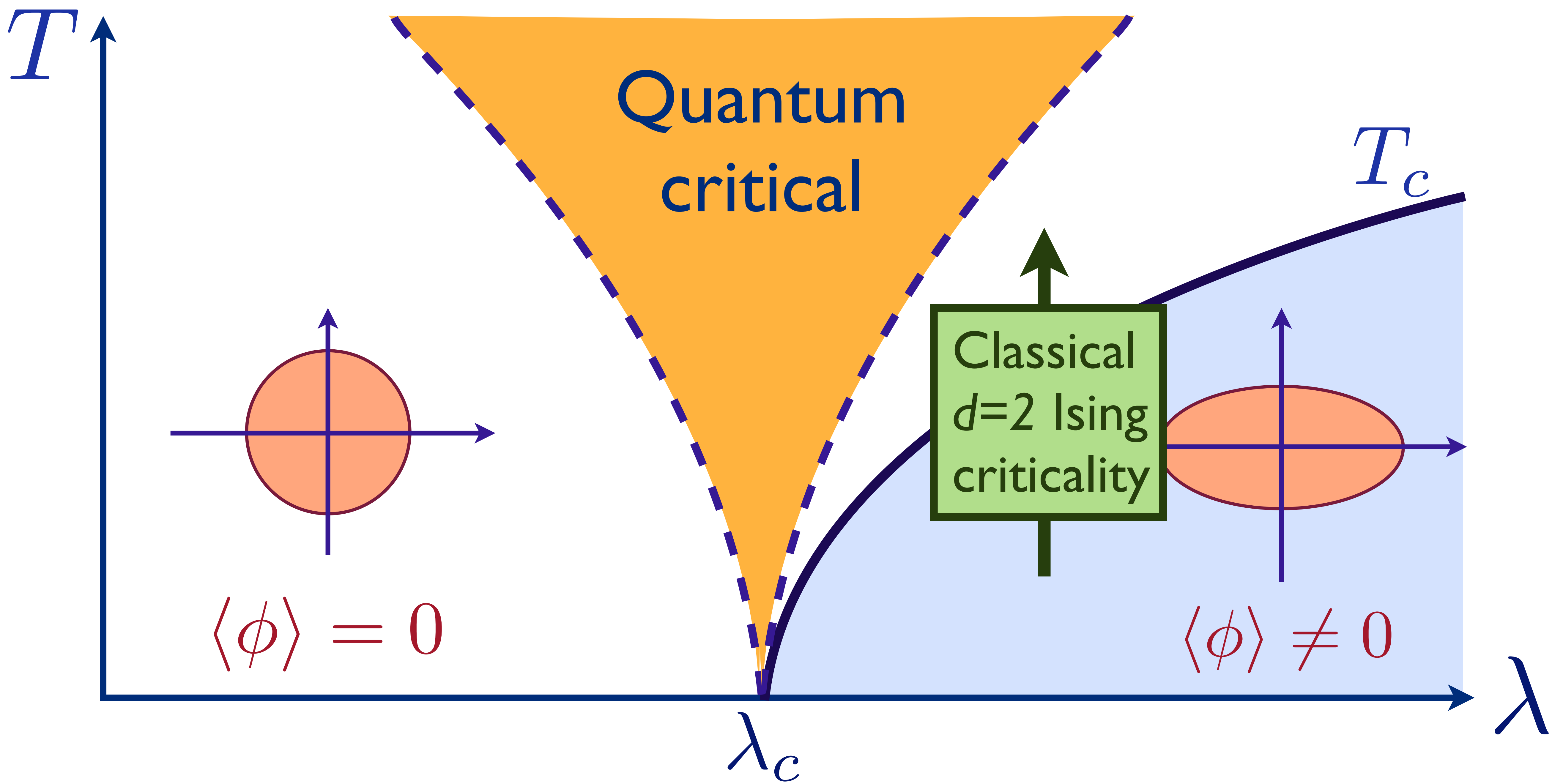
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Pomeranchuk instability



