Statistical mechanics of strange metals and black holes

May 17, 24, 31, June 7, 2022

Subir Sachdev

Talk online: qpt.physics.harvard.edu/talks

Corrections to scaling at the SYK saddle point

SYK model	Charged black holes
$$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \frac{3}{2}\ln\left(\frac{N^{1/3}U}{k_B T}\right) + \ldots$$	$$\frac{S(T)}{k_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2}\frac{k_B T}{\hbar}\right) - \frac{3}{2}\ln\left(\frac{c^2}{A_0^{1/6}(\hbar G)^{1/3}(k_B T/\hbar)}\right) + \ldots$$
$$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$$	$$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$$
$$\lim_{T\to 0}\frac{1}{N k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$$	$$\lim_{T\to 0}\frac{1}{k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$$
$$D(E) \sim \exp(N s_0)\sinh\left(\sqrt{2N\gamma E}\right)$$	$$D(E) \sim \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\sinh\left(\left[\sqrt{\pi} A_0^{3/2}\frac{c^3}{\hbar G}\frac{E}{\hbar c}\right]^{1/2}\right)$$

Conformal Perturbation theory

$$S = S_{\rm CFT} + \sum_h \lambda_h \int_0^\beta d\tau O_h(\tau)$$

where $G_{\rm CFT} = G_* \sim {\rm sgn}(\tau)/\sqrt{|\tau|}$ and $\langle O_h(\tau) O_h(0)\rangle \sim 1/|\tau|^{2h}$

$$G(\tau) \sim \frac{{\rm sgn}(\tau)}{\sqrt{|\tau|}}\left(1 + \sum_h \frac{g_h}{|\tau|^{h-1}} + \ldots\right) \quad , \quad S(T) = N\left[s_0 + \sum_h s_h T^{h-1} + \ldots\right]$$

Conformal Perturbation theory

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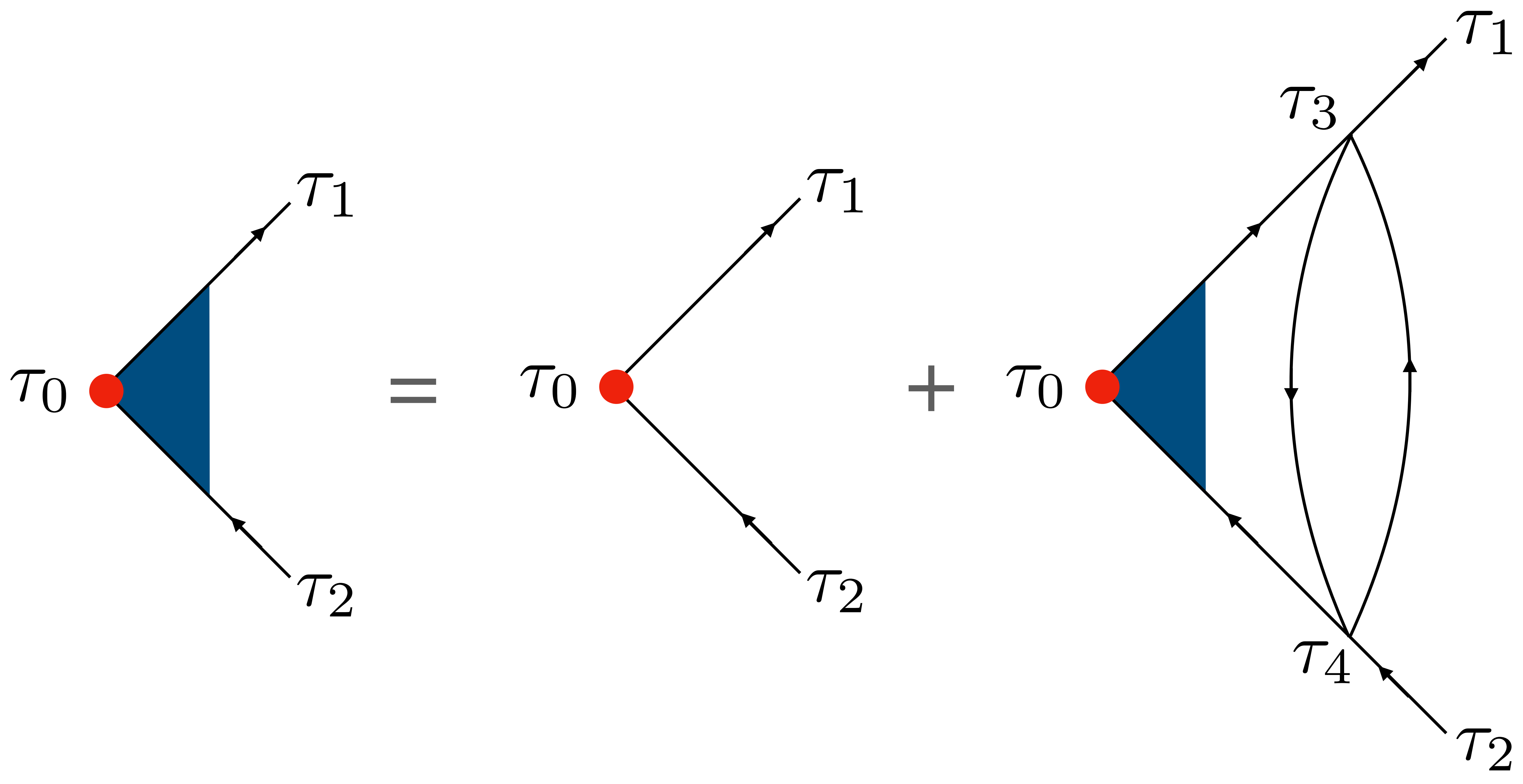

Gross, Rosenhaus (2017)
Klebanov, Tarnopolsky (2017)

Conformal Perturbation theory

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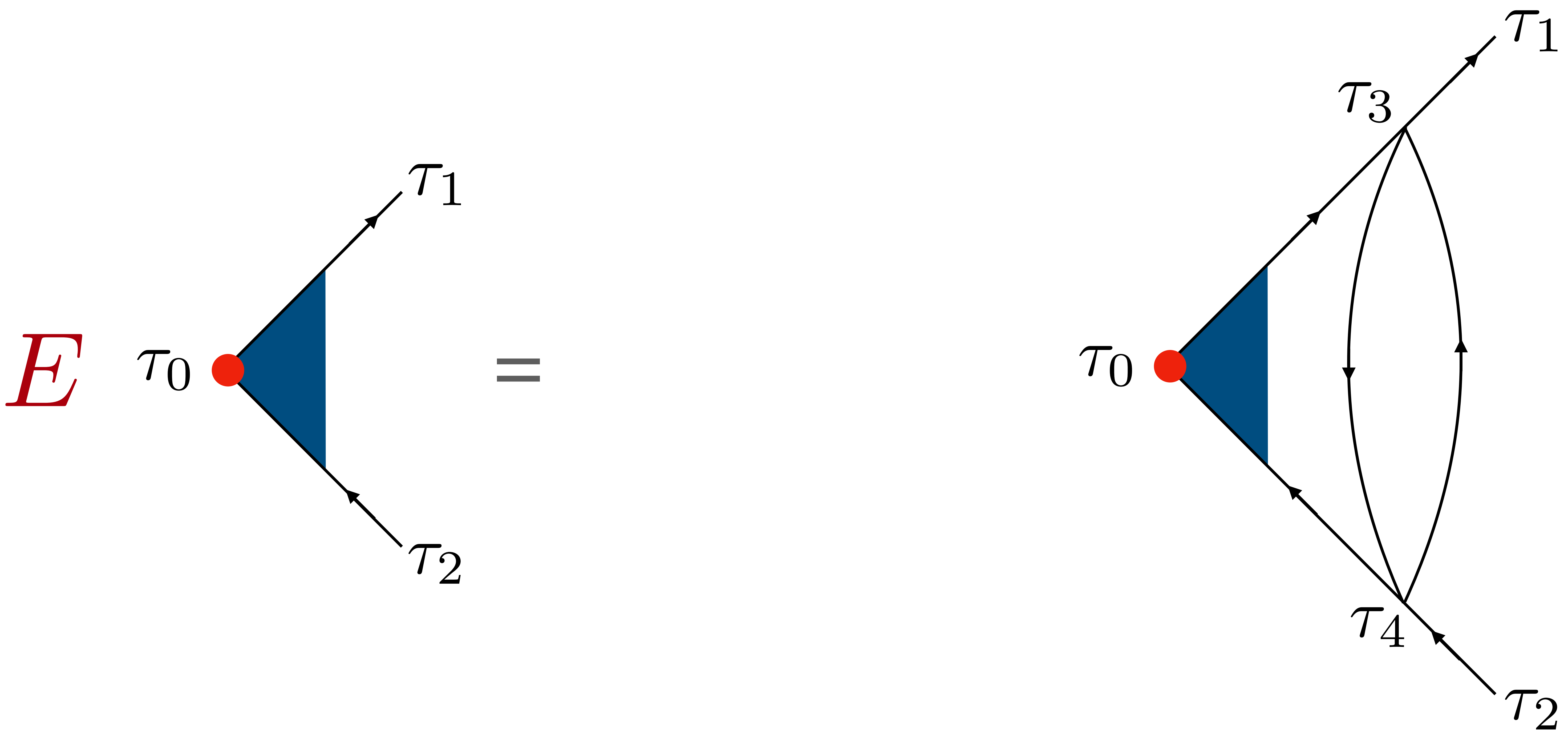


Solution of eigenvalue equation with $E = 1$ yields a tower of O_h.

Gross, Rosenhaus (2017)
Klebanov, Tarnopolsky (2017)

Conformal Perturbation theory

We define the three point function

$$v_h(\tau_1, \tau_2, \tau_0) = \langle c(\tau_1) c^\dagger(\tau_2) O_h(\tau_0) \rangle \,.$$

In the long time scaling limit, we can drop the bare first time on the right hand side, and obtain the eigenvalue equation

$$E v(\tau_1, \tau_2, \tau_0) = \int d\tau_3 d\tau_4 K(\tau_1, \tau_2; \tau_3, \tau_4) v_h(\tau_3, \tau_4, \tau_0) \,,$$

where the kernel K is

$$K(\tau_1, \tau_2; \tau_3, \tau_4) = -3U^2 G_*(\tau_{13}) G_*(\tau_{24}) G_*(\tau_{34})^2 \,,$$

with $\tau_{ij} \equiv \tau_i - \tau_j$, and we are interested in the eigenvalue $E = 1$.

