Statistical mechanics of strange metals and black holes

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Black hole thermodynamics

The Einstein action for gravity in 3+1 dimensions is

$$I_E = \int d^4x \sqrt{g} \left[-\frac{1}{2\kappa^2} \mathcal{R}_4 \right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g \exp(-I_E) \, ,$$

where $\kappa^2 = 8\pi G_N$ is the gravitational constant, $\mathcal{R}_4$ is the Ricci scalar. The Schwarzschild solution of the saddle-point equations is

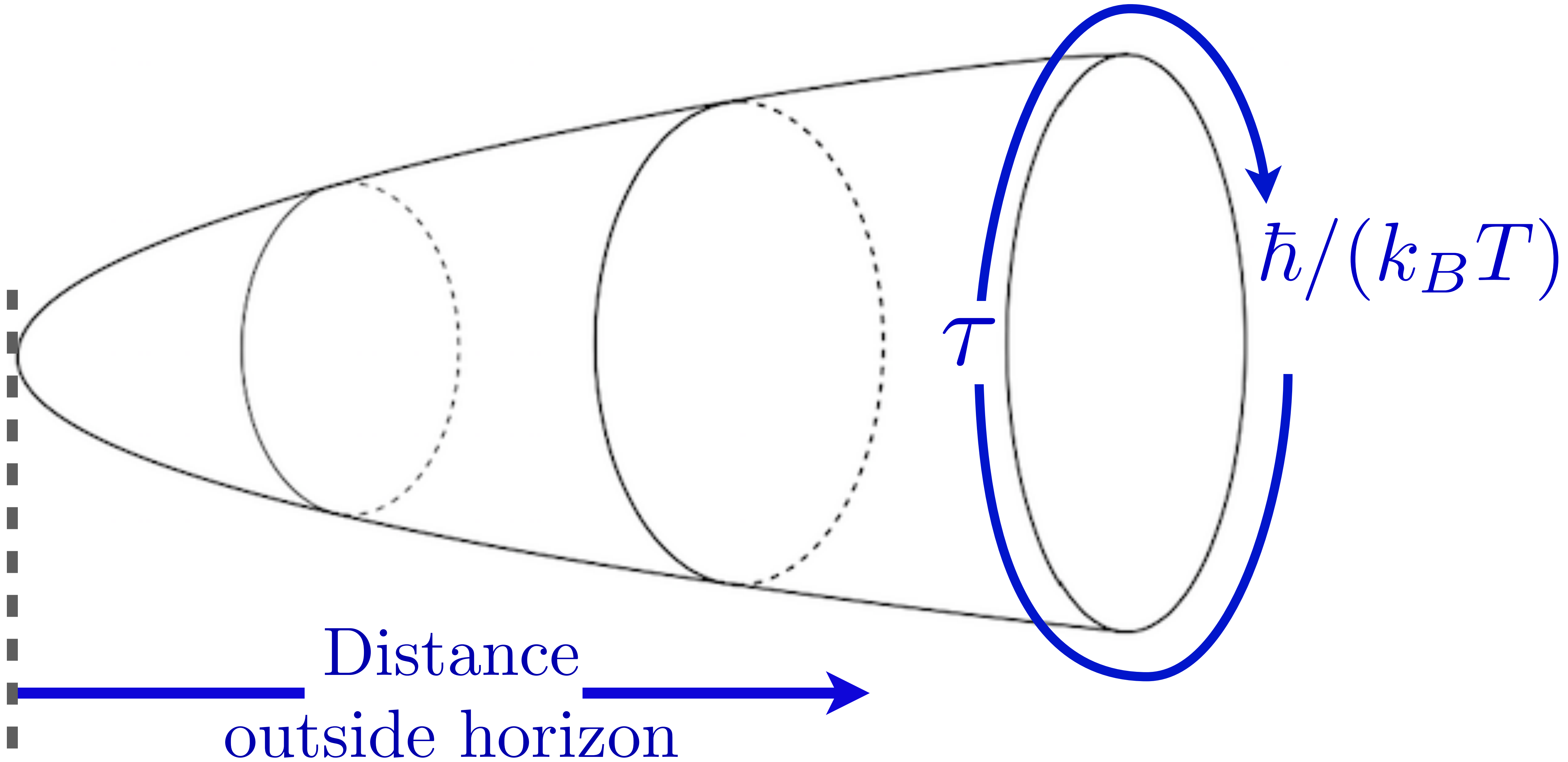
$$ds^2 = V(r) d\tau^2 + r^2 d\Omega_2^2 + \frac{dr^2}{V(r)}$$

