Strong coupling problems in condensed matter and the AdS/CFT correspondence

Reviews:

arXiv:0910.1139
arXiv:0901.4103

Talk online: sachdev.physics.harvard.edu

PHYSICS
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Max Metlitski, Harvard

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1. Quantum-critical transport

   Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

   Constraints from duality relations

3. Quantum criticality of Dirac fermions

   "Vector" 1/N expansion

4. Quantum criticality of Fermi surfaces

   The genus expansion

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The Superfluid-Insulator transition

Boson Hubbard model

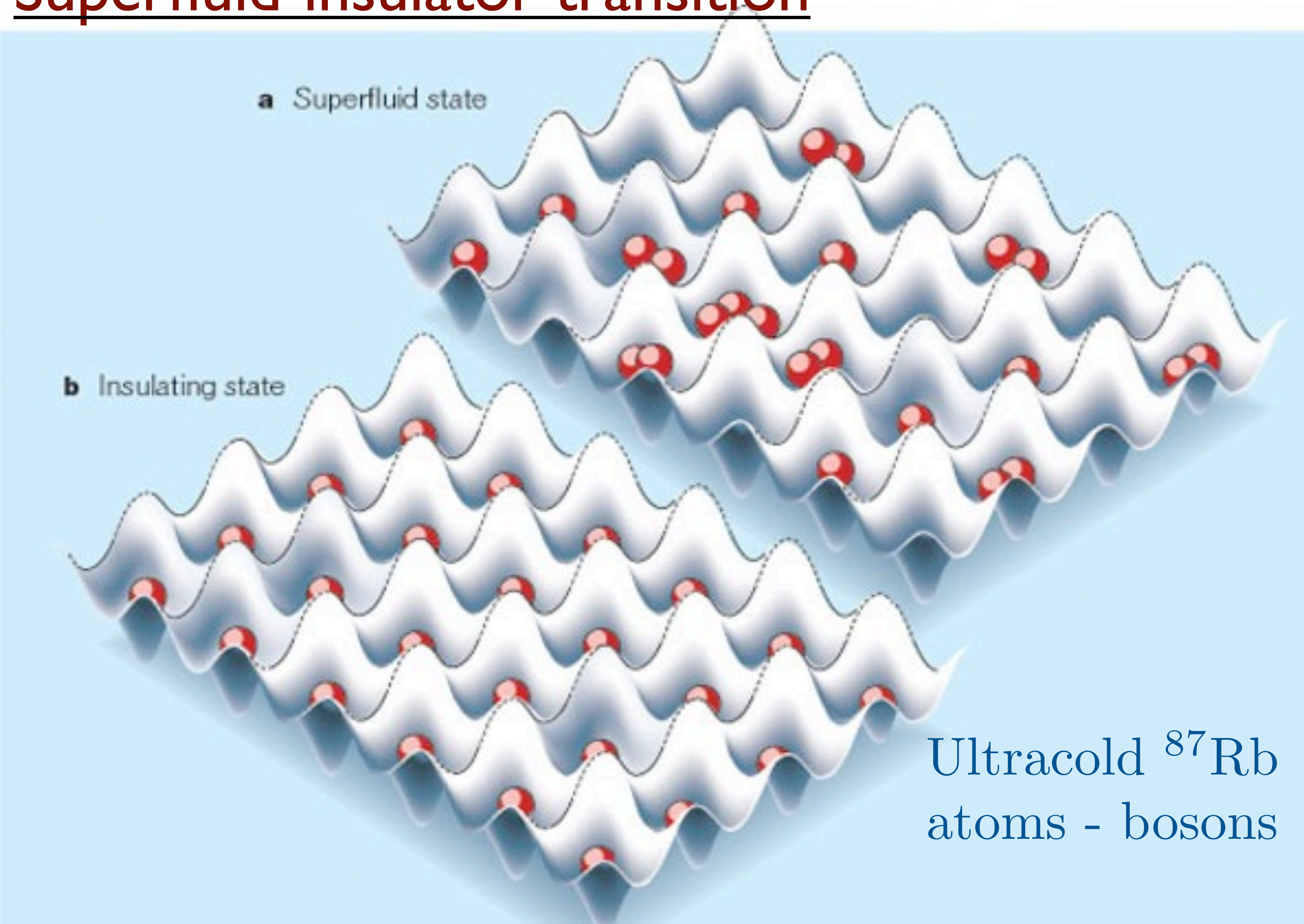
Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, j, of a lattice, with short-range repulsive interactions.

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \cdots$$

$$n_j \equiv b_j^\dagger b_j$$

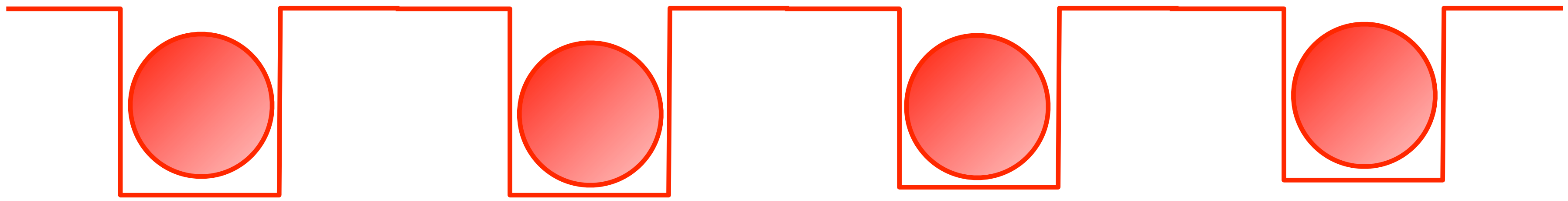
M.P. A. Fisher, P.B. Weichmann, G. Grinstein, and D.S. Fisher, *Phys. Rev. B* **40**, 546 (1989).

Superfluid-insulator transition



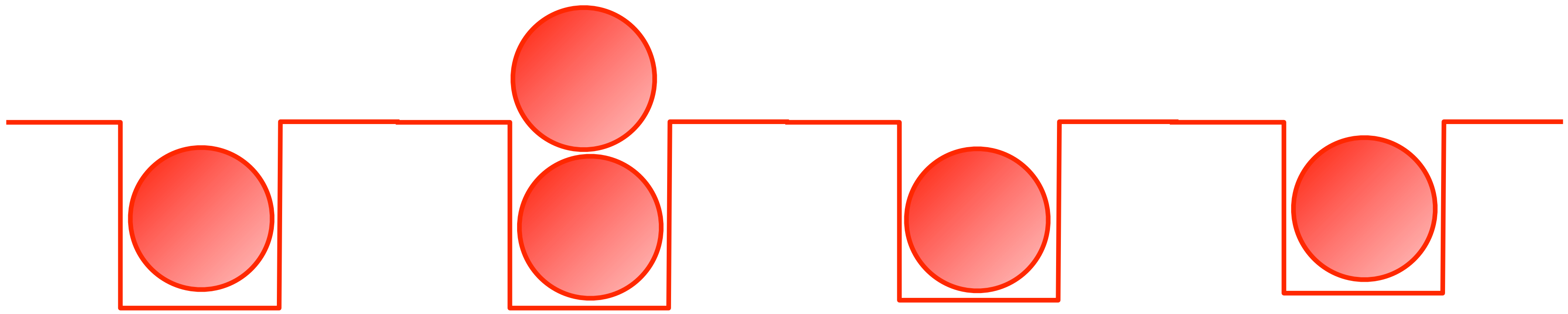
Ultracold ^{87}Rb atoms - bosons

M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).