Conformal Perturbation theory

It is sufficient to solve the eigenvalue equation as $\tau_0 \to \infty$. Then, we can use the operator product expansion to write

$$c(\tau_1)c^\dagger(\tau_2) \sim \mathrm{sgn}(\tau_{12}) \left[\frac{1}{|\tau_{12}|^{1/2}} + \sum_h \frac{c_h}{|\tau_{12}|^{1/2-h}} O_h(\tau_1) + \ldots \right]$$

Inserting this into the defintion of v, we conclude that $v \sim \mathrm{sgn}(\tau_{12})/|\tau_{12}|^{1/2-h}$ as $\tau_0 \to \infty$. Then the eigenvalue equation simplifies to

$$E = -\frac{3\tan(\pi h/2 - \pi/4)}{2h-1} = 1\,.$$

There are an infinite number of solutions, and the lowest values are $h = 2, 3.77354\ldots, 5.567946\ldots, 7.63197\ldots, \ldots$. Consequently, the low T behavior of the entropy is

$$S(T) = N\left[s_0 + \gamma\, T + \gamma_2\, T^{2.77354\ldots} + \ldots \right]\,.$$

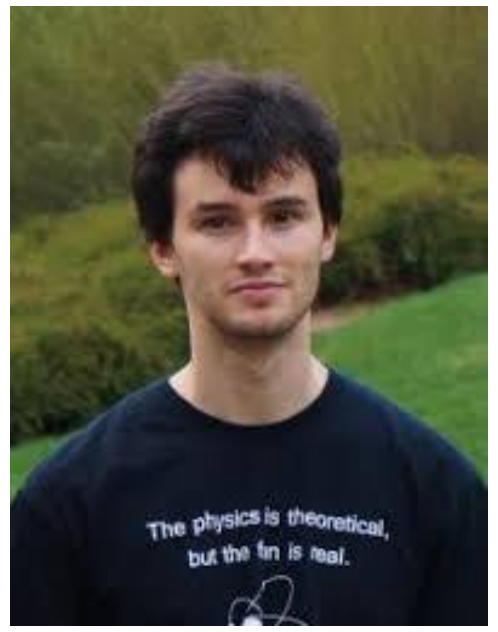
We will have a particular interest in the $h = 2$ operator in the remaining discussion.

Dynamic spin susceptibility of the spin liquid at $M = \infty$

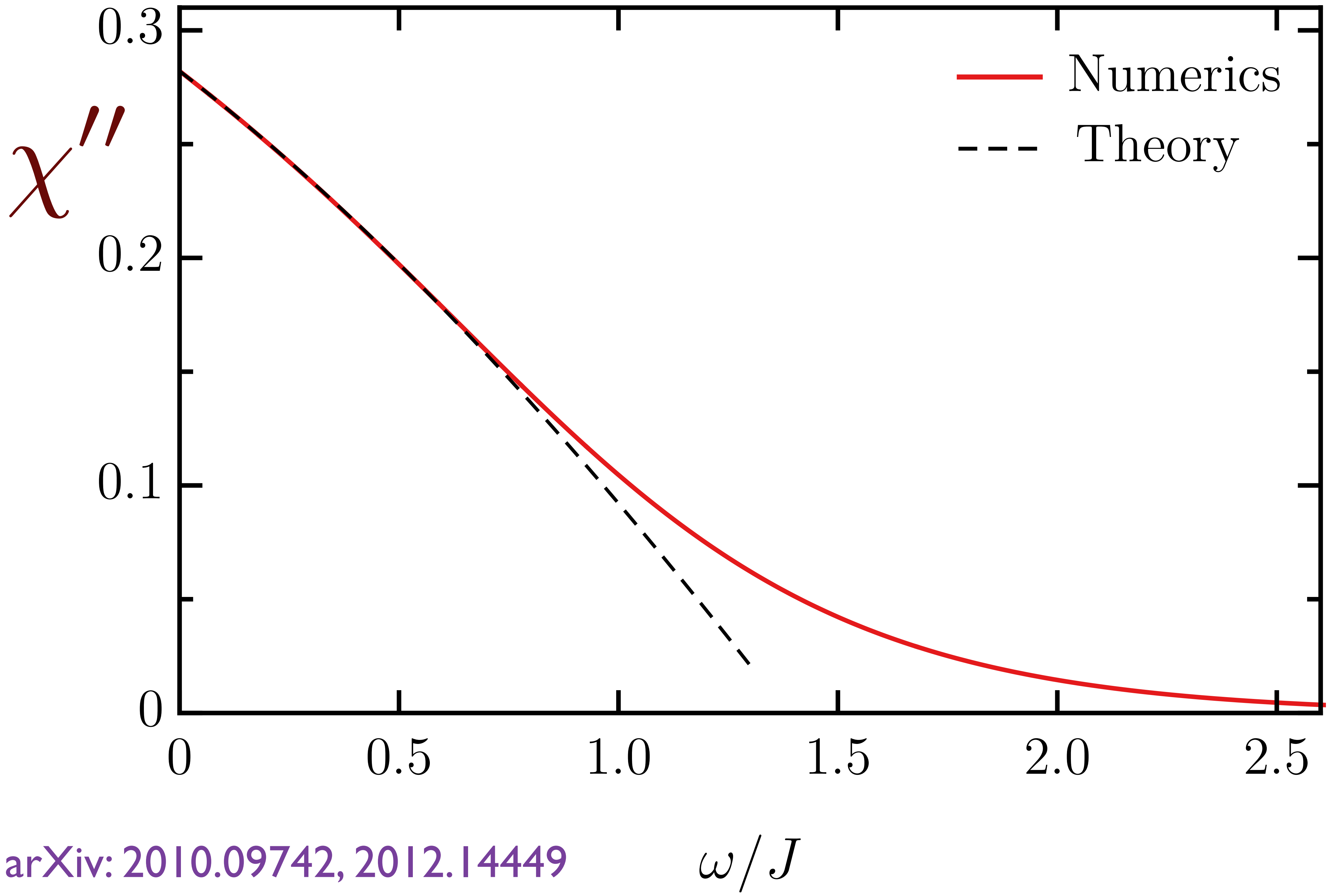
$$Q(\tau) = \int_0^\infty \frac{d\omega}{\pi} \chi''(\omega) e^{-\omega\tau}$$

$$\chi''(\omega) \sim \mathrm{sgn}(\omega)\left[1 - \mathcal{C}\gamma|\omega| - \frac{7}{16}(\mathcal{C}\gamma)^2|\omega|^2 - \mathcal{C}'|\omega|^{2.77354\ldots} + \frac{37}{48}(\mathcal{C}\gamma)^3|\omega|^3 - \ldots\right]$$





Numerical solution of SYK equations (SY, PRL 1993), compared with conformal perturbation theory. $\mathcal{C}$ is a known number, and γ is the co-efficient of the action for the 'boundary graviton' in holographic dual.



Maria Tikhanovskaya, Haoyu Guo, S. Sachdev, G. Tarnopolsky, arXiv: 2010.09742, 2012.14449

Dynamic spin susceptibility of the spin liquid at $M = \infty$

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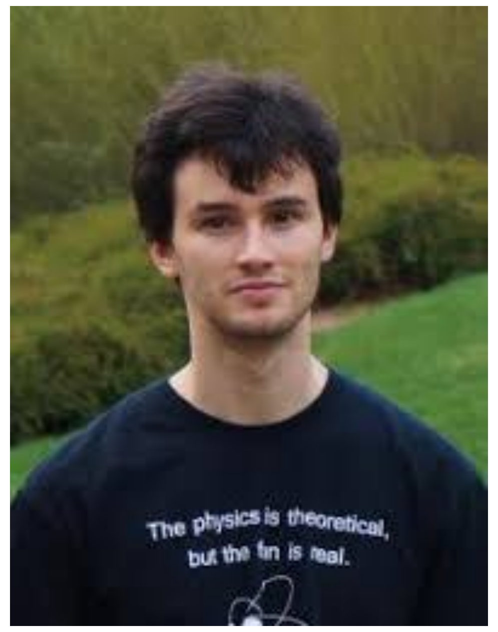
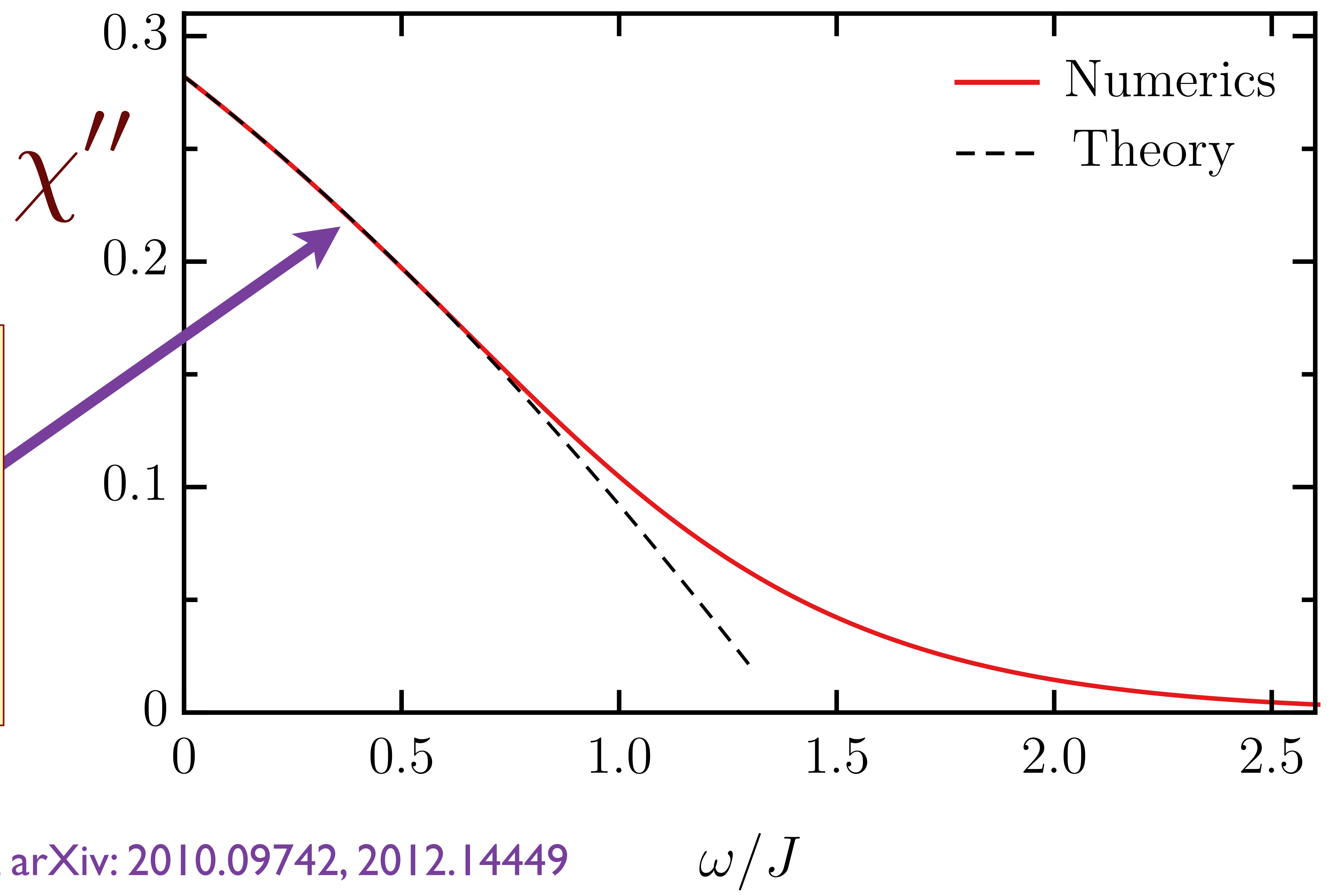