where $d\Omega_2^2$ is the metric of the 2-sphere, and

$$V(r) = 1 - \frac{m}{r} .$$

The gravitational mass of the black hole is $M = 2G_N m$. The black hole horizon is at $r = r_0$ where $V(r_0) = 0$; so

$$r_0 = m$$



The $T > 0$ quantum partition function is obtained in a spacetime which is periodic as a function of τ with period $\hbar/(k_B T)$. We have to ensure that there is no singularity at the horizon r_0 where $V(r_0) = 0$. Let us change radial co-ordinates to y, where $r = r_0 + y^2$. Then for small y

$$ds^2 = \frac{4}{V'(r_0)} \left[\frac{(V'(r_0))^2}{4} y^2 d\tau^2 + dy^2 \right] + r_0^2 d\Omega_2^2 = \frac{4}{V'(r_0)} \left[y^2 d\theta^2 + dy^2 \right] + r_0^2 d\Omega_2^2$$

The expression in the square brackets is the metric of the flat plane in polar co-ordinates, with radial co-ordinate y and angular co-ordinate $\theta = V'(r_0)\tau/2$. Smoothness requires periodicity in θ with period 2π, and so

$$4\pi T = V'(r_0) = \frac{1}{m} \,.$$

The free energy $\beta F = I_E$, where $\beta = 1/T$. So the entropy is

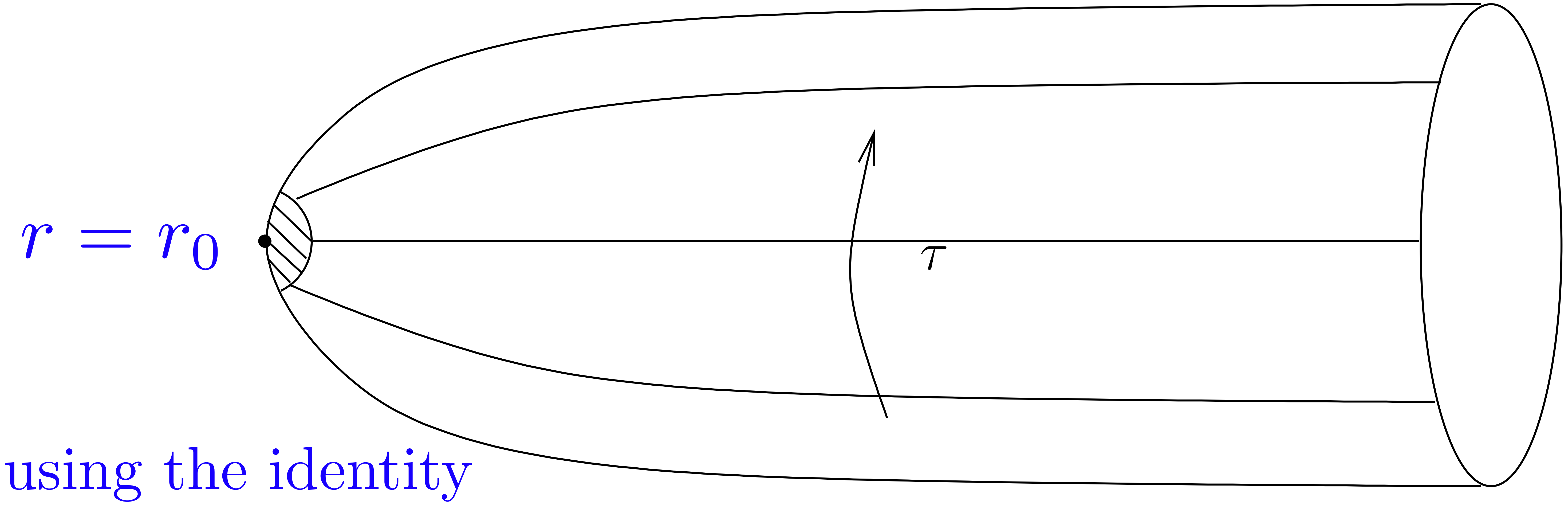
$$S = -\frac{\partial F}{\partial T} = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I_E$$

However, the metric is τ-independent, and the only explicit dependence of the action is via $I_E = \beta H$. Such an action implies $S = 0$.