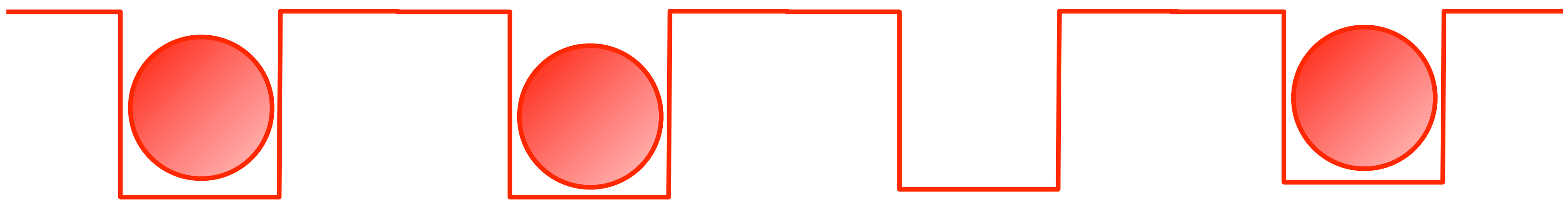
Insulator (the vacuum) at large U

Excitations:



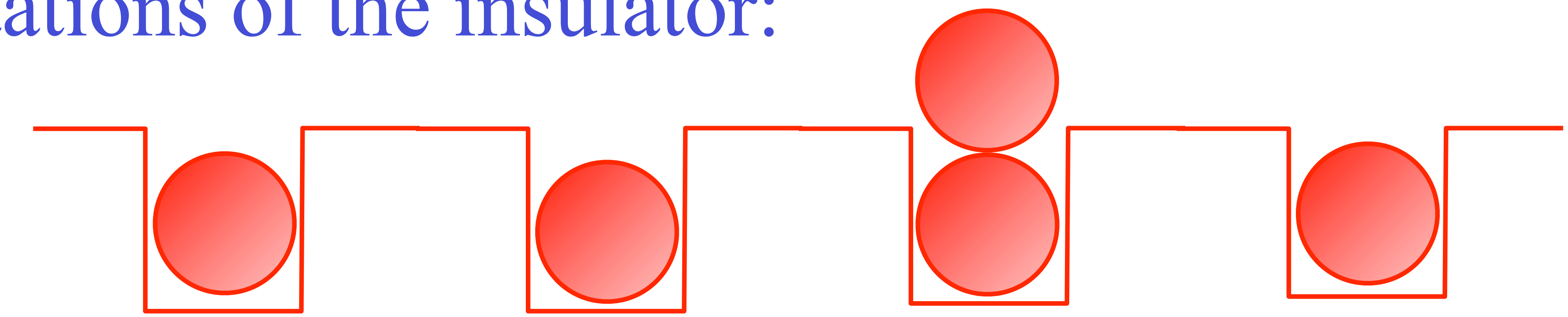
Particles $\sim \psi^\dagger$

Excitations:



Holes $\sim \psi$

Excitations of the insulator:



Particles $\sim \psi^{\dagger}$

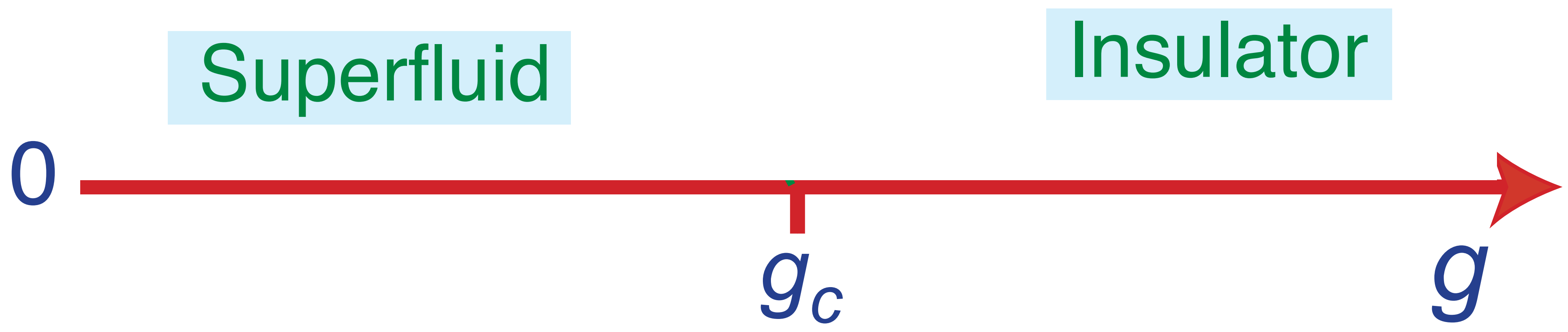
Holes $\sim \psi$

Density of particles = density of holes $\Rightarrow$
"relativistic" field theory for ψ:

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

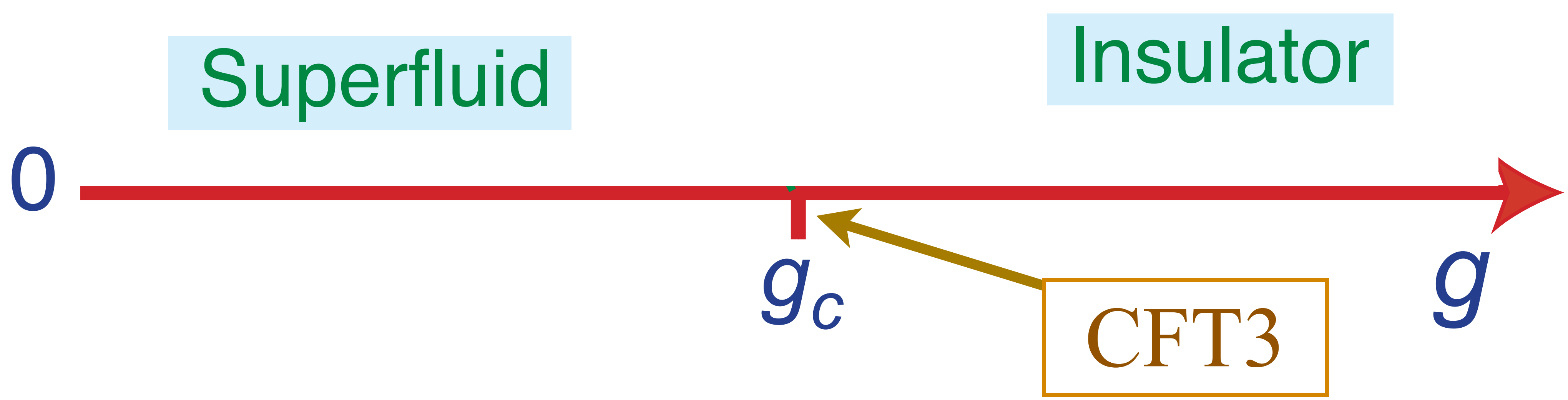
$$\text{Insulator} \Leftrightarrow \langle \psi \rangle = 0$$