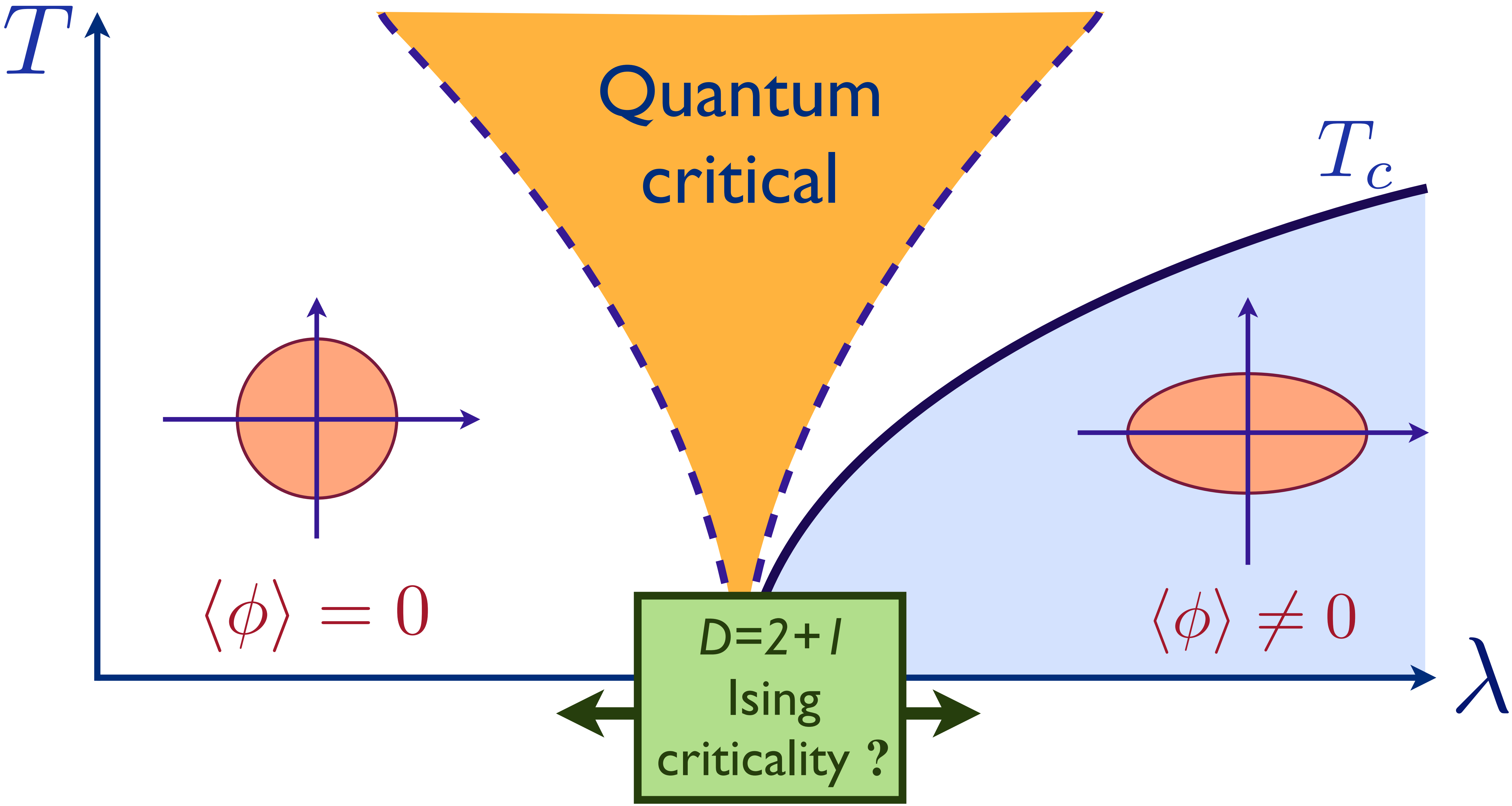
Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability



Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability



Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

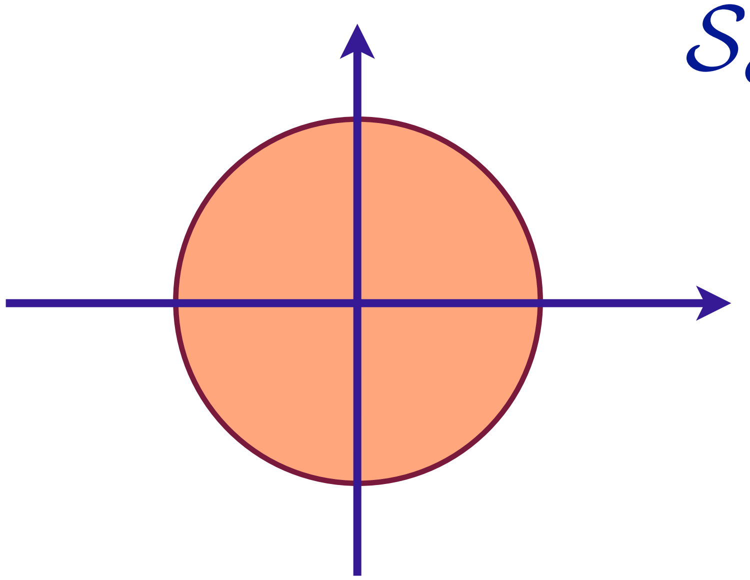
$$\mathcal{S}_\phi = \int d^2 r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right]$$

Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right]$$

Effective action for electrons:



$$\mathcal{S}_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[\sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right]$$

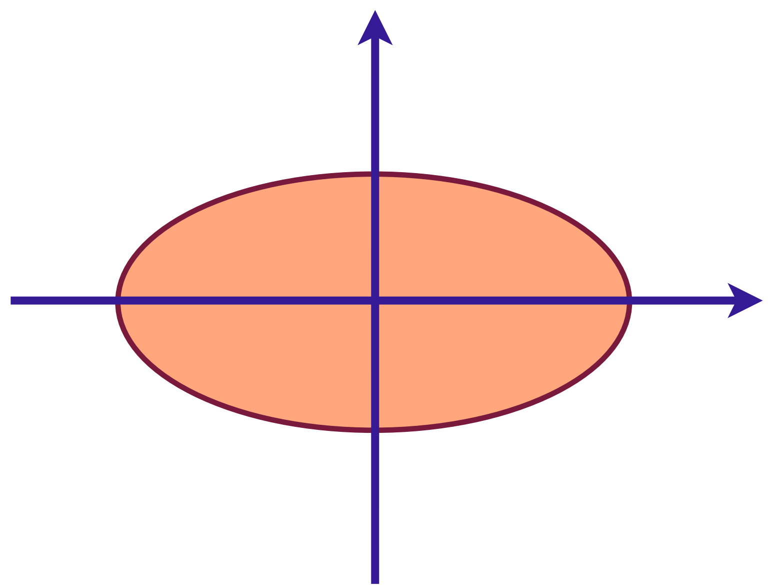
$$\equiv \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