The entire contribution to the entropy comes from the vicinity of the co-ordinate singularity at $r = r_0$. We evaluate the action is the small region around this point

$$I_{\rm grav} = I_E + I_{GH} \quad , \quad I_{GH} = \int_\partial d^3x \sqrt{g_b} \left[-\frac{1}{\kappa^2} \mathcal{K}_3 \right] \quad , \quad \mathcal{Z} = \int \mathcal{D}g \exp(-I_{\rm grav}) \, ,$$

where $\mathcal{K}_3$ is the extrinsic scalar curvature of the 3-dimensional boundary of spacetime. I_{GH} is the Gibbons-Hawking boundary term, deduced by the requirement that the Euler-Lagrange equations of $I_{\rm grav}$ co-incide with the Einstein equations, with no additional boundary terms. The entire contribution to the entropy will come from I_{GH}.



We evaluate I_{GH} by using the identity

$$\int_\partial d^3x\sqrt{g_b}\,\mathcal{K}_3 = \frac{\partial}{\partial n}\int_\partial d^3x\sqrt{g_b}$$

where n is the Gaussian normal co-ordinate of the boundary. Evaluating at $y = \epsilon$, we have

$$\int_\partial d^3x\sqrt{g_b} = 2\pi\epsilon\mathcal{A}$$

where $\mathcal{A} = 4\pi r_0^2$ is the area of the horizon. Combining everything, we have the famous result of Hawking

$$S = \frac{2\pi\mathcal{A}}{\kappa^2} = \frac{\mathcal{A}}{4G_N}.$$

Charged black holes

We consider a charged black hole in Einstein-Maxwell theory of g and a U(1) gauge flux $F = dA$

$$I_{EM} = \int d^4x \sqrt{g} \left[-\frac{1}{2\kappa^2} \mathcal{R}_4 + \frac{1}{4g_F^2} F^2 \right] \quad , \quad \mathcal{Z}_{\mathcal{Q}} = \int \mathcal{D}g \mathcal{D}A \exp(-I_{EM} - I_{GH}) \,.$$

The saddle-point equations now yield a solution as before with

$$V(r) = 1 + \frac{\Theta^2}{r^2} - \frac{m}{r} \quad ; \quad A_\tau = i\mu \left(1 - \frac{r_0}{r} \right) \quad ; \quad \Theta = \frac{\kappa r_0}{\sqrt{2} g_F} \mu \quad ; \quad \mathcal{Q} = \frac{4\pi \mu r_0}{g_F^2} \quad ; \quad S = \frac{2\pi \mathcal{A}}{\kappa^2}$$

where $\mathcal{Q}$ is the total charge, the chemical potential is μ, and as before the horizon is where $V(r_0) = 0$, the temperature $T = V'(r_0)/(4\pi)$, and $\mathcal{A} = 4\pi r_0^2$.

This defines a two parameter family of charged black hole solutions of I_{EM} determined by T and $\mathcal{Q}$.

Charged black holes

Now we take the limit $T \to 0$ at fixed $\mathcal{Q}$. Then we find the remarkable feature that the horizon radius remains finite

$$R_h \equiv r_0(T \to 0, \mathcal{Q}) = \frac{\mathcal{Q}\kappa g_F}{4\pi}$$

In this limit, entropy becomes

$$S(T \to 0, \mathcal{Q}) = \frac{4\pi R_h^2}{G_N} + \gamma T \quad , \quad \gamma \equiv \frac{4\pi^2 R_h^3}{G_N}$$