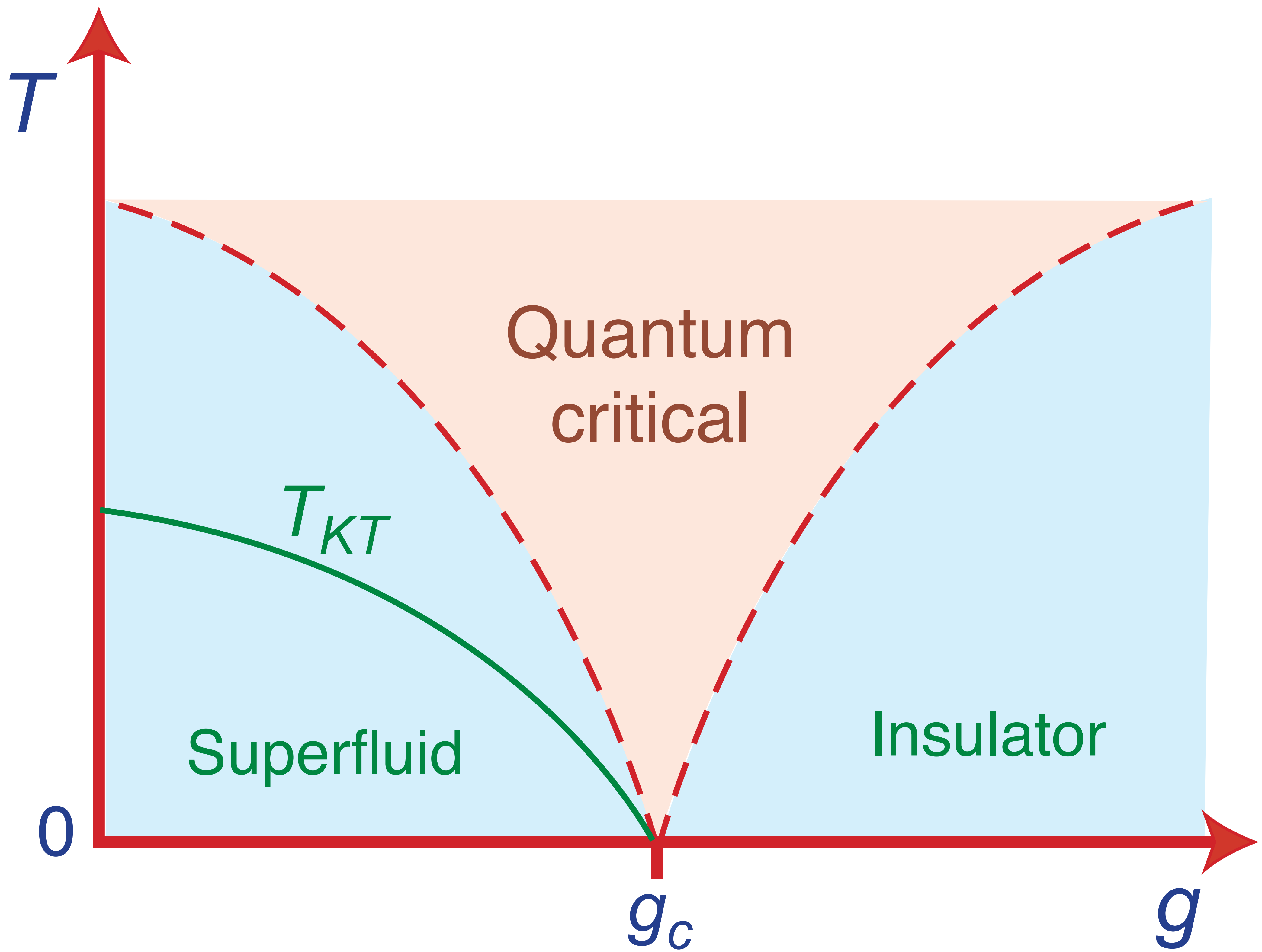
$$\text{Superfluid} \Leftrightarrow \langle \psi \rangle \neq 0$$