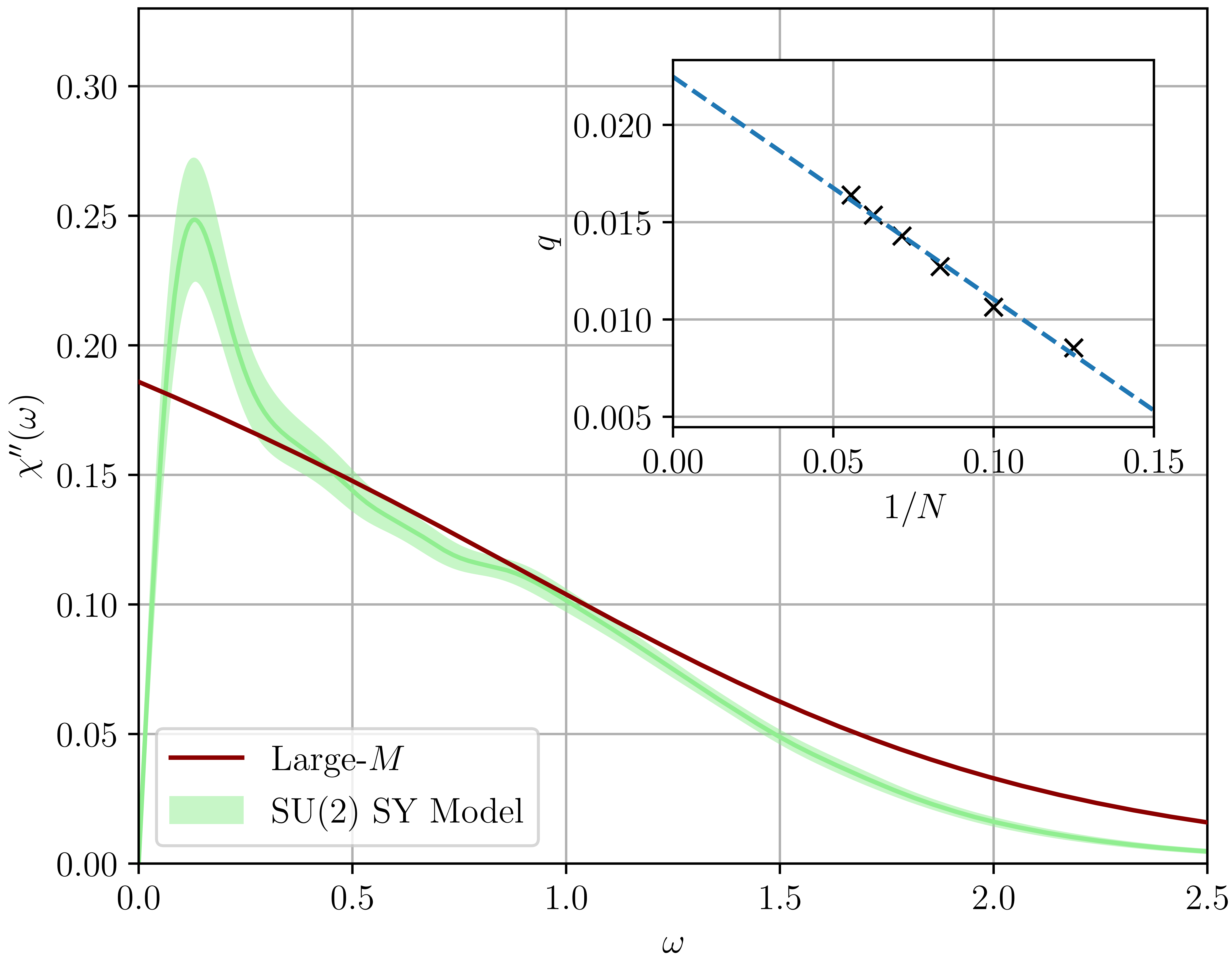
Numerical solution of SYK equations (SY, PRL 1993), compared with conformal perturbation theory. $\mathcal{C}$ is a known number, and γ is the co-efficient of the action for the 'boundary graviton' in holographic dual.

Correction from the boundary graviton



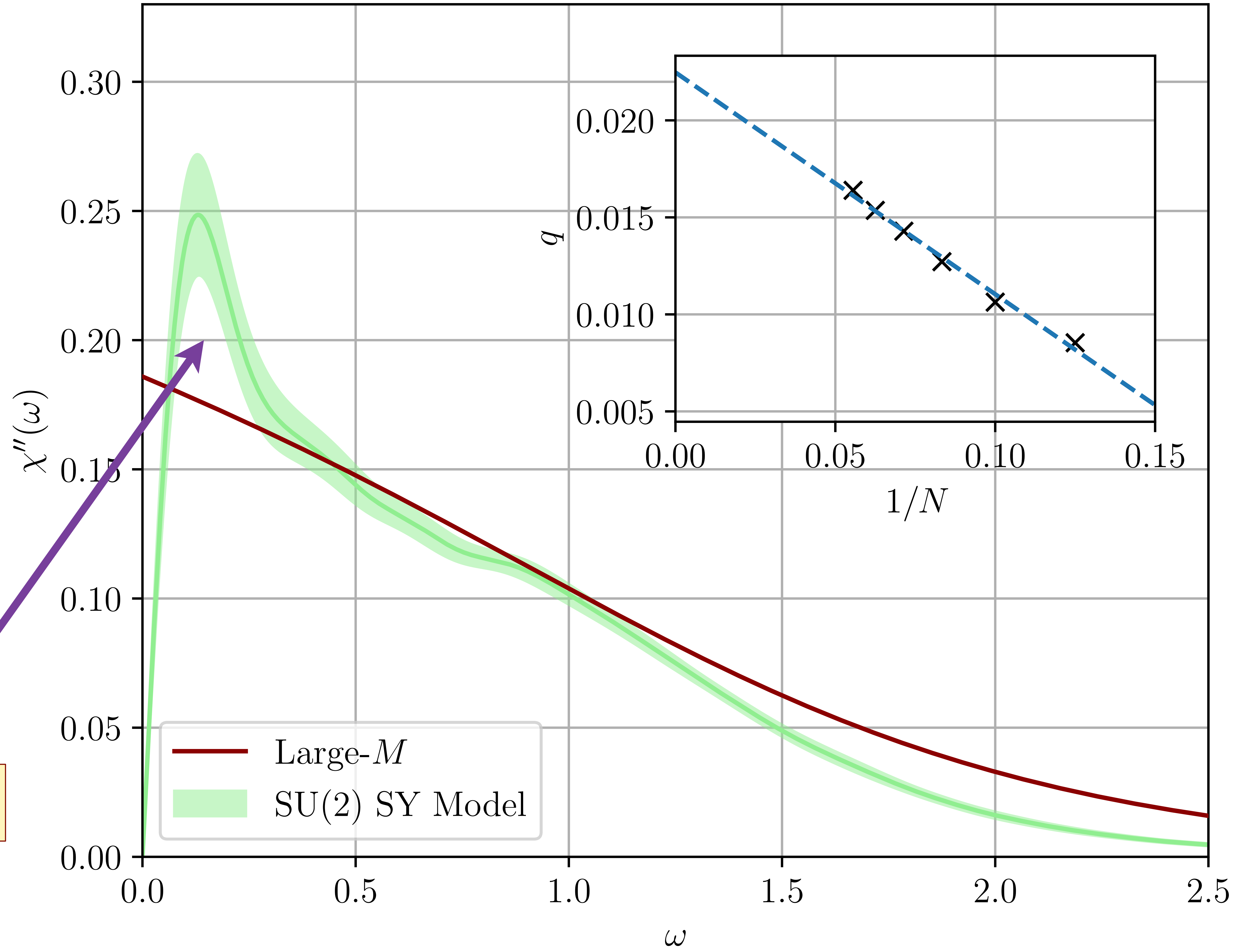
Maria Tikhanovskaya, Haoyu Guo, S. Sachdev, G. Tarnopolsky, arXiv: 2010.09742, 2012.14449