Quantum criticality of Pomeranchuk instability

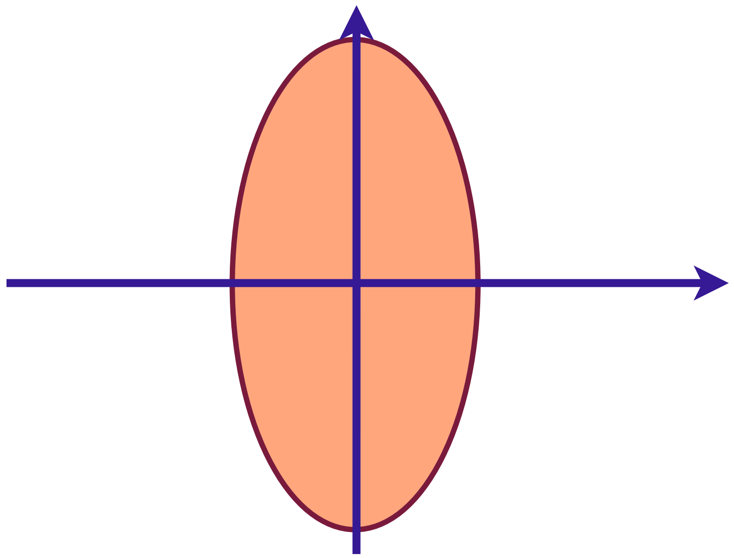
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \, \phi \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} (\cos k_x - \cos k_y) c^{\dagger}_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}$$

for spatially independent ϕ



$$\langle \phi \rangle > 0$$



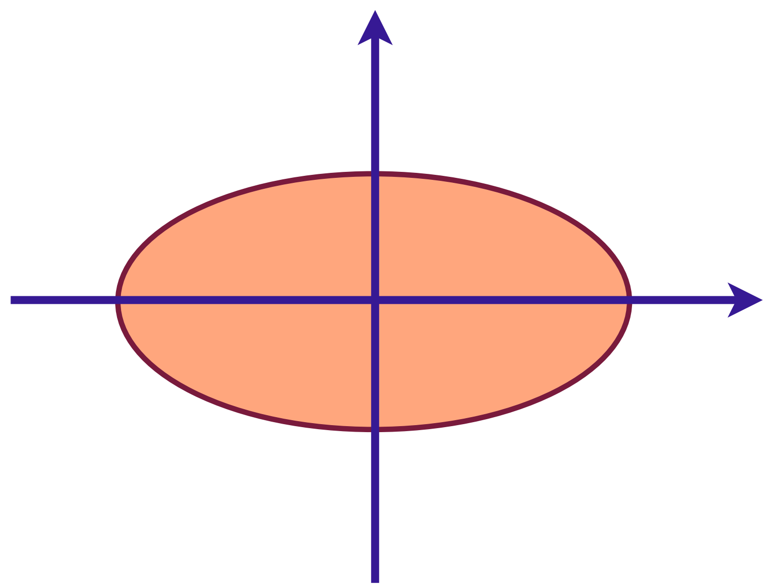
$$\langle \phi \rangle < 0$$

Quantum criticality of Pomeranchuk instability

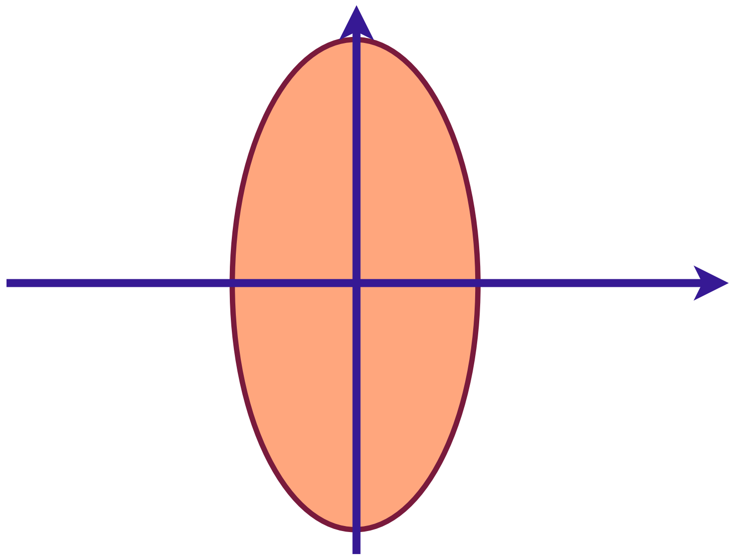
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{q}} \left(\cos k_x - \cos k_y\right) c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\alpha} c_{\mathbf{k}-\mathbf{q}/2,\alpha}$$