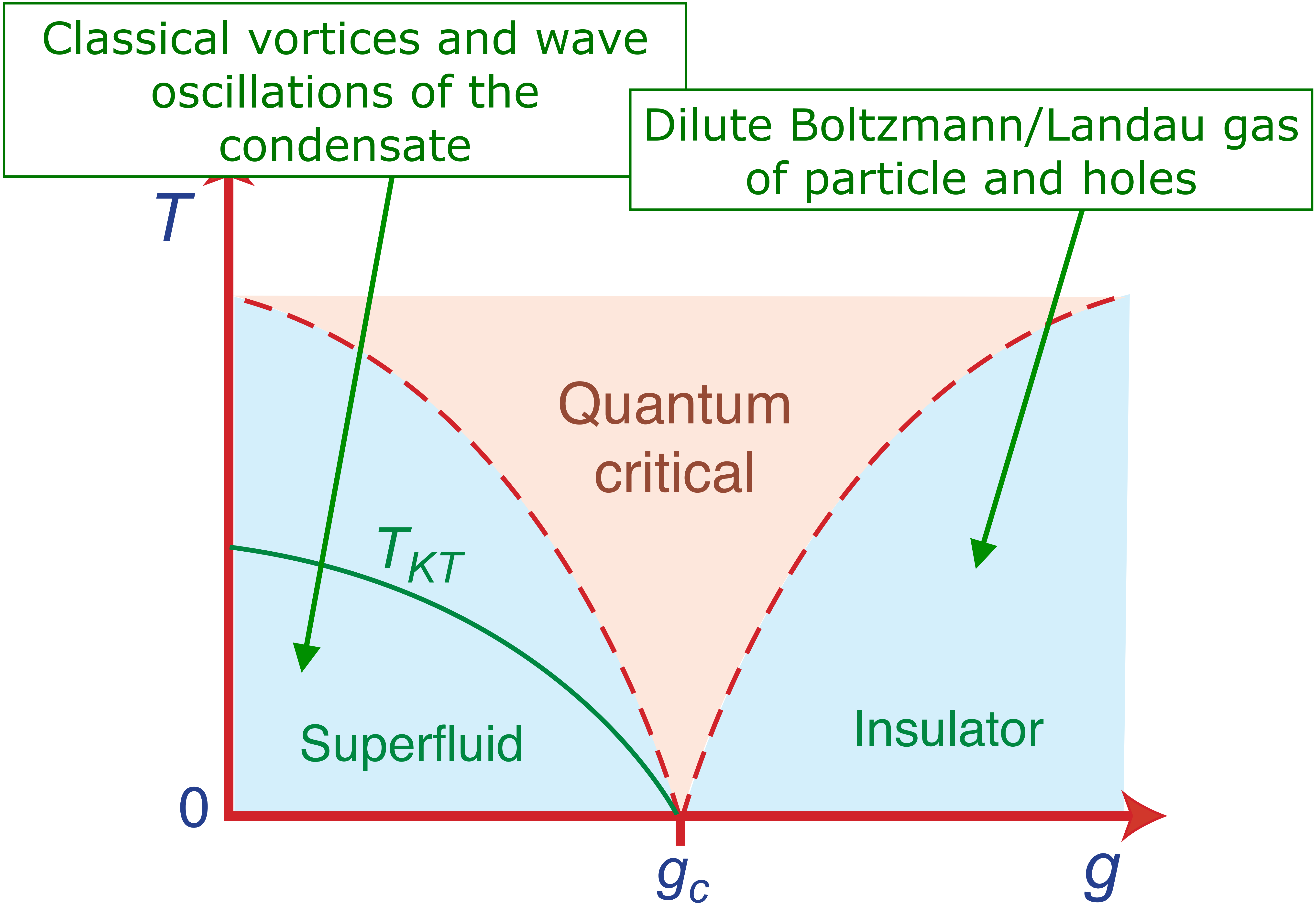

Superfluid
Insulator
0
g_c
g

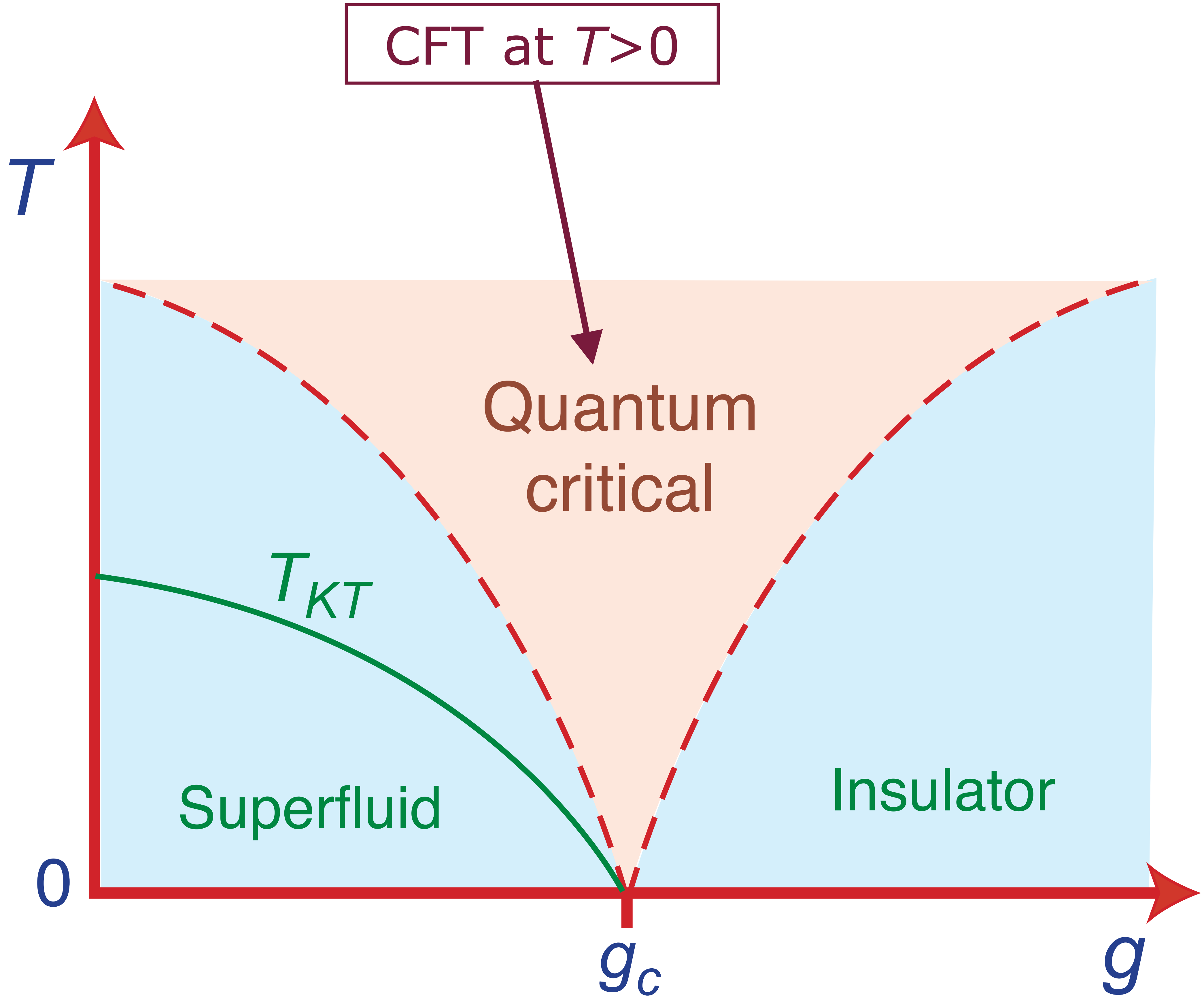
$$\mathcal{S} = \int d^2 r d\tau \left[|\partial_\tau \psi|^2 + v^2 |\vec{\nabla} \psi|^2 + (g - g_c)|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

$$\langle \psi \rangle \neq 0 \qquad \qquad \langle \psi \rangle = 0$$




T
Quantum
critical
T_KT
Superfluid
Insulator
0
g_c
g


Classical vortices and wave oscillations of the condensate
Dilute Boltzmann/Landau gas of particle and holes
T
Quantum critical
T_{KT}
Superfluid
Insulator
0
g_c
g



Monday, October 26, 2009

Resistivity of Bi films

Conductivity σ

$$\sigma_{\text{Superconductor}}(T \to 0) = \infty$$
$$\sigma_{\text{Insulator}}(T \to 0) = 0$$
$$\sigma_{\text{Quantum critical point}}(T \to 0) \approx \frac{4e^2}{h}$$

D. B. Haviland, Y. Liu, and A. M. Goldman, *Phys. Rev. Lett.* **62**, 2180 (1989)

M. P. A. Fisher, *Phys. Rev. Lett.* **65**, 923 (1990)

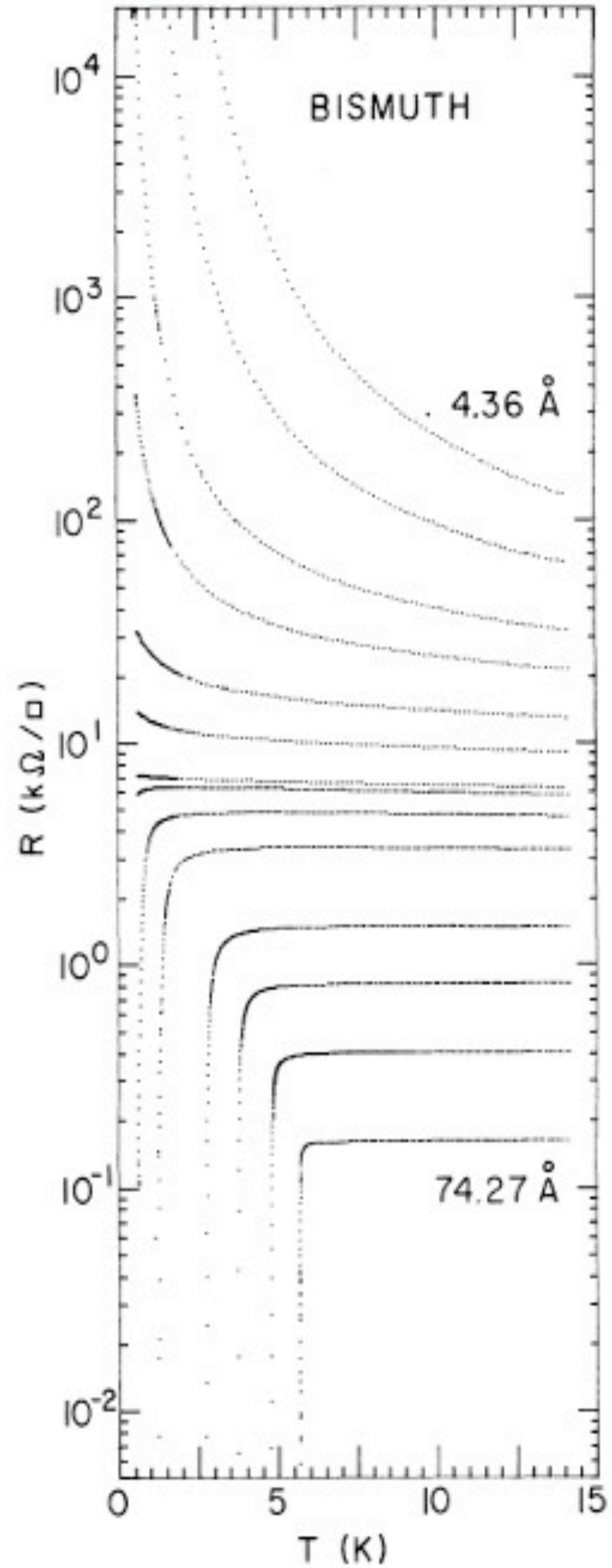


FIG. 1. Evolution of the temperature dependence of the sheet resistance $R(T)$ with thickness for a Bi film deposited onto Ge. Fewer than half of the traces actually acquired are shown. Film thicknesses shown range from 4.36 to 74.27 Å.