For the near-horizon metric, it is useful to introduce the co-ordinate ζ

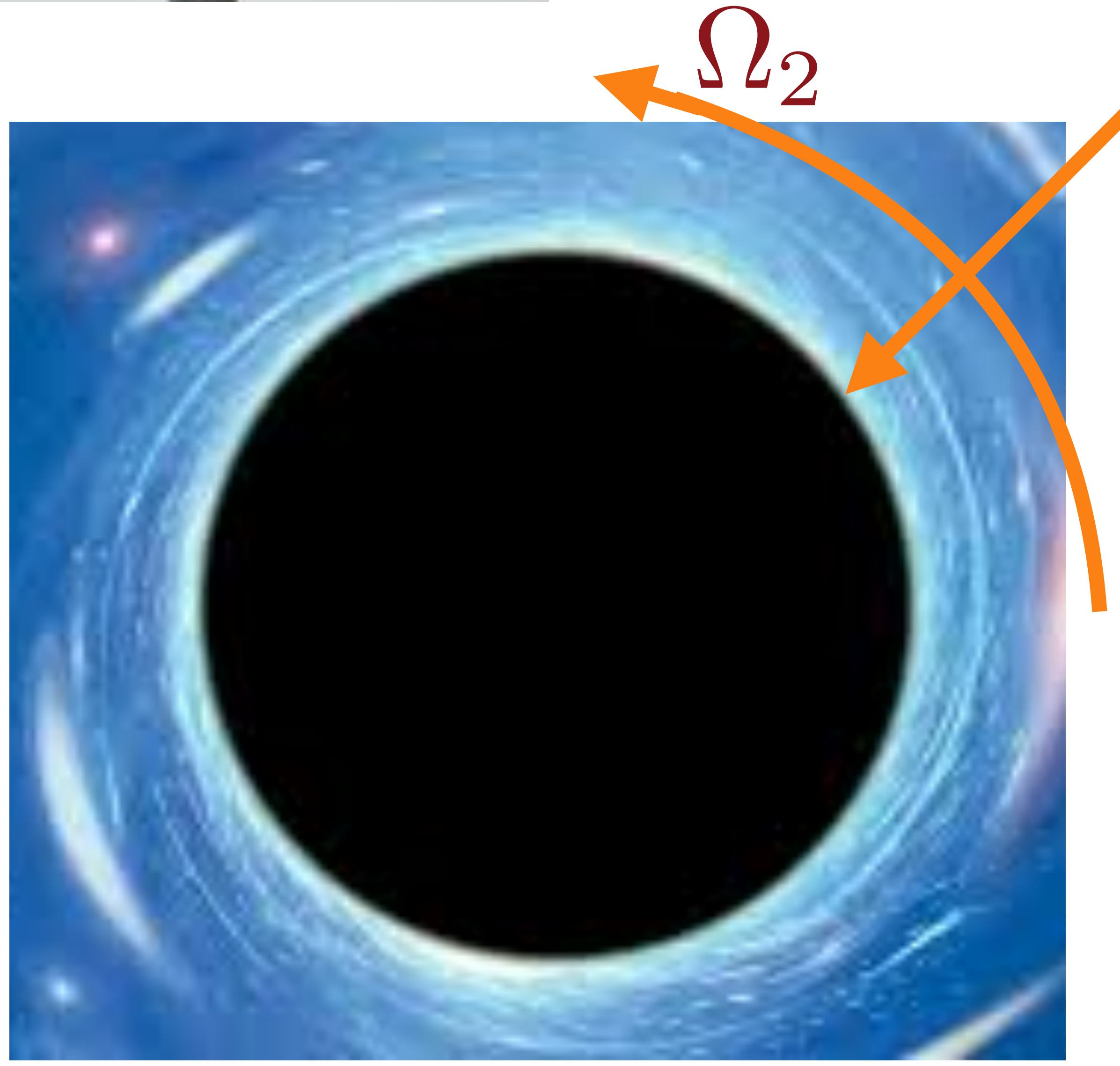
$$r = R_h + \frac{R_h^2}{\zeta}$$

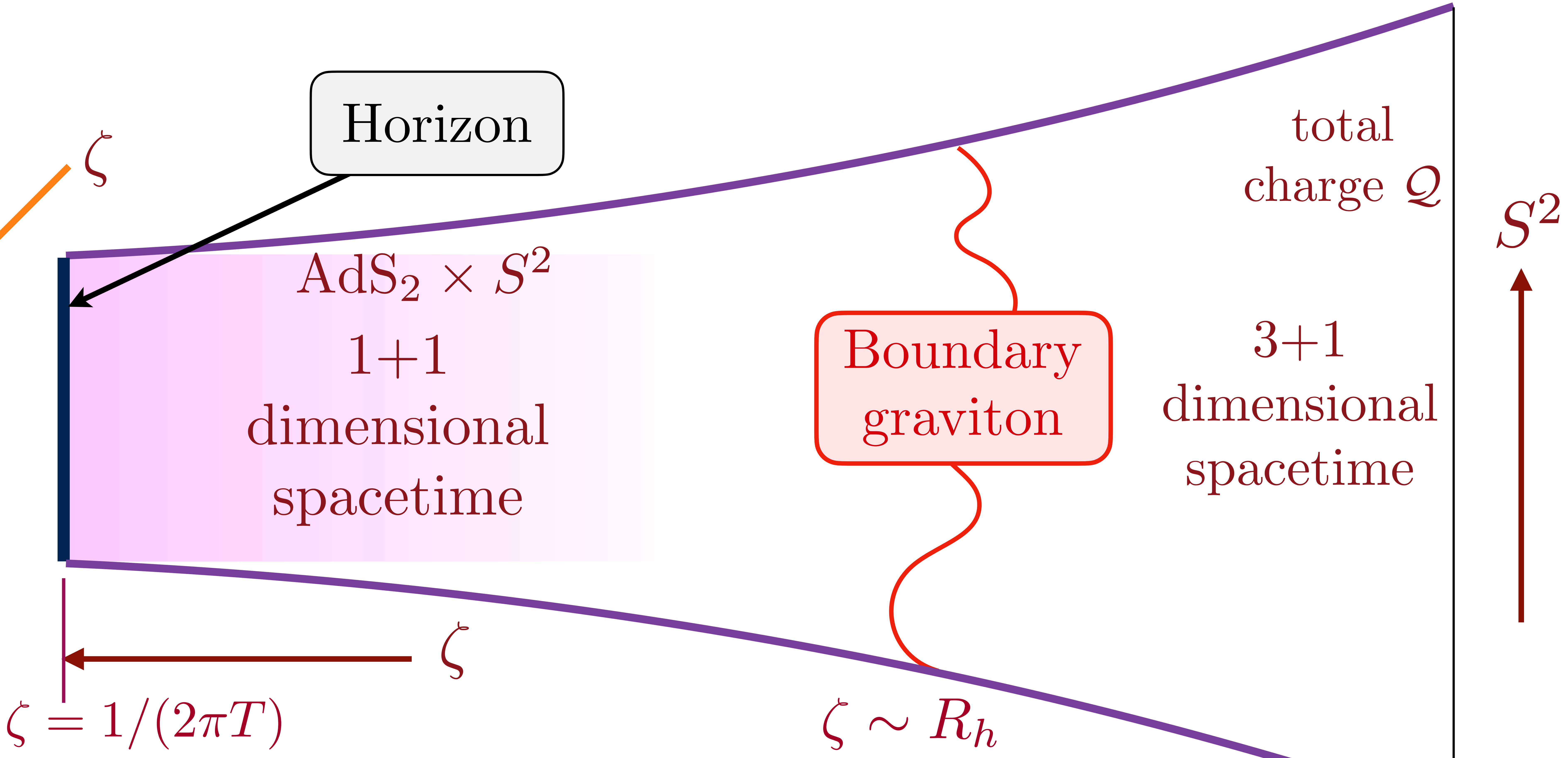
so that the horizon at $T = 0$ is at $\zeta = \infty$. Then in the near-horizon regime $R_h \ll \zeta < \infty$ the $T = 0$ metric is

$$ds^2 = R_h^2 \frac{d\tau^2 + d\zeta^2}{\zeta^2} + R_h^2 d\Omega_2^2$$

This spacetime is $\mathrm{AdS}_2 \times S^2$.

Reissner-Nordstrom black hole of Einstein-Maxwell theory





Dimensional reduction from 3+1 dimensions to 1+1 dimensions (AdS_2) at low energies!