for spatially dependent ϕ



$\langle \phi \rangle > 0$



$\langle \phi \rangle < 0$

Quantum criticality of Pomeranchuk instability

$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right]$$

$$\mathcal{S}_c = \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{q}} \left(\cos k_x - \cos k_y \right) c_{\mathbf{k}+\mathbf{q}/2,\alpha}^\dagger c_{\mathbf{k}-\mathbf{q}/2,\alpha}$$

Quantum critical field theory

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}c_{i\alpha} \exp\left(-\mathcal{S}_\phi - \mathcal{S}_c - \mathcal{S}_{\phi c} \right)$$

Quantum criticality of Pomeranchuk instability

Hertz theory

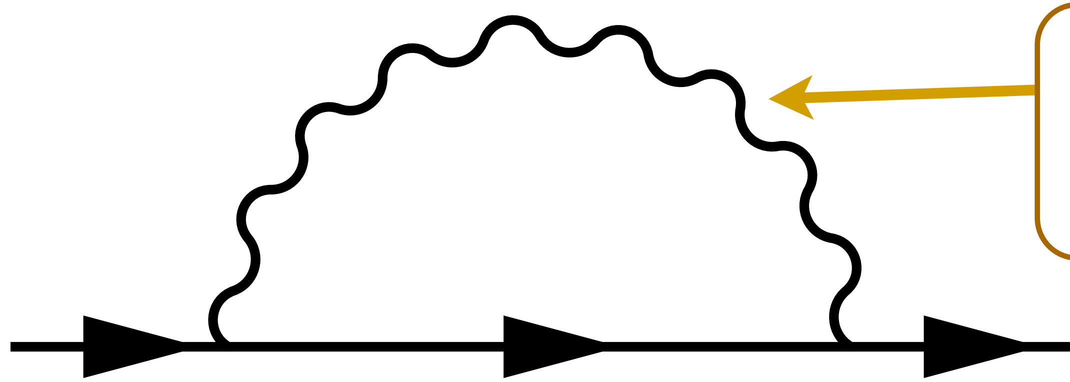
Integrate out c_α fermions and obtain non-local corrections to ϕ action

$$\delta\mathcal{S}_\phi \sim N_f \gamma^2 \int \frac{d^2 q}{4\pi^2} \int \frac{d\omega}{2\pi} |\phi(\mathbf{q},\omega)|^2 \left[\frac{|\omega|}{q} + q^2 \right] + \ldots$$

This leads to a critical point with dynamic critical exponent $z = 3$ and quantum criticality controlled by the Gaussian fixed point.