Exact diagonalization of clusters of SU(2) spins



H. Shackleton, A. Wietek, A. Georges, and S. Sachdev, PRL **126**, 136602 (2021)

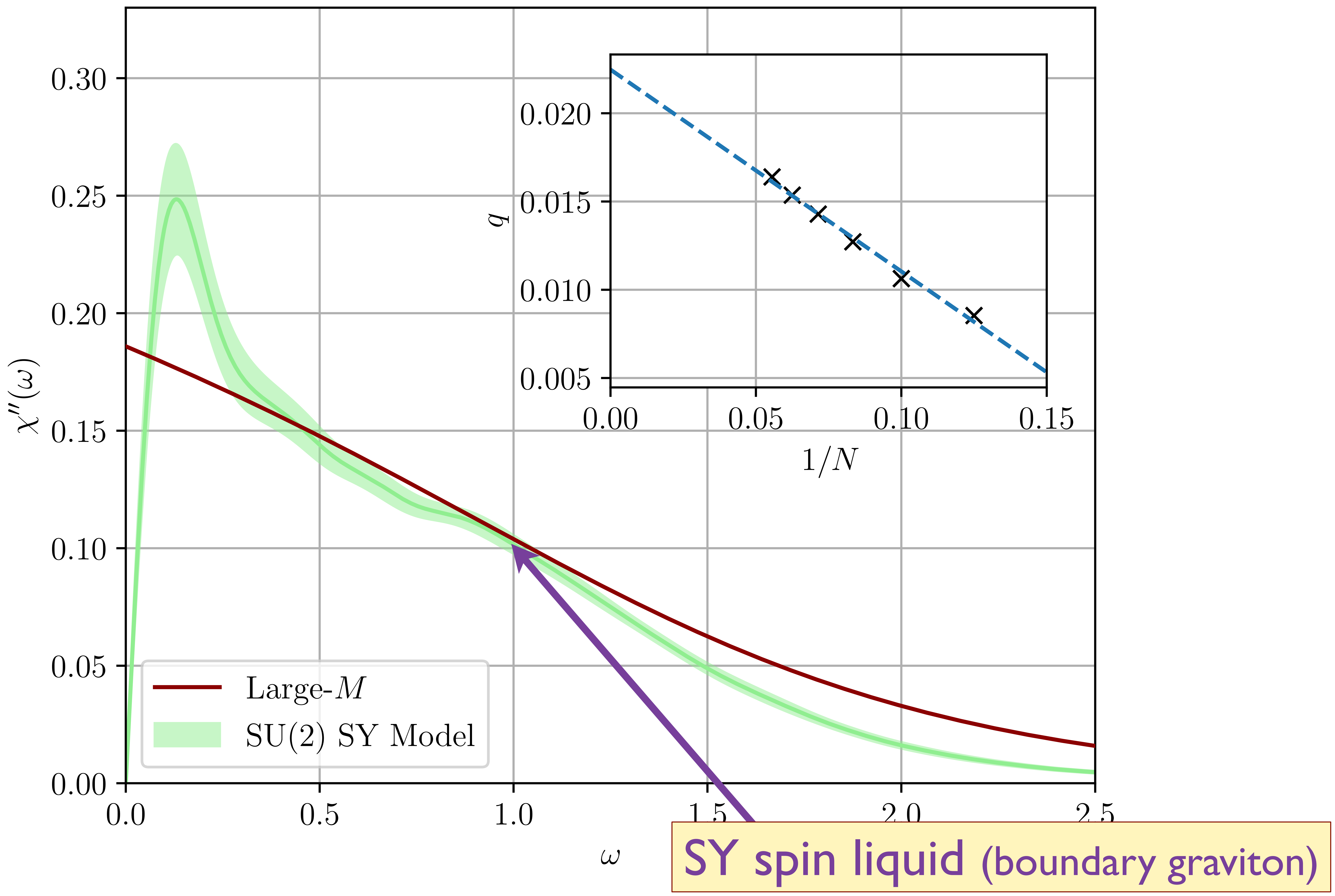
Exact diagonalization of clusters of SU(2) spins



Spin glass

H. Shackleton, A. Wietek, A. Georges, and S. Sachdev, PRL **126**, 136602 (2021)

Exact diagonalization of clusters of SU(2) spins



H. Shackleton, A. Wietek, A. Georges, and S. Sachdev, PRL **126**, 136602 (2021)

Adding spin glass order to the SU($M \to \infty$) equations:

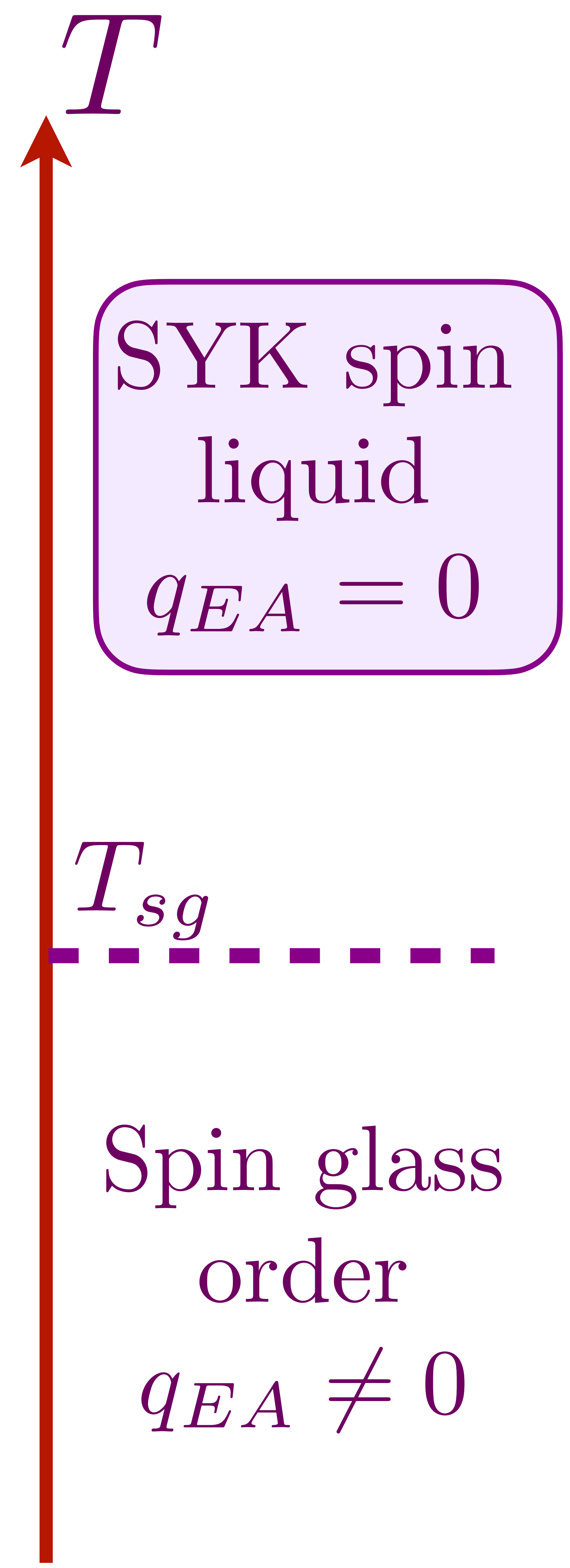
Maine Christos

Felix Haehl

arXiv:2110.00007

$$T_{\rm sg} \sim J \exp\left(-\sqrt{M\pi}\right) \quad , \quad e^{-\sqrt{2\pi}} = 0.0815\ldots$$

Georges, Parcollet, S.S. (2001)



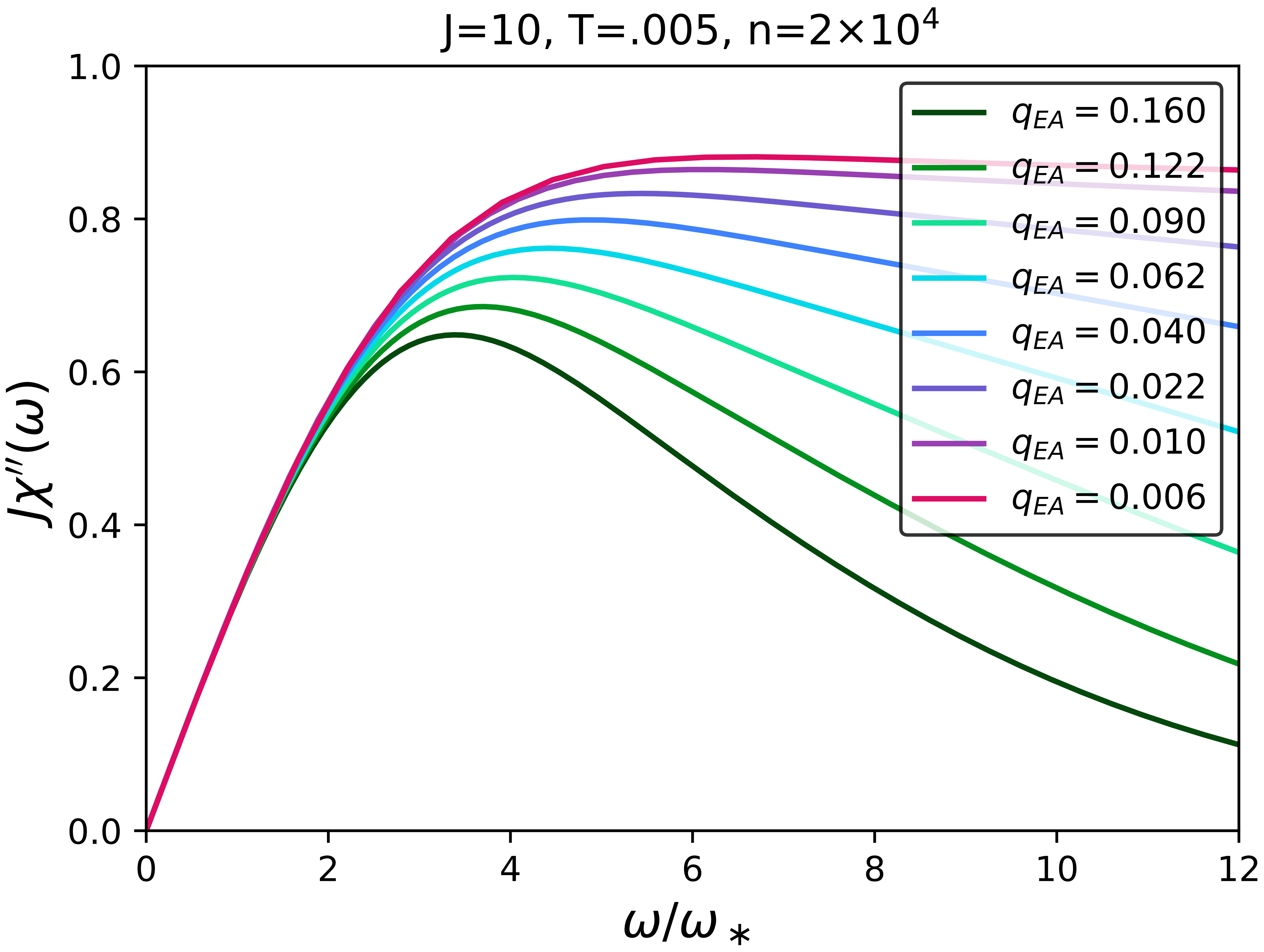
Adding spin glass order to the SU($M \to \infty$) equations:

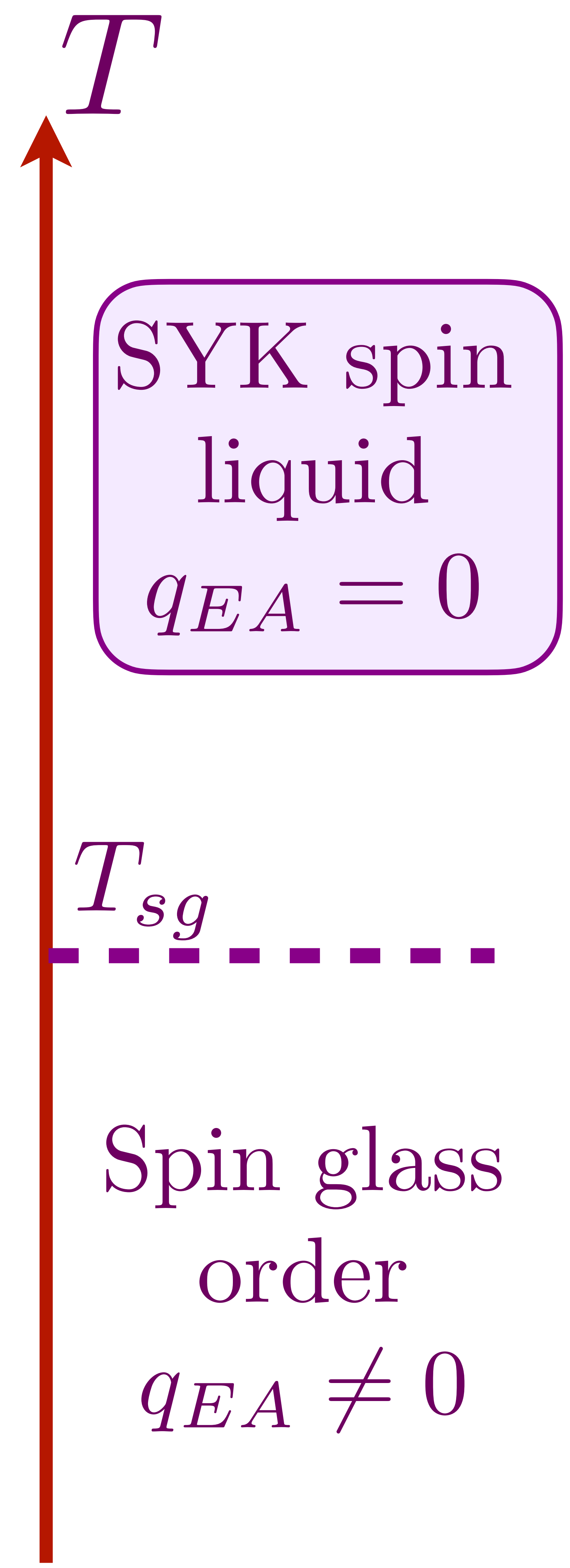
$$\Sigma(\tau) = J^2 Q_{aa}(\tau) G(\tau)$$
$$G(i\omega) = [i\omega - \Sigma(i\omega)]^{-1}$$
$$Q_{ab}(\tau) = -G(\tau)G(-\tau)\delta_{ab} + q_{ab}$$

Need only add the static spin glass order parameter q_{ab}, which is determined by the $1/M$ corrections.

$$T_{\mathrm{sg}} \sim J \exp\left(-\sqrt{M\pi}\right) \quad , \quad e^{-\sqrt{2\pi}} = 0.0815\ldots$$

$$\chi''(\omega) = \frac{\pi\omega}{T} q_{EA}\, \delta(\omega) + \frac{1}{J} \Phi_\chi \left(\frac{\omega}{J q_{EA}} \right) + \ldots, \quad T \to 0$$





**The Schwarzian theory:
accounting for the
h=2 operator exactly
in the SYK model as a
time reparameterization soft-mode**

SYK model	Charged black holes
$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \frac{3}{2}\ln\left(\frac{N^{1/3}U}{k_B T}\right) + \ldots$	$\frac{S(T)}{k_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi}A_0^{3/2}c^2}{2}\frac{k_B T}{\hbar}\right) - \frac{3}{2}\ln\left(\frac{c^2}{A_0^{1/6}(\hbar G)^{1/3}(k_B T/\hbar)}\right) + \ldots$
$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$	$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$
$\lim_{T\to 0}\frac{1}{Nk_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$	$\lim_{T\to 0}\frac{1}{k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$
$D(E) \sim \exp(Ns_0)\sinh\left(\sqrt{2N\gamma E}\right)$	$D(E) \sim \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\sinh\left(\left[\sqrt{\pi}A_0^{3/2}\frac{c^3}{\hbar G}\frac{E}{\hbar c}\right]^{1/2}\right)$

G-Σ path integral

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp \left[- \sum_{ia} \int_0^\beta d\tau \, c_{ia}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{ia} \right.$$
$$\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}G(\tau_1, \tau_2) \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) \right. \right.$$
$$\left. \left. + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \right) \right] .$$

G-Σ path integral

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2)\exp(-NS)$$

$$S = \ln\det\left[\delta(\tau_1-\tau_2)(\partial_{\tau_1}+\mu)-\Sigma(\tau_1,\tau_2)\right] + \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1,\tau_2)G(\tau_2,\tau_1)+(U^2/2)G^2(\tau_2,\tau_1)G^2(\tau_1,\tau_2)\right]$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$\tau = f(\sigma)$$

$$G(\tau_1,\tau_2) = \left[f'(\sigma_1)f'(\sigma_2)\right]^{-1/4}\frac{g(\sigma_1)}{g(\sigma_2)}G(\sigma_1,\sigma_2)$$

$$\Sigma(\tau_1,\tau_2) = \left[f'(\sigma_1)f'(\sigma_2)\right]^{-3/4}\frac{g(\sigma_1)}{g(\sigma_2)}\Sigma(\sigma_1,\sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

G-Σ path integral

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2)e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4}G_s(f(\tau_1) - f(\tau_2))e^{i\phi(\tau_1)-i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau)e^{-E_0/T+Ns_0-NS_{\rm eff}[f,\phi]},$$

where $E_0 \propto N$ is the ground state energy.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5,** 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Symmetries of the large N saddle point

Let us write the large N saddle point solutions of S as

$$\begin{aligned} G_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-1/2} \\ \Sigma_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-3/2}. \end{aligned}$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for Σ). It turns out this is true only for the SL(2, R) transformations under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

A. Kitaev

Symmetries of the large N saddle point

- The saddle-point

$$G(\tau_1 - \tau_2) = -A \frac{e^{-2\pi\mathcal{E}T(\tau_1-\tau_2)}}{\sqrt{1+e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T(\tau_1 - \tau_2))} \right)^{2\Delta}$$

is invariant only under PSL(2, R) transformations which map the thermal circle onto itself, and an associated gauge transformation

$$\frac{\tan(\pi T f(\tau))}{\pi T} = \frac{a \dfrac{\tan(\pi T \tau)}{\pi T} + b}{c \dfrac{\tan(\pi T \tau)}{\pi T} + d} \quad , \quad ad - bc = 1,$$

$$-i\phi(\tau) = -i\phi_0 + 2\pi\mathcal{E}T(\tau - f(\tau))$$

A. Kitaev, 2015

R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017)

G-Σ path integral

Symmetry arguments, and explicit computations, show that the effective action is

$$S_{\text{eff}}[f,\phi] = \frac{NK}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi\mathcal{E}T)\partial_\tau f)^2 - \frac{N\gamma}{4\pi^2}\int_0^{1/T} d\tau \,\{\tan(\pi T f(\tau)), \tau\},$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g,\tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\text{eff}}\left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, PRB **95**, 155131 (2017);
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746

Low temperature thermodynamics: for $k_B T \ll U$

$$\begin{aligned}\mathcal{Z} &= \mathrm{Tr}\exp\left(-\frac{\mathcal{H}}{k_B T}\right) \\ &= \exp\left(N\frac{S_0}{k_B}\right)\int\frac{\mathcal{D}f(\tau)\mathcal{D}\phi(\tau)}{||\mathrm{SL(2,R)}||}\exp\left(-\frac{1}{\hbar}S_{\text{eff}}\left[f(\tau),\phi(\tau)\right]\right)\end{aligned}$$

- Feynman path integral over $f(\tau)$, the reparameterization of the time of the SYK model, and $\phi(\tau)$ a phase conjugate to the total charge $\mathcal{Q}$.

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)
A. Kitaev (2015)
J. Maldacena and D. Stanford, Phys. Rev. D **94,** 106002 (2016)

The Schwarzian theory: boundary graviton (a time reparameterization soft-mode) in Einstein-Maxwell theory of charged black holes

SYK model	Charged black holes
$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \frac{3}{2}\ln\left(\frac{N^{1/3}U}{k_B T}\right) + \ldots$	$\frac{S(T)}{k_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2}\frac{k_B T}{\hbar}\right) - \frac{3}{2}\ln\left(\frac{c^2}{A_0^{1/6}(\hbar G)^{1/3}(k_B T/\hbar)}\right) + \ldots$
$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$	$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$
$\lim_{T\to 0}\frac{1}{N k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$	$\lim_{T\to 0}\frac{1}{k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$
$D(E) \sim \exp(N s_0)\sinh\left(\sqrt{2N\gamma E}\right)$	$D(E) \sim \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\sinh\left(\left[\sqrt{\pi} A_0^{3/2}\frac{c^3}{\hbar G}\frac{E}{\hbar c}\right]^{1/2}\right)$

Charged black holes

We consider a charged black hole in Einstein-Maxwell theory of g and a U(1) gauge flux $F = dA$

$$I_{EM} = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa^2}\mathcal{R}_4 + \frac{1}{4g_F^2}F^2\right] \quad , \quad \mathcal{Z}_{\mathcal{Q}} = \int \mathcal{D}g\mathcal{D}A\, \exp(-I_{EM} - I_{GH})\,.$$

The saddle-point equations now yield a solution as before with

$$V(r) = 1 + \frac{\Theta^2}{r^2} - \frac{m}{r} \quad ; \quad A_\tau = i\mu\left(1 - \frac{r_0}{r}\right) \quad ; \quad \Theta = \frac{\kappa r_0}{\sqrt{2}g_F}\mu \quad ; \quad \mathcal{Q} = \frac{4\pi\mu r_0}{g_F^2} \quad ; \quad S = \frac{2\pi\mathcal{A}}{\kappa^2}$$

where $\mathcal{Q}$ is the total charge, the chemical potential is μ, and as before the horizon is where $V(r_0) = 0$, the temperature $T = V'(r_0)/(4\pi)$, and $\mathcal{A} = 4\pi r_0^2$.

This defines a two parameter family of charged black hole solutions of I_{EM} determined by T and $\mathcal{Q}$.