Quantum critical transport

Quantum *"perfect fluid"* with shortest possible relaxation time, τ_R

$$\tau_R \gtrsim \frac{\hbar}{k_B T}$$

S. Sachdev, *Quantum Phase Transitions,* Cambridge (1999).

Quantum critical transport

Transport co-oefficients not determined by collision rate, but by universal constants of nature

Electrical conductivity

$$\sigma = \frac{e^2}{h} \times [\text{Universal constant } \mathcal{O}(1)\ ]$$

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Quantum critical transport

Transport co-oefficients not determined by collision rate, but by universal constants of nature

Momentum transport

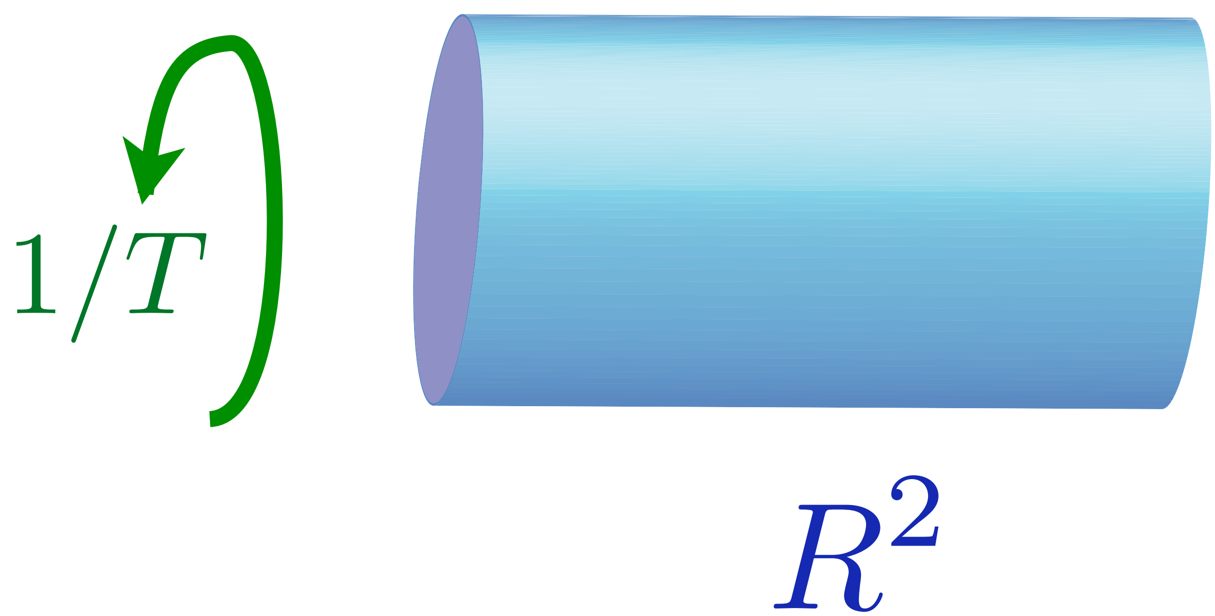
$$\frac{\eta}{s} \equiv \frac{\text{viscosity}}{\text{entropy density}}$$

$$= \frac{\hbar}{k_B} \times [\text{Universal constant } \mathcal{O}(1)\ ]$$

P. Kovtun, D. T. Son, and A. Starinets, *Phys. Rev. Lett.* **94,** 11601 (2005)

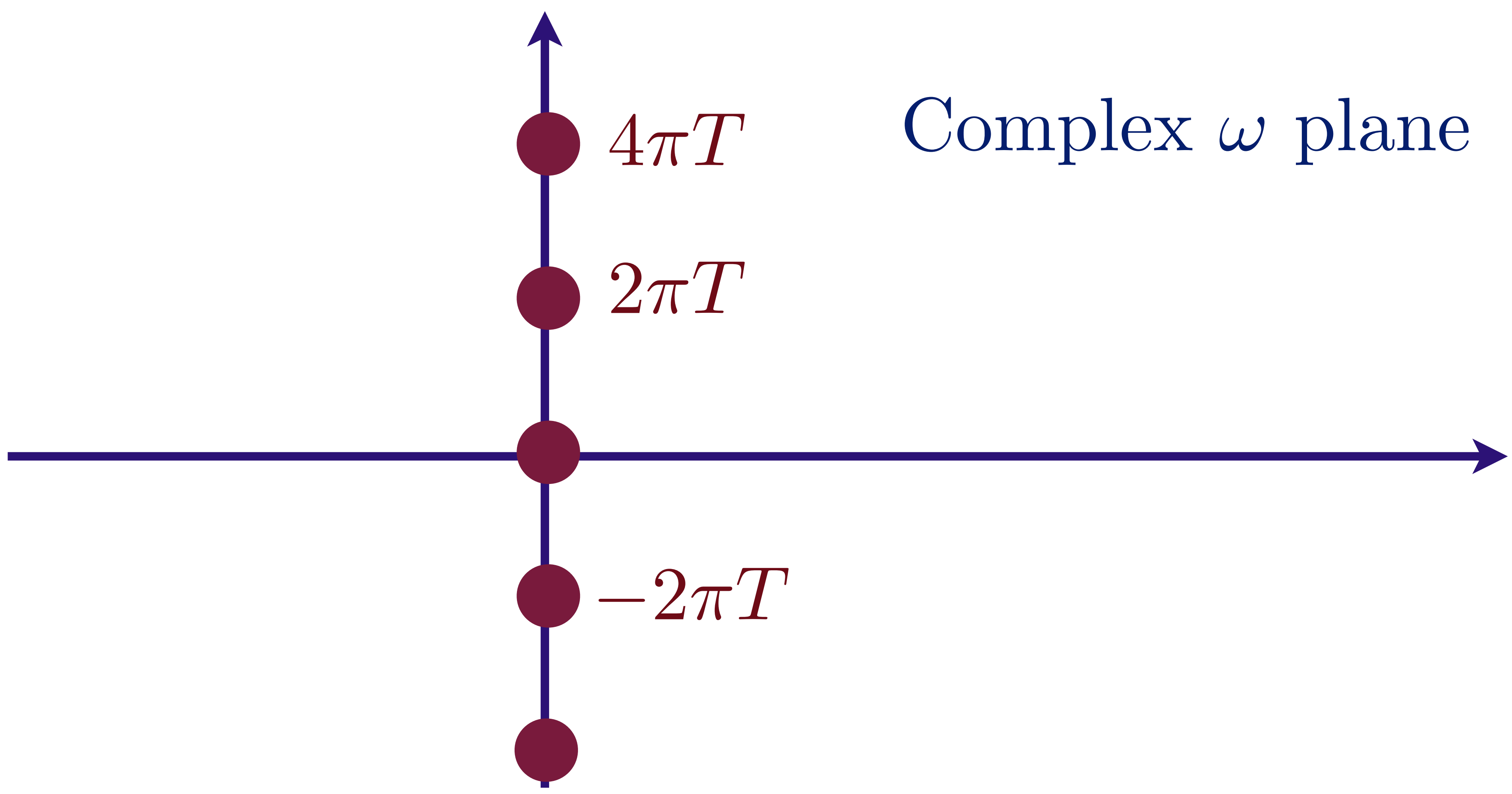
Quantum critical transport

Euclidean field theory: Compute current correlations on $R^2 \times S^1$ with circumference $1/T$