Quantum criticality of Pomeranchuk instability

Hertz theory



$$\frac{1}{N_f(q^2 + |\omega|/q)}$$

Self energy of c_α fermions to order $1/N_f$

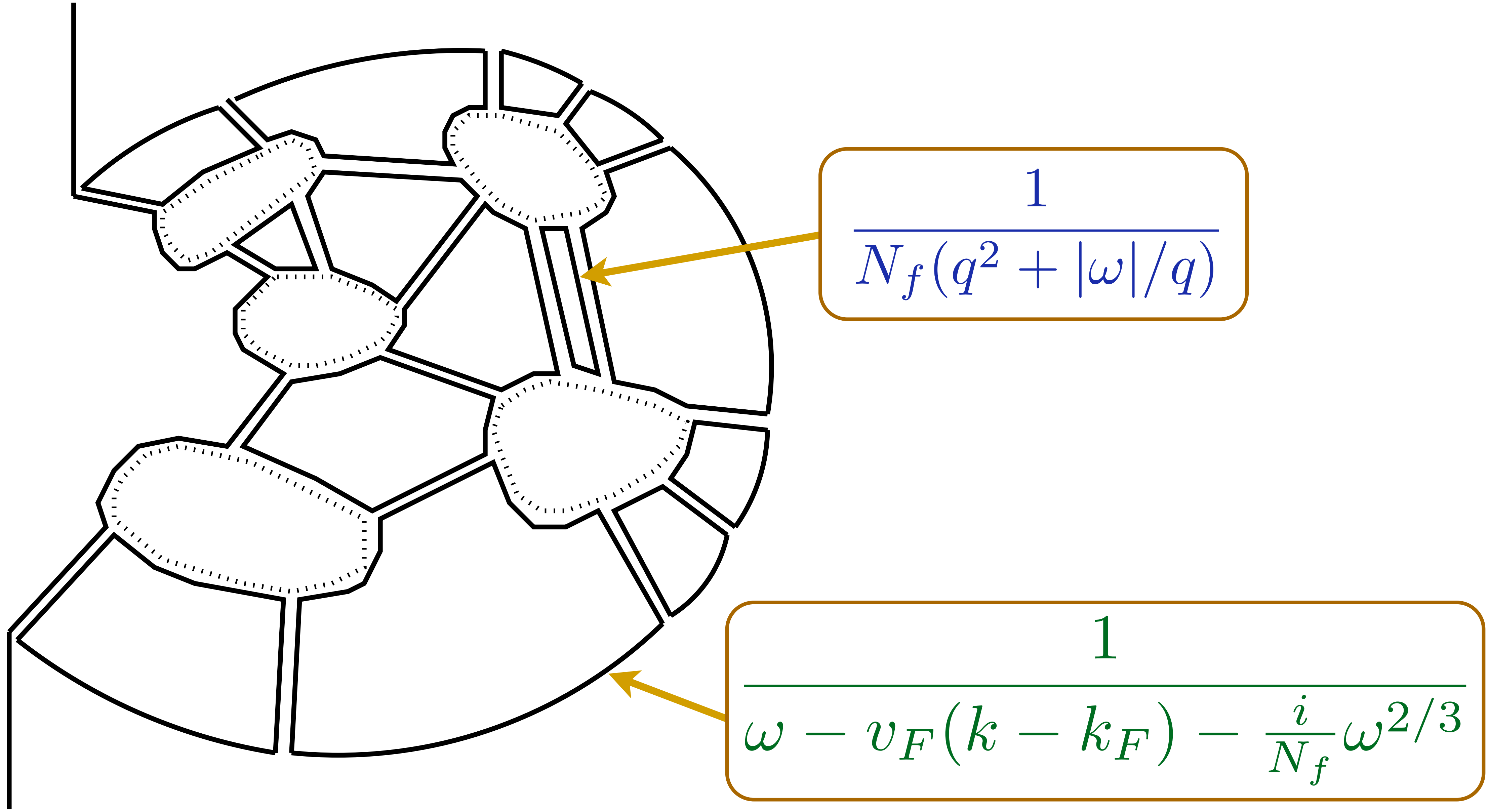
$$\Sigma_c(k,\omega) \sim \frac{i}{N_f}\omega^{2/3}$$

This leads to the Green's function

$$G(k,\omega) \approx \frac{1}{\omega - v_F(k-k_F) - \frac{i}{N_f}\omega^{2/3}}$$

Note that the order $1/N_f$ term is more singular in the infrared than the bare term; this leads to problems in the bare $1/N_f$ expansion in terms that are dominated by low frequency fermions.