SYK model	Charged black holes
$\frac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \frac{3}{2}\ln\left(\frac{N^{1/3}U}{k_B T}\right) + \ldots$	$\frac{S(T)}{k_B} = \frac{1}{\hbar G}\left(\frac{A_0 c^3}{4} + \frac{\sqrt{\pi} A_0^{3/2} c^2}{2}\frac{k_B T}{\hbar}\right) - \frac{3}{2}\ln\left(\frac{c^2}{A_0^{1/6}(\hbar G)^{1/3}(k_B T/\hbar)}\right) + \ldots$
$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$	$G(\tau) \sim e^{-2\pi\mathcal{E}T\tau}\left(\frac{T}{\sin(\pi T\tau)}\right)^{2\Delta}$
$\lim_{T\to 0}\frac{1}{N k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$	$\lim_{T\to 0}\frac{1}{k_B}\frac{\partial S}{\partial Q} = 2\pi\mathcal{E}$
$D(E) \sim \exp(N s_0)\sinh\left(\sqrt{2N\gamma E}\right)$	$D(E) \sim \exp\left(\frac{A_0 c^3}{4\hbar G}\right)\sinh\left(\left[\sqrt{\pi} A_0^{3/2}\frac{c^3}{\hbar G}\frac{E}{\hbar c}\right]^{1/2}\right)$

Charged black holes

Now we take the limit $T \to 0$ at fixed $\mathcal{Q}$. Then we find the remarkable feature that the horizon radius remains finite

$$R_h \equiv r_0(T \to 0, \mathcal{Q}) = \frac{\mathcal{Q}\kappa g_F}{4\pi}$$

In this limit, entropy becomes

$$S(T \to 0, \mathcal{Q}) = \frac{4\pi R_h^2}{G_N} + \gamma T \quad , \quad \gamma \equiv \frac{4\pi^2 R_h^3}{G_N}$$

For the near-horizon metric, it is useful to introduce the co-ordinate ζ

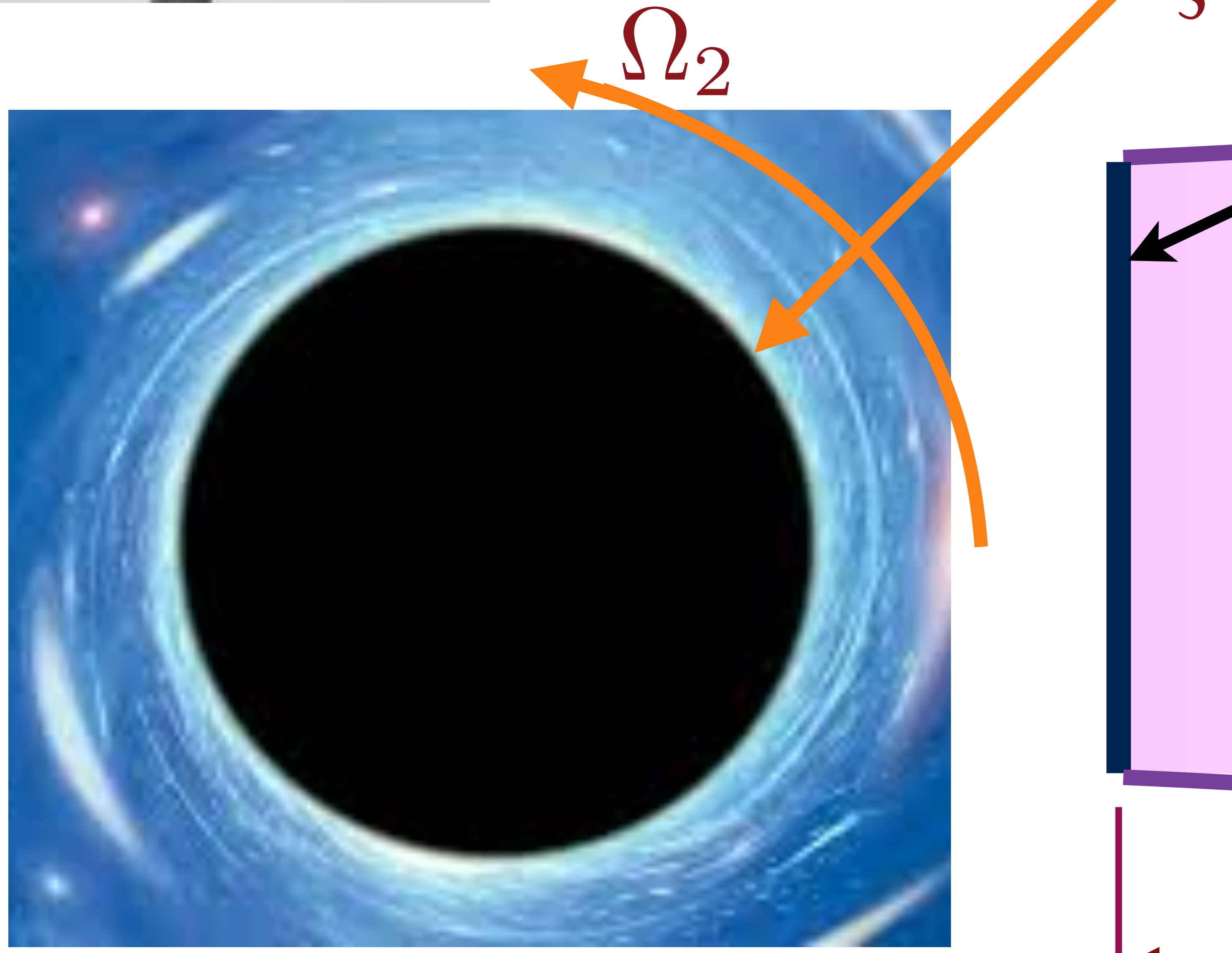
$$r = R_h + \frac{R_h^2}{\zeta}$$

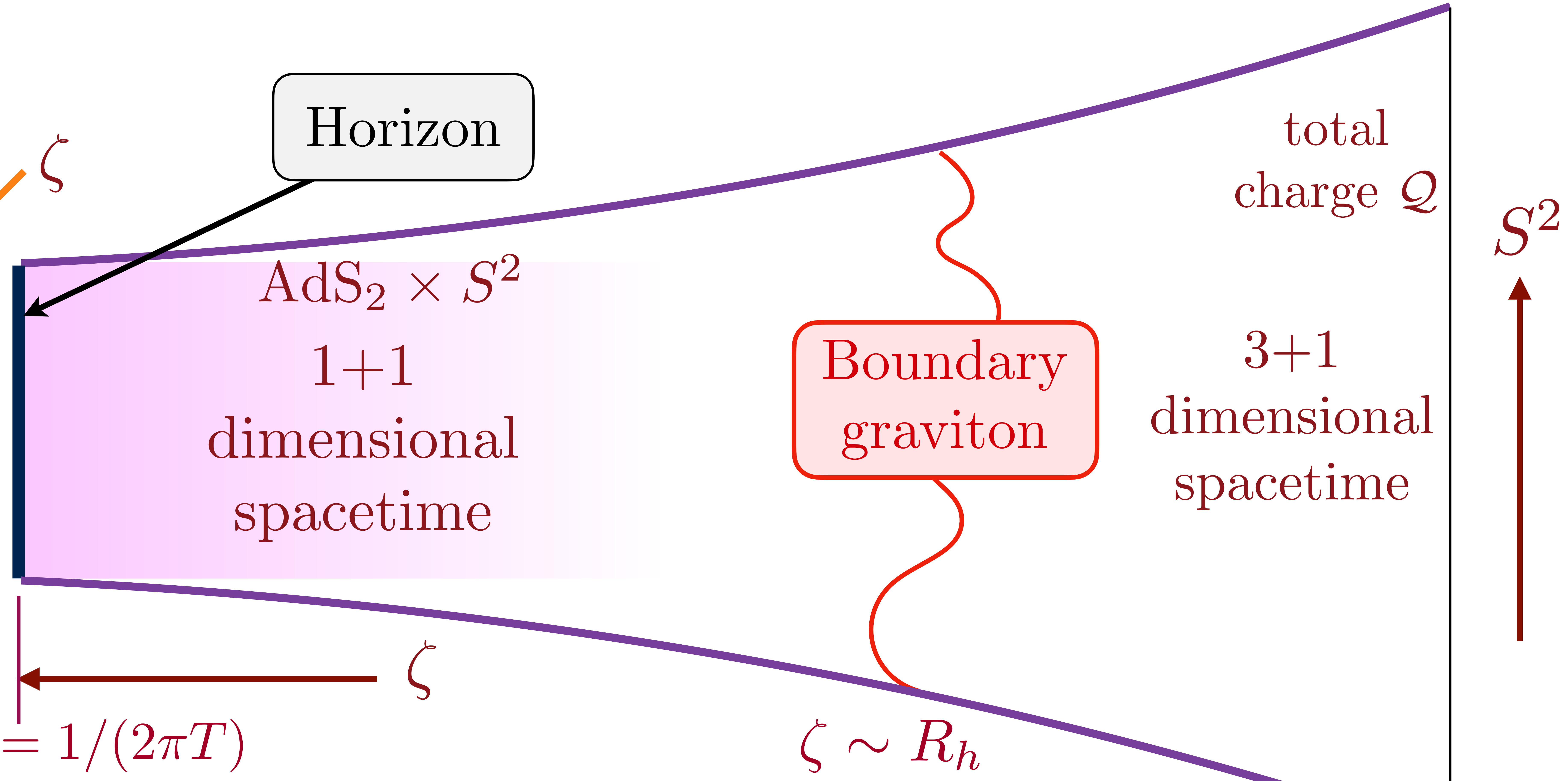
so that the horizon at $T = 0$ is at $\zeta = \infty$. Then in the near-horizon regime $R_h \ll \zeta < \infty$ the $T = 0$ metric is

$$ds^2 = R_h^2 \frac{d\tau^2 + d\zeta^2}{\zeta^2} + R_h^2 d\Omega_2^2$$

This spacetime is $\mathrm{AdS}_2 \times S^2$.

Reissner-Nordstrom black hole of Einstein-Maxwell theory





Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!

Quantum path integral for charged black holes

1. Reduce the 4-spacetime dimensional theory in I_{EM} to a 1+1 dimensional theory $I_{EM,2}$ by taking all fields dependent only upon the radial co-ordinate r and imaginary time τ.

2. Take the low energy limit of $I_{EM,2}$ by mapping it to a near-horizon theory, I_{JT}, in a 1+1 dimensional spacetime with a boundary.