Quantum critical transport

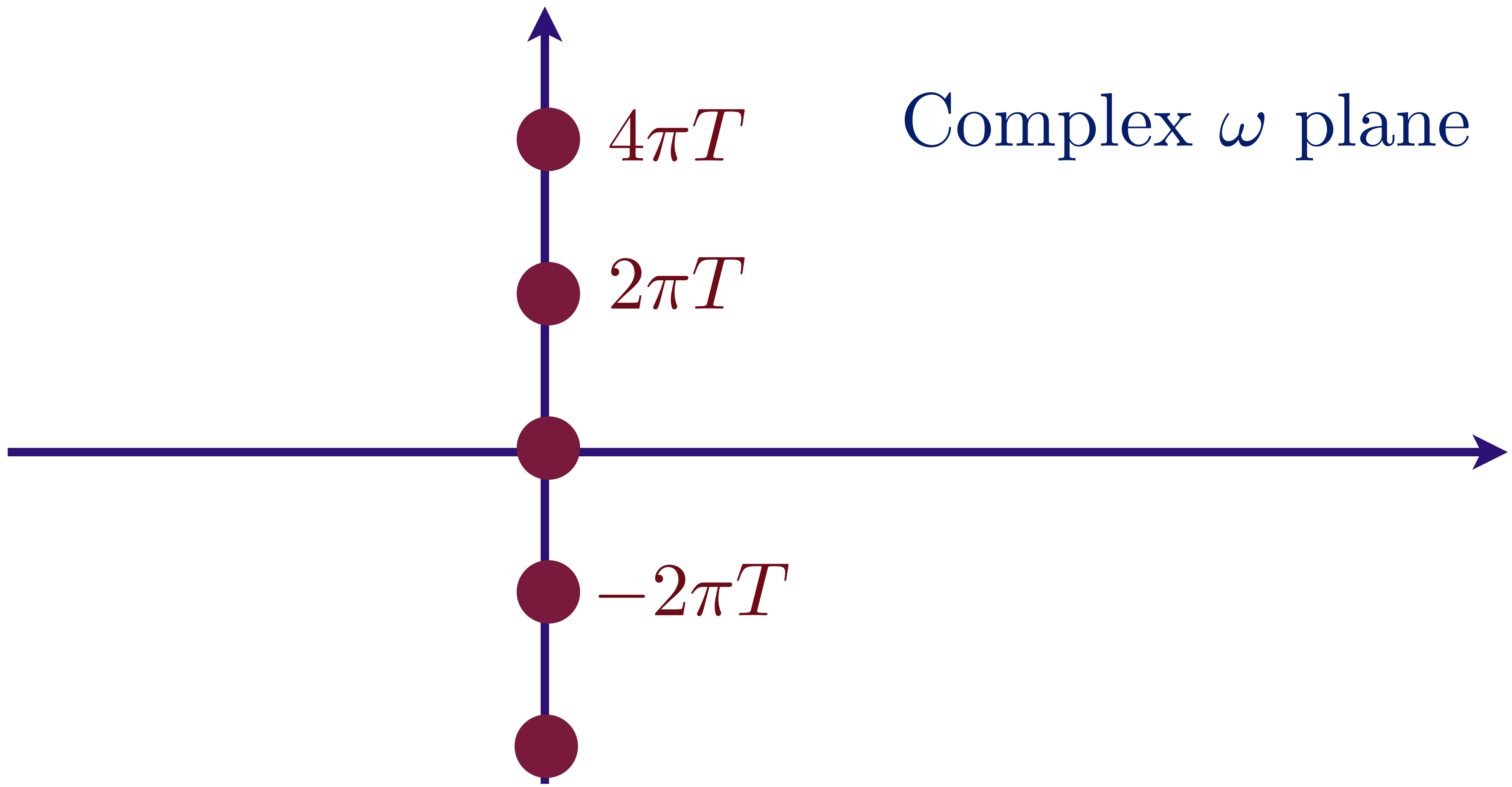
Euclidean field theory: Compute current correlations on $R^2 \times S^1$ with circumference $1/T$



Direct $1/N$ or $\epsilon = 4 - d$ expansion for correlators at $\omega_n = 2\pi n T i$, with n integer

Quantum critical transport

Euclidean field theory: Compute current correlations on $R^2 \times S^1$ with circumference $1/T$

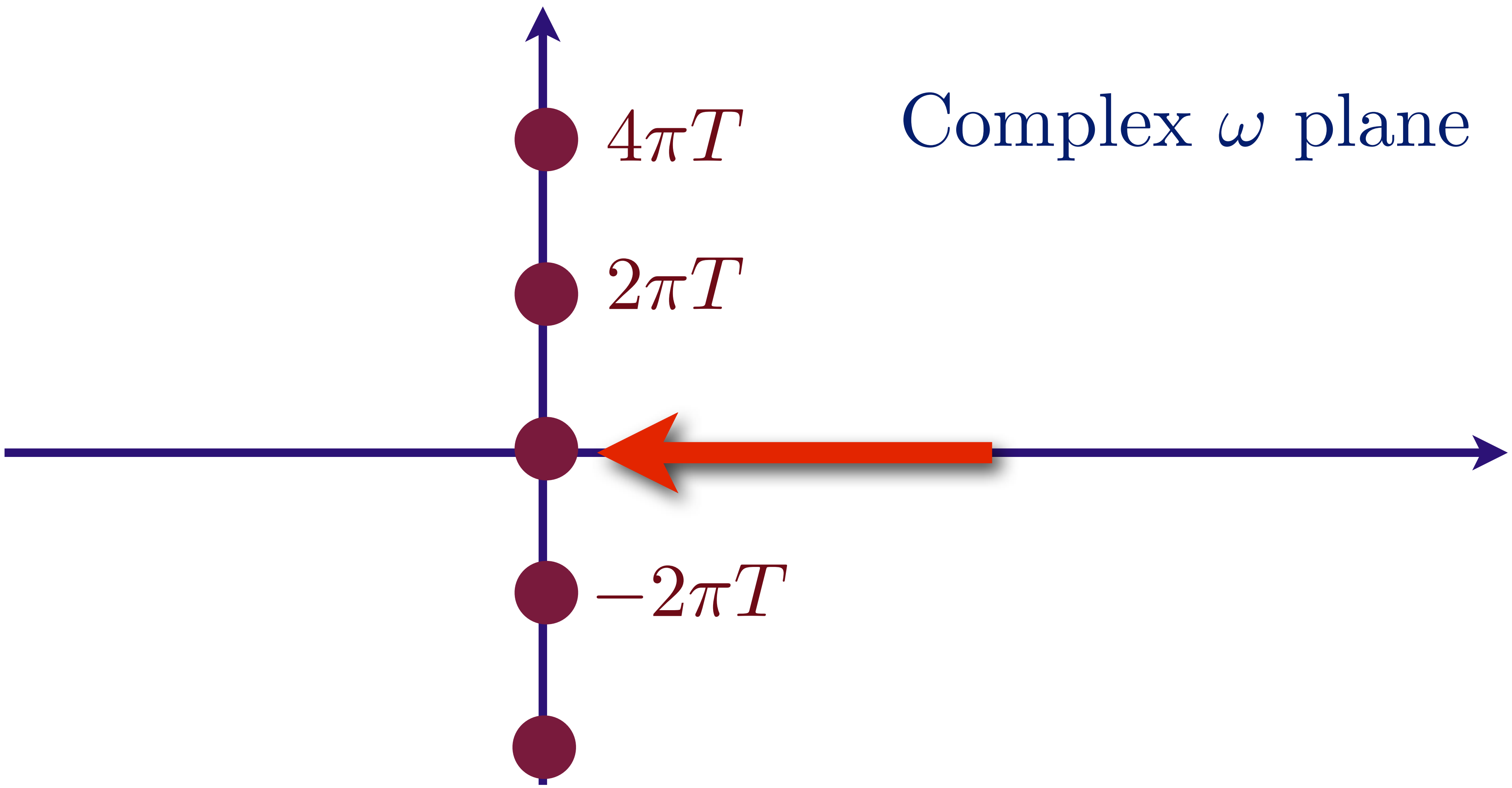


Strong coupling problem:

Correlators at $\omega \to 0$, along the real axis.