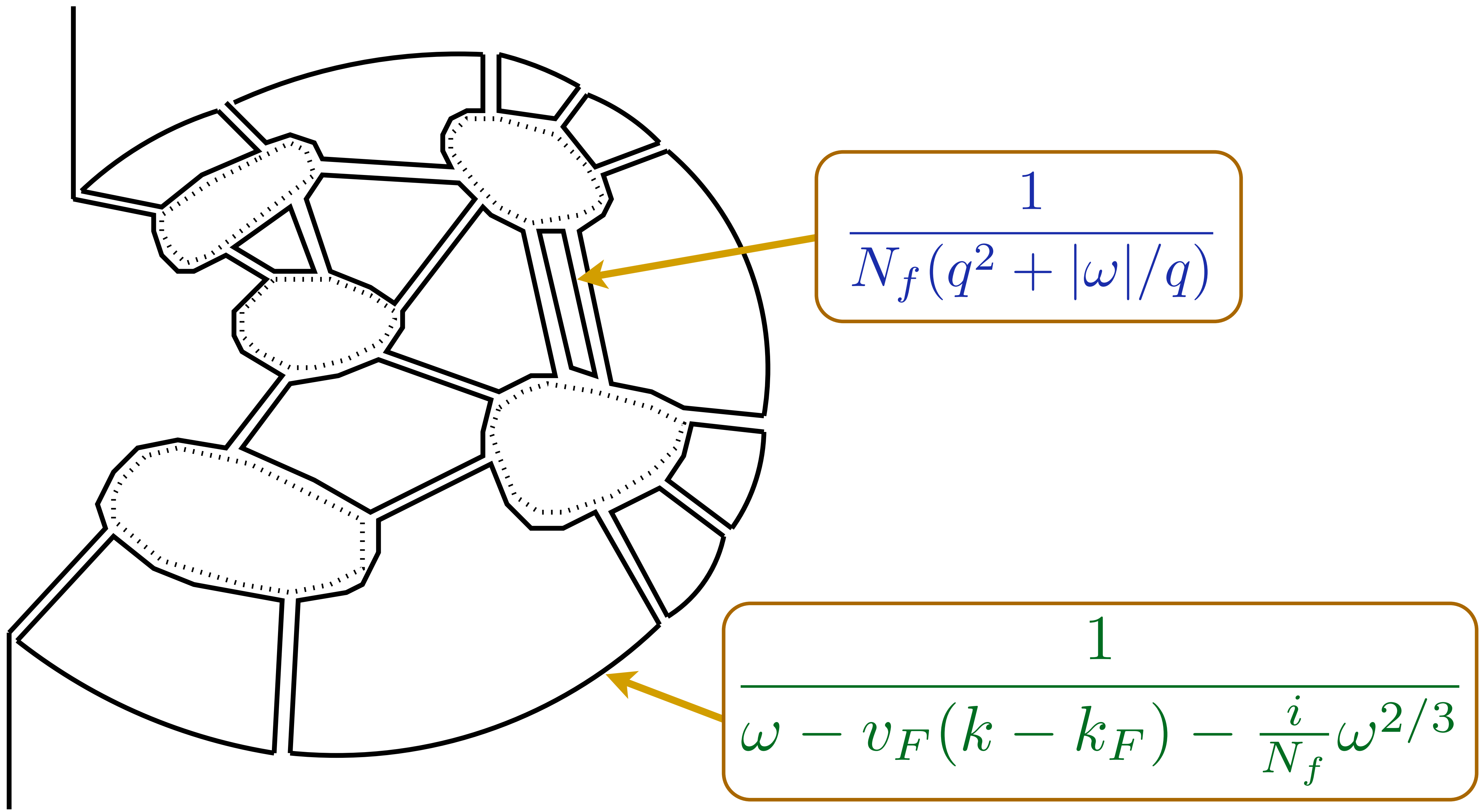
Quantum criticality of Pomeranchuk instability



The infrared singularities of fermion particle-hole pairs are most severe on planar graphs: these all contribute at leading order in $1/N_f$.

Sung-Sik Lee, *Physical Review* B **80**, 165102 (2009)

Quantum criticality of Pomeranchuk instability



A string theory for the Fermi surface ?