3. Compute fluctuations about the AdS_2 saddle point of I_{JT}. Einstein gravity in 1+1 dimensions has no graviton, and is 'pure gauge'. In the JT-gravity theory with boundary, there is a remnant degree of freedom which is a boundary graviton. The action for this boundary graviton is the Schwarzian theory. The partition function of this Schwarzian theory can be evaluated exactly.

Quantum path integral for charged black holes

1. Make the metric ansatz

$$ds^2 = \frac{ds_2^2}{\Phi(\zeta,\tau)} + [\Phi(\zeta,\tau)]^2 \, d\Omega_2^2$$

where ds_2^2 is an arbitrary metric in the (ζ,τ) spacetime, and Φ is a scalar field in the (ζ,τ) spacetime.

2. The low energy theory on the (ζ,τ) spacetime involves a metric h, and a scalar field Φ_1 given by $\lim_{\zeta\to\infty}[\Phi(\zeta,\tau)]^2 = R_h^2 + \Phi_1(\zeta,\tau)$, obeying the action

$$I_{JT} = -\frac{2\pi\mathcal{A}_0}{\kappa^2} + \int d^2x\sqrt{h}\left[-\frac{2\pi}{\kappa^2}\,\Phi_1\left(\mathcal{R}_2 + \frac{2}{R_h^3}\right)\right] - \frac{4\pi}{\kappa^2}\int_\partial dx\sqrt{h_b}\Phi_1\,\mathcal{K}_1$$

where $\mathcal{A}_0 = 4\pi R_h^2$ is the area of the horizon at $T=0$, and $\mathcal{K}_1$ is the extrinsic curvature of the one-dimensional boundary $\zeta \to 0$ where

$$h_{\tau\tau}(\zeta\to 0) = \frac{R_h^3}{\zeta^2} \quad , \quad \Phi_1(\zeta\to 0) = \frac{2R_h^3}{\zeta}$$

Quantum path integral for charged black holes

3. Remarkably, the partition function of the $1+1$ dimensional JT gravity theory can be evaluated exactly (here we are ignoring the gauge field path integral, which is subdominant at fixed $\mathcal{Q}$)

$$\mathcal{Z}_{\mathcal{Q}} = \int \mathcal{D}h\mathcal{D}\Phi_1 \exp\left(-I_{JT}\right)$$

The action is linear in Φ_1, and the integral over Φ_1 yields a constraint $\mathcal{R}_2 = -2/R_h^3$ *i.e.* the metric h is rigidly AdS_2. The only dynamical degree of freedom in JT gravity is a time reparameterization along the boundary $\tau \to f(\tau)$. To ensure that the bulk metric obeys its boundary condition, we also have to make the spatial co-ordinate ζ a function of τ, so we map $(\tau, \zeta) \to (f(\tau), \zeta(\tau))$. Then the metric obeys its boundary condition provided $\zeta(\tau)$ is related to $f(\tau)$ by (here ζ_b is a small constant whose value cancels in the final result)

$$\zeta(\tau) = \zeta_b f'(\tau) + \zeta_b^3 \frac{\left[f''(\tau)\right]^2}{2f'(\tau)} + \mathcal{O}(\zeta_b^4)$$

Finally, we evaluate I_{GH} along this boundary curve. In this manner we obtain the action

$$I_{1,\text{eff}}[f] = -\frac{2\pi\mathcal{A}_0}{\kappa^2} - \frac{\gamma}{4\pi^2}\int d\tau \left\{f(\tau), \tau\right\} \quad , \quad \{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

where $\gamma = 32\pi^3 R_h^3/\kappa^2$ is precisely the linear-T co-efficient in the black hole entropy.

Quantum path integral for charged black holes

3. After a conformal map to finite temperature (and ignoring the contribution of the gauge field fluctuation), we can write the low energy partition function of a 3+1-dimensional black hole with charge $\mathcal{Q} = 4\pi R_h/(\kappa g_F)$, as a path integral over a single field $f(\tau)$ in one time dimension:

$$\mathcal{Z}_{\mathcal{Q}} = \exp\left(\frac{2\pi\mathcal{A}_0}{\kappa^2}\right)\int \frac{\mathcal{D}f}{||\text{SL(2,R)}||}\exp\left(\frac{\gamma}{4\pi^2}\int_0^{1/T} d\tau\,\{\tan(\pi T f(\tau)),\tau\}\right)$$

where $\gamma = 32\pi^3 R_h^3/\kappa^2$, $\mathcal{A}_0 = 4\pi R_h^2$, and $f(\tau)$ is a monotonic function of τ obeying

$$f(\tau + 1/T) = f(\tau) + 1/T\,.$$

We divide by the (infinite) volume of the SL(2,R) group because

$$\{f,\tau\} = \{\frac{af+b}{cf+d},\tau\}$$

where a, b, c, d are constants with $ad - bc = 1$.

The Schwarzian theory: linear-T resistivity in the random *t*-*J* model

Random *t-J* model

$$H = -\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} t_{ij}\, c_{i\alpha}^\dagger c_{j\alpha} + \frac{1}{\sqrt{N}} \sum_{i<j=1}^{N} J_{ij}\, \vec{S}_i \cdot \vec{S}_j$$

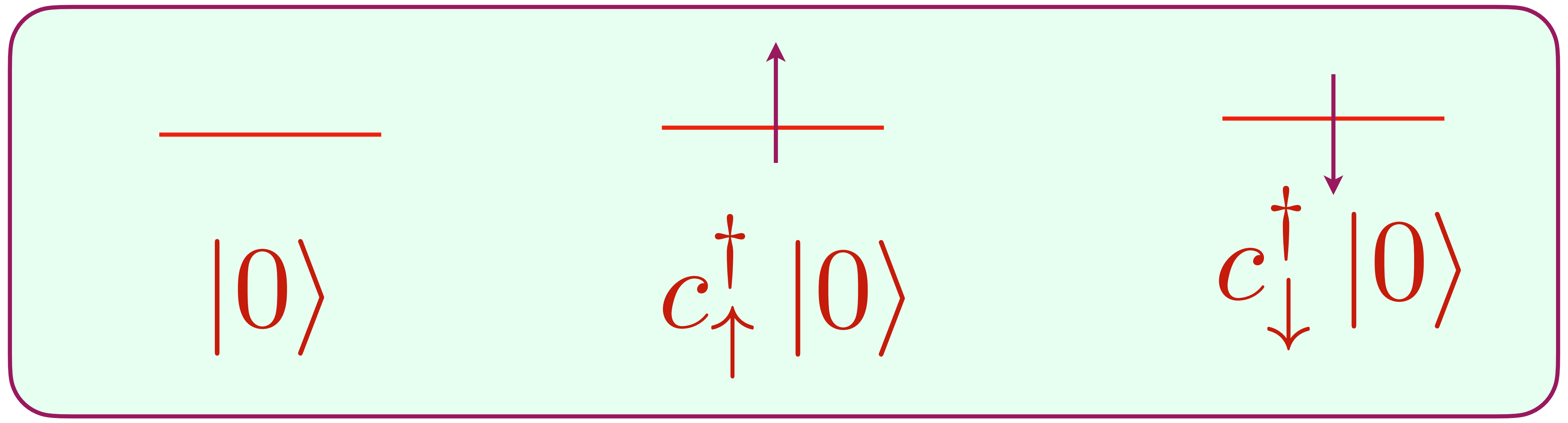
We consider the hole-doped case, with no double occupancy.

$$\alpha = \uparrow, \downarrow, \quad \{c_{i\alpha}, c_{j\beta}^\dagger\} = \delta_{ij}\delta_{\alpha\beta}, \quad \{c_{i\alpha}, c_{j\beta}\} = 0$$

$$\vec{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{i\beta}, \quad \sum_{\alpha} c_{i\alpha}^\dagger c_{i\alpha} \leq 1, \quad \frac{1}{N}\sum_{i\alpha} c_{i\alpha}^\dagger c_{i\alpha} = 1 - p$$