Quantum critical transport

Euclidean field theory: Compute current correlations on $R^2 \times S^1$ with circumference $1/T$



Strong coupling problem:

Correlators at $\omega \to 0$, along the real axis.

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k, \omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} \chi(k, \omega)$

For *all* CFT2s, at all $\hbar\omega / k_B T$

$$\chi(k, \omega) = \frac{4e^2}{h} K \frac{vk^2}{v^2 k^2 - \omega^2} \quad ; \quad \sigma(\omega) = \frac{4e^2}{h} \frac{Kv}{-i\omega}$$

where K is a universal number characterizing the CFT2 (the level number), and v is the velocity of "light".

This follows from the conformal mapping of the plane to the cylinder, which relates correlators at $T = 0$ to those at $T > 0$.

CFT correlator of $\mathrm{U}(1)$ current J_μ in 1+1 dimensions

Charge density correlation at $T = 0$:

$$\left\langle J_R(x,\tau) J_R(0) \right\rangle \sim \frac{1}{(\tau + ix)^2}$$

$$\left\langle J_t(k,\omega) J_t(-k,-\omega) \right\rangle \sim \frac{k^2}{k^2 - \omega^2}$$

CFT correlator of $U(1)$ current J_μ in 1+1 dimensions

Charge density correlation at $T \geq 0$:

$$\left\langle J_R(x,\tau) J_R(0) \right\rangle \sim \frac{\pi^2 T^2}{\sin^2\left(\pi T(\tau + ix)\right)}$$

$$\left\langle J_t(k, i\omega_n) J_t(-k, -i\omega_n) \right\rangle \sim \frac{k^2}{k^2 + \omega_n^2}$$

Conformal mapping of plane to cylinder with circumference *1/T*

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Conformal mapping of plane to cylinder with circumference *1/T*

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k,\omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k\to 0} \frac{-i\omega}{k^2}\chi(k,\omega)$

For *all* CFT2s, at all $\hbar\omega/k_B T$

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where K is a universal number characterizing the CFT2 (the level number), and v is the velocity of "light".
This follows from the conformal mapping of the plane to the cylinder, which relates correlators at $T=0$ to those at $T>0$.

No hydrodynamics in CFT2s.

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k, \omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} \chi(k, \omega)$

For *all* CFT3s, at $\hbar\omega \gg k_B T$

$$\chi(k, \omega) = \frac{4e^2}{h} K \frac{k^2}{\sqrt{v^2 k^2 - \omega^2}} \quad ; \quad \sigma(\omega) = \frac{4e^2}{h} K$$

where K is a universal number characterizing the CFT3, and v is the velocity of "light".

Density correlations in CFTs at $T > 0$

Two-point density correlator, $\chi(k, \omega)$

Kubo formula for conductivity $\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} \chi(k, \omega)$

However, for *all* CFT3s, at $\hbar\omega \ll k_B T$, we have the Einstein relation

$$\chi(k, \omega) = 4e^2 \chi_c \frac{Dk^2}{Dk^2 - i\omega} \quad ; \quad \sigma(\omega) = 4e^2 D \chi_c = \frac{4e^2}{h} \Theta_1 \Theta_2$$

where the **compressibility**, χ_c, and the **diffusion constant** D obey

$$\chi = \frac{k_B T}{(hv)^2} \Theta_1 \quad ; \quad D = \frac{hv^2}{k_B T} \Theta_2$$

with Θ_1 and Θ_2 universal numbers characteristic of the CFT3

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

Density correlations in CFTs at $T > 0$

In CFT3s collisions are "phase" randomizing, and lead to relaxation to local thermodynamic equilibrium. So there is a crossover from collisionless behavior for $\hbar\omega \gg k_B T$, to hydrodynamic behavior for $\hbar\omega \ll k_B T$.

$$\sigma(\omega) = \begin{cases} \dfrac{4e^2}{h} K & , \quad \hbar\omega \gg k_B T \\ \dfrac{4e^2}{h} \Theta_1 \Theta_2 \equiv \sigma_Q & , \quad \hbar\omega \ll k_B T \end{cases}$$

and in general we expect $K \neq \Theta_1 \Theta_2$ (verified for Wilson-Fisher fixed point).

K. Damle and S. Sachdev, *Phys. Rev. B* **56**, 8714 (1997).

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SU(N) SYM3 with $\mathcal{N} = 8$ supersymmetry

- Has a single dimensionful coupling constant, e_0, which flows to a strong-coupling fixed point $e_0 = e_0^*$ in the infrared.
- The CFT3 describing this fixed point resembles "critical spin liquid" theories.
- This CFT3 is the low energy limit of string theory on an M2 brane. The AdS/CFT correspondence provides a dual description using 11-dimensional supergravity on $\text{AdS}_4 \times S_7$.
- The CFT3 has a global SO(8) R symmetry, and correlators of the SO(8) charge density can be computed exactly in the large N limit, even at $T > 0$.

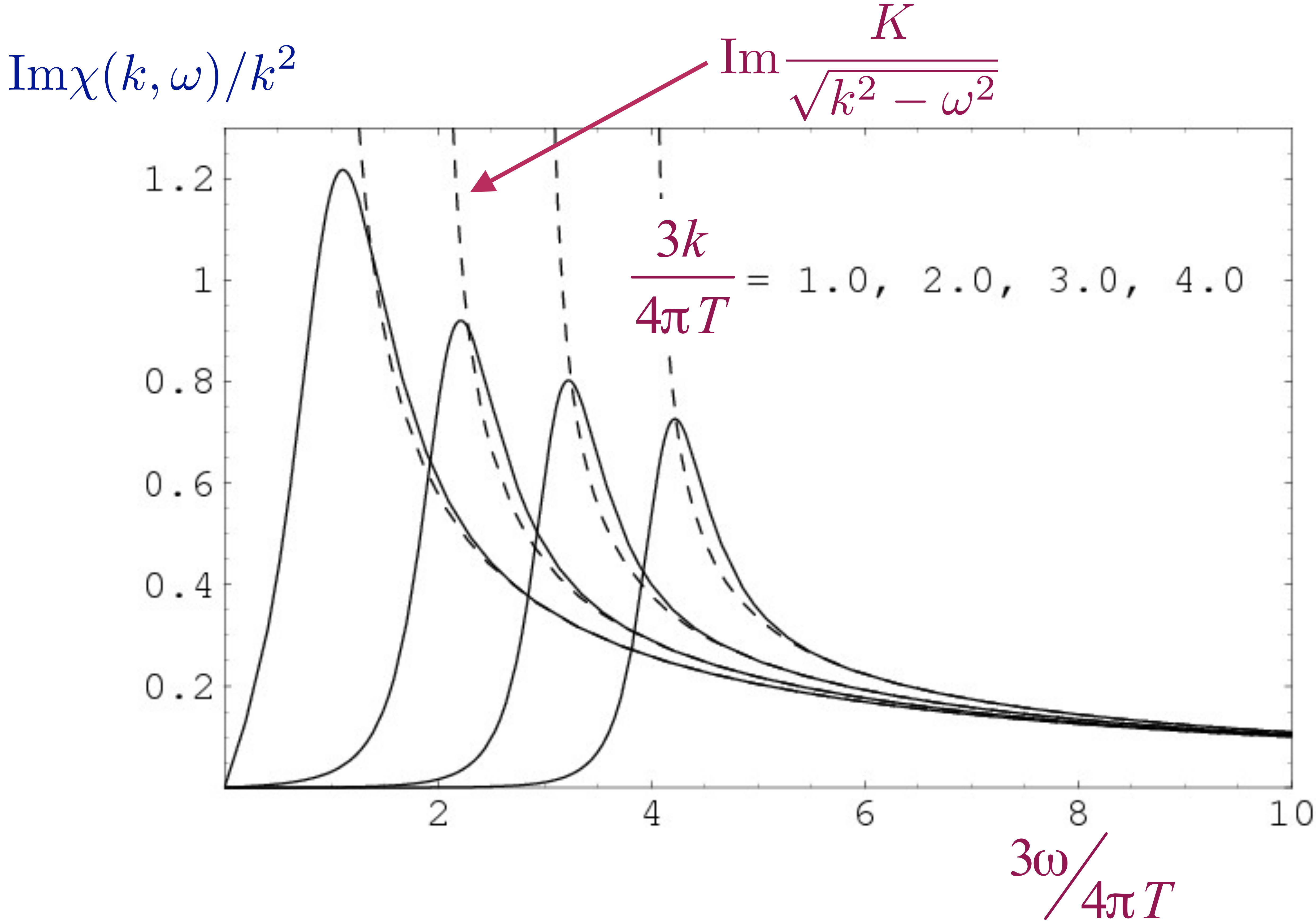
SU(N) SYM3 with $\mathcal{N} = 8$ supersymmetry