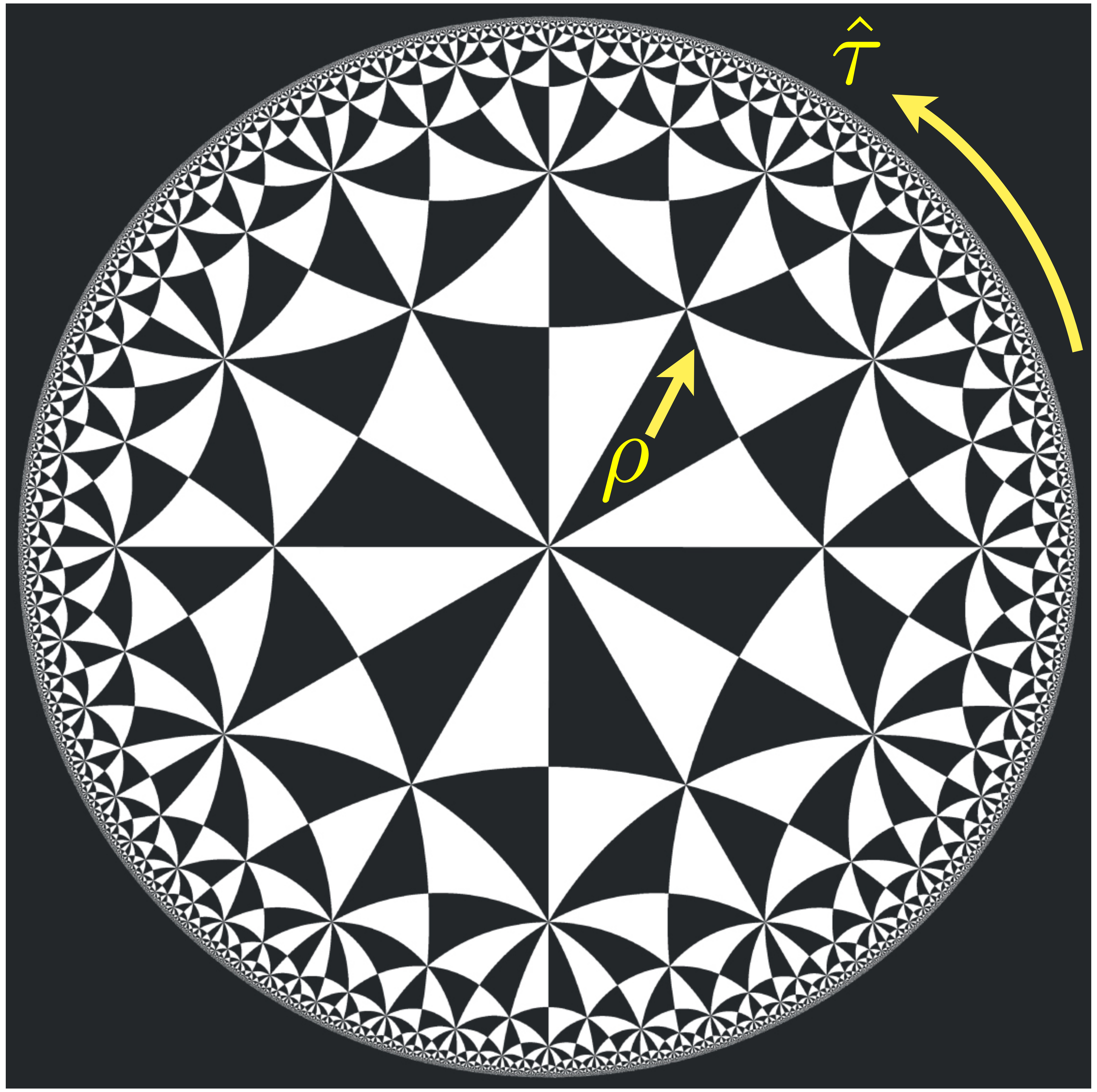
The AdS_2 metric

$$ds^2 = \frac{d\tau^2 + d\zeta^2}{\zeta^2}$$

is invariant under isometries which are SL(2,R) transformations. Verify that the co-ordinate change

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d}, \quad ad - bc = 1,$$

with a,b,c,d real, leaves the AdS_2 metric invariant.



The co-ordinate transformation

$$\zeta = \frac{1}{\cosh(2\pi T\rho) - \sinh(2\pi T\rho)\cos(2\pi T\hat{\tau})} \quad , \quad \tau = \frac{\sinh(2\pi T\rho)\sin(2\pi T\hat{\tau})}{\cosh(2\pi T\rho) - \sinh(2\pi T\rho)\cos(2\pi T\hat{\tau})}$$

maps the metric to

$$ds^2 = 4\pi^2 T^2 \left[d\rho^2 + \sinh^2(2\pi T\rho) d\hat{\tau}^2\right]$$