$$J_{ij} \text{ random}, \ \overline{J_{ij}} = 0, \ \overline{J_{ij}^2} = J^2$$

$$t_{ij} \text{ random}, \ \overline{t_{ij}} = 0, \ \overline{t_{ij}^2} = t^2$$

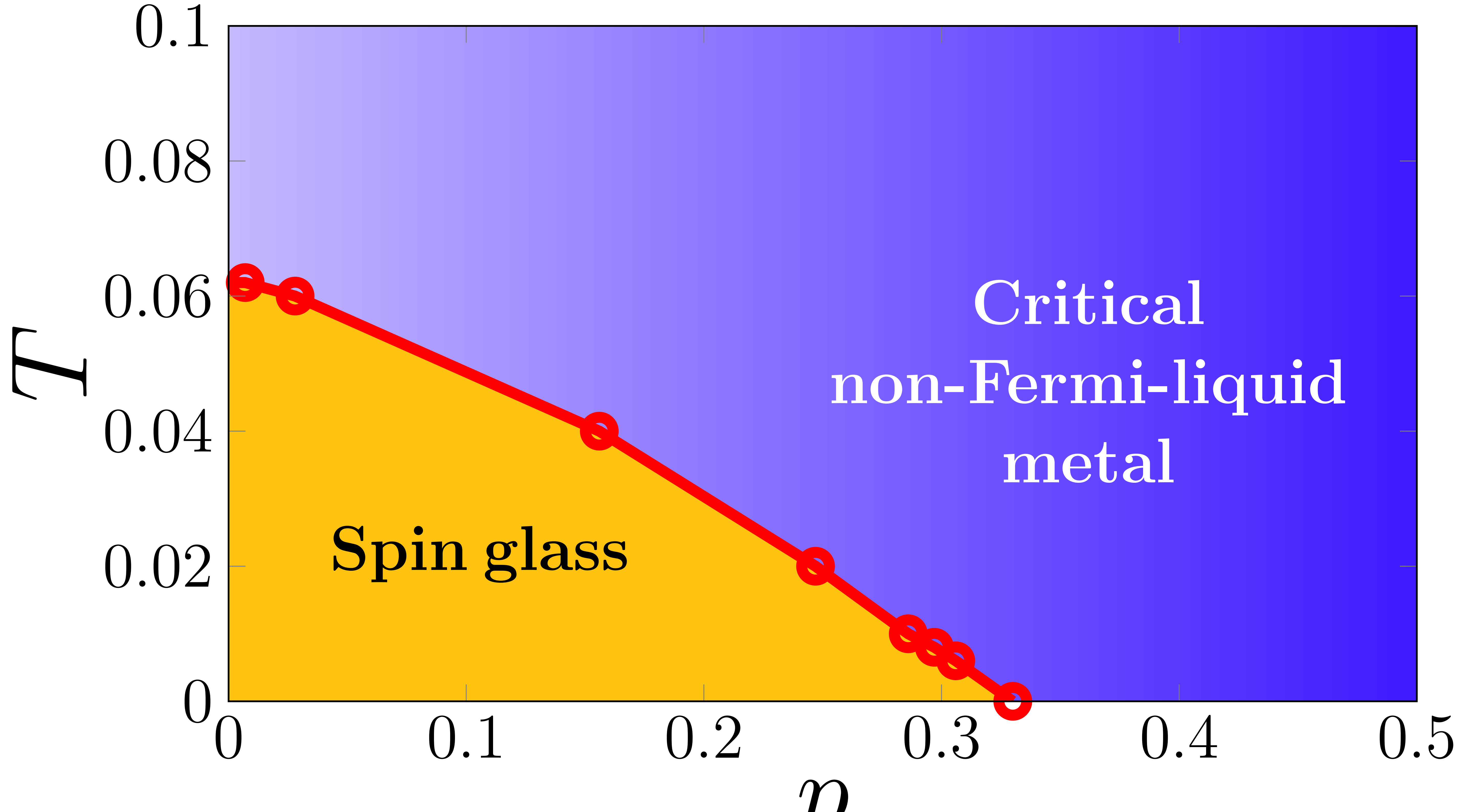


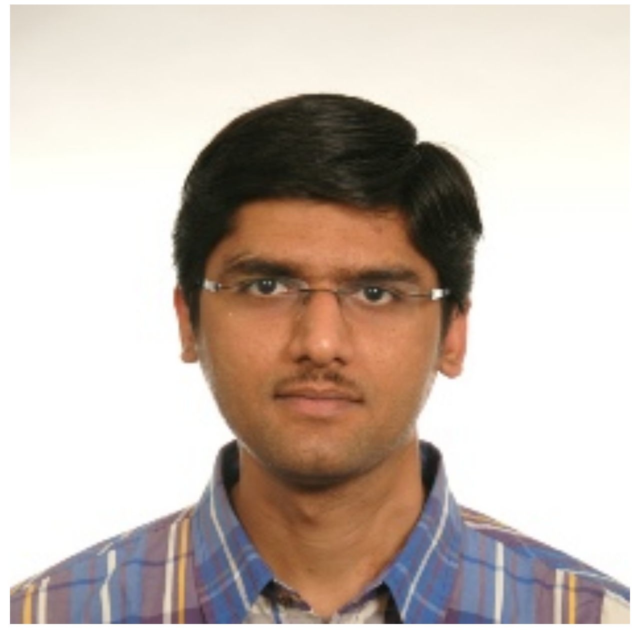
Random *t*-*J* model

Solvable in a SYK-like large M and M' limit after fractionalizing $c_\alpha = f_\alpha b^\dagger$ into fermionic spinons and bosonic holons ($c_{a\alpha} = f_\alpha b_a^\dagger$, $\alpha = 1 \ldots M$, $a = 1 \ldots M'$)

$$G_b(i\omega_n) = \frac{1}{i\omega_n + \mu_b - \Sigma_b(i\omega_n)}, \quad G_f(i\omega_n) = \frac{1}{i\omega_n + \mu_f - \Sigma_f(i\omega_n)}$$

$$\Sigma_b(\tau) = -t^2 G_f(\tau) G_f(-\tau) G_b(\tau), \quad \Sigma_f(\tau) = -J^2 G_f(\tau)^2 G_f(-\tau) + k t^2 G_f(\tau) G_b(\tau) G_b(-\tau)$$





M. Christos,
D. G. Joshi,
S. S. and
M. Tikhanovskaya,
arXiv:2203.16548

Random *t-J* model

Solvable in a SYK-like large M and M' limit after fractionalizing
$c_\alpha = f_\alpha b^\dagger$ into fermionic spinons and bosonic holons ($c_{a\alpha} = f_\alpha b_a^\dagger$, $\alpha = 1 \dots M$, $a = 1 \dots M'$)
or $c_\alpha = \mathfrak{b}_\alpha \mathfrak{f}^\dagger$ into bosonic spinons and fermionic holons ($c_{a\alpha} = \mathfrak{b}_\alpha \mathfrak{f}_a^\dagger$, $\alpha = 1 \dots M$, $a = 1 \dots M'$)

Metallic spin glass

Condense spinon $\mathfrak{b}_\alpha$.

$$\langle \boldsymbol{S}(\tau) \cdot \boldsymbol{S}(0) \rangle \sim \text{constant}$$

Critical metal

$$\text{Holon: } \left\langle b(\tau) b^\dagger(0) \right\rangle \sim \frac{1}{\tau^{2\Delta_b}}$$

$$\text{Spinon: } \left\langle f_\alpha(\tau) f_\alpha^\dagger(0) \right\rangle \sim \frac{1}{\tau^{2\Delta_f}}$$

$$\Delta_b + \Delta_f = 1/2, \quad 0 < \Delta_b < 1/4.$$

$$\langle \boldsymbol{S}(\tau) \cdot \boldsymbol{S}(0) \rangle \sim \frac{1}{\tau^{4\Delta_f}}$$

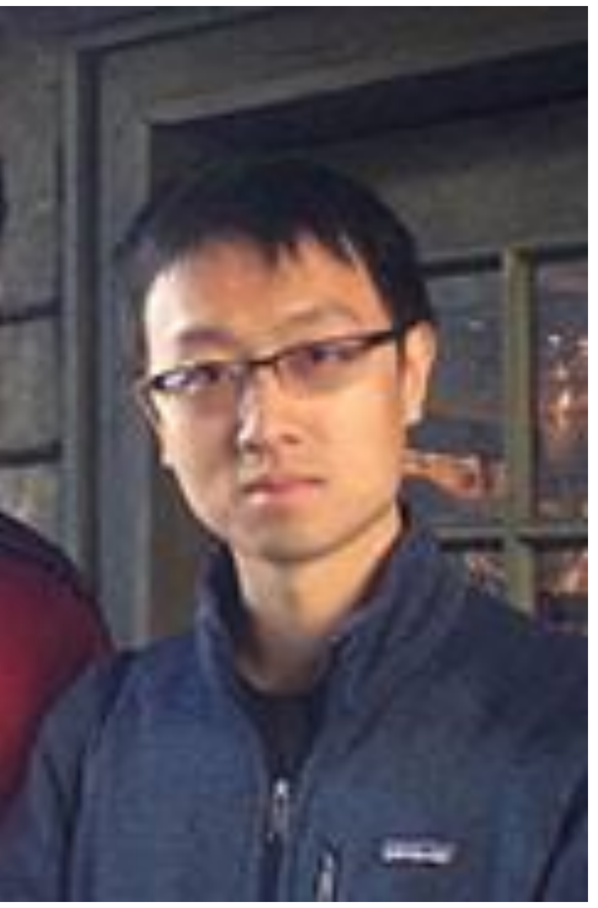
$$\left\langle c_\alpha(\tau) c_\alpha^\dagger(0) \right\rangle \sim \frac{1}{\tau}$$

p_c $\qquad$ p

M. Christos, D. G. Joshi, S. S. and M. Tikhanovskaya, arXiv:2203.16548

Random *t-J* model

Solvable in a SYK-like large M and M' limit after fractionalizing $c_\alpha = f_\alpha b^\dagger$ into fermionic spinons and bosonic holons ($c_{a\alpha} = f_\alpha b_a^\dagger$, $\alpha = 1 \ldots M$, $a = 1 \ldots M'$)



Haoyu Guo,
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S. Sachdev,
Annals of Physics **418**, 168202 (2020)

The $h = 2$ operator now leads to corrections to the Green's functions of the partons

$$G_{f,b}(\tau) \sim \frac{\pm 1}{|\tau|^{2\Delta_{f,b}}} \left(1 + \frac{\alpha_{f,b}}{|\tau|} + \ldots \right)$$

We can compute the resistivity from this in a large-d model, and find as $T \to 0$ that

$$\rho(T) = \rho(0) \left(1 + \alpha_\rho \frac{T}{J} + \ldots \right).$$

The linear-T term arises from the $h = 2$ operator, which we will see is a 'time reparameterization soft-mode', and a 'boundary graviton' in the charged black hole.

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D. G. Joshi,
S. S. and
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arXiv:2203.16548

SYK model	Charged black holes
$\dfrac{S(T)}{k_B} = N(s_0 + \gamma\, k_B T) - \dfrac{3}{2} \ln\left(\dfrac{N^{1/3} U}{k_B T}\right) + \ldots$	$\dfrac{S(T)}{k_B} = \dfrac{1}{\hbar G}\left(\dfrac{A_0 c^3}{4} + \dfrac{\sqrt{\pi} A_0^{3/2} c^2}{2} \dfrac{k_B T}{\hbar}\right) - \dfrac{3}{2} \ln\left(\dfrac{c^2}{A_0^{1/6} (\hbar G)^{1/3} (k_B T/\hbar)}\right) + \ldots$
$G(\tau) \sim e^{-2\pi \mathcal{E} T \tau}\left(\dfrac{T}{\sin(\pi T \tau)}\right)^{2\Delta}$	$G(\tau) \sim e^{-2\pi \mathcal{E} T \tau}\left(\dfrac{T}{\sin(\pi T \tau)}\right)^{2\Delta}$
$\lim\limits_{T \to 0} \dfrac{1}{N k_B} \dfrac{\partial S}{\partial Q} = 2\pi \mathcal{E}$	$\lim\limits_{T \to 0} \dfrac{1}{k_B} \dfrac{\partial S}{\partial Q} = 2\pi \mathcal{E}$
$D(E) \sim \exp(N s_0) \sinh\left(\sqrt{2 N \gamma E}\right)$	$D(E) \sim \exp\left(\dfrac{A_0 c^3}{4 \hbar G}\right) \sinh\left(\left[\sqrt{\pi} A_0^{3/2} \dfrac{c^3}{\hbar G} \dfrac{E}{\hbar c}\right]^{1/2}\right)$

Large d t-J model with random J_{ij} and SU$(M \to \infty)$ symmetry: resistivity $\rho(T) = \rho(0)\left[1 + \alpha_\rho\,(T/J) + \ldots\right]$ in a critical metal phase as $T \to 0$.

Fermions at non-zero density coupled to a critical boson

Yukawa-SYK models

$$H = \sum_{ij} t_{ij} \psi_i^\dagger \psi_j + \sum_{\ell} \varepsilon_\ell b_\ell^\dagger b_\ell + \sum_{ij\ell} g_{ij\ell} \psi_i^\dagger \psi_j (b_\ell + b_\ell^\dagger)$$

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