- The SO(8) charge correlators of the CFT3 are given by the usual AdS/CFT prescription applied to the following gauge theory on AdS4:

$$\mathcal{S} = -\frac{1}{4g_{4D}^2} \int d^4x \sqrt{-g} g^{MA} g^{NB} F^a_{MN} F^a_{AB}$$

where $a = 1 \ldots 28$ labels the generators of SO(8). Note that in large N theory, this looks like 28 copies of an Abelian gauge theory.

Collisionless to hydrodynamic crossover of SYM3



P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

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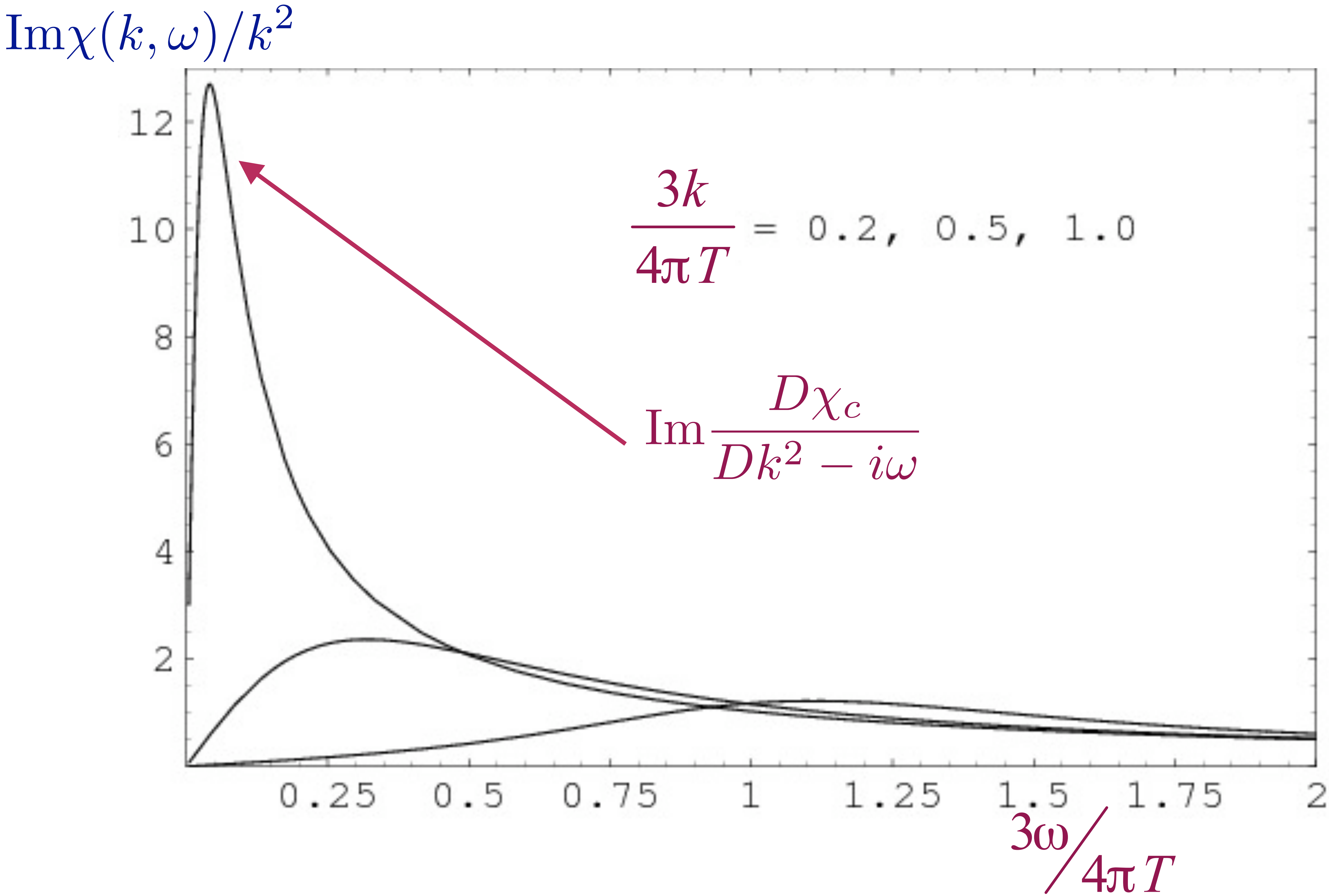


P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

Universal constants of SYM3

$$\chi_c = \frac{k_B T}{(hv)^2}\Theta_1$$

$$D = \frac{hv^2}{k_B T}\Theta_2$$

$$\sigma(\omega) = \begin{cases} \frac{4e^2}{h}K & , \quad \hbar\omega \gg k_B T \\ \frac{4e^2}{h}\Theta_1\Theta_2 & , \quad \hbar\omega \ll k_B T \end{cases}$$

$$K = \frac{\sqrt{2}N^{3/2}}{3}$$

$$\Theta_1 = \frac{8\pi^2\sqrt{2}N^{3/2}}{9}$$

$$\Theta_2 = \frac{3}{8\pi^2}$$

C. Herzog, JHEP **0212**, 026 (2002)

P. Kovtun, C. Herzog, S. Sachdev, and D.T. Son, Phys. Rev. D **75**, 085020 (2007)

Electromagnetic self-duality

- Unexpected result, $K = \Theta_1 \Theta_2$.
- This is traced to a *four*-dimensional electromagnetic self-duality of the theory on AdS_4. In the large N limit, the SO(8) currents decouple into 28 U(1) currents with a Maxwell action for the U(1) gauge fields on AdS_4.
- This special property is not expected for generic CFT3s.

CFT correlator of $U(1)$ current J_μ at $T = 0$

$$\left\langle J_\mu(p) J_\nu(-p) \right\rangle = K \sqrt{p^2} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$$

K: a universal number analogous to the level number of the Kac-Moody algebra in 1+1 dimensions

Application of Kubo formula shows that

$$\sigma\left(\omega / T = \infty \right) = \frac{4e^2}{h} 2\pi K$$

CFT correlator of $\mathrm{U}(1)$ current J_μ at $T > 0$

$$\langle J_\mu(k,\omega) J_\nu(-k,-\omega) \rangle = \sqrt{k^2 - \omega^2} \left(P^T_{\mu\nu} K^T(k,\omega) + P^L_{\mu\nu} K^L(k,\omega) \right)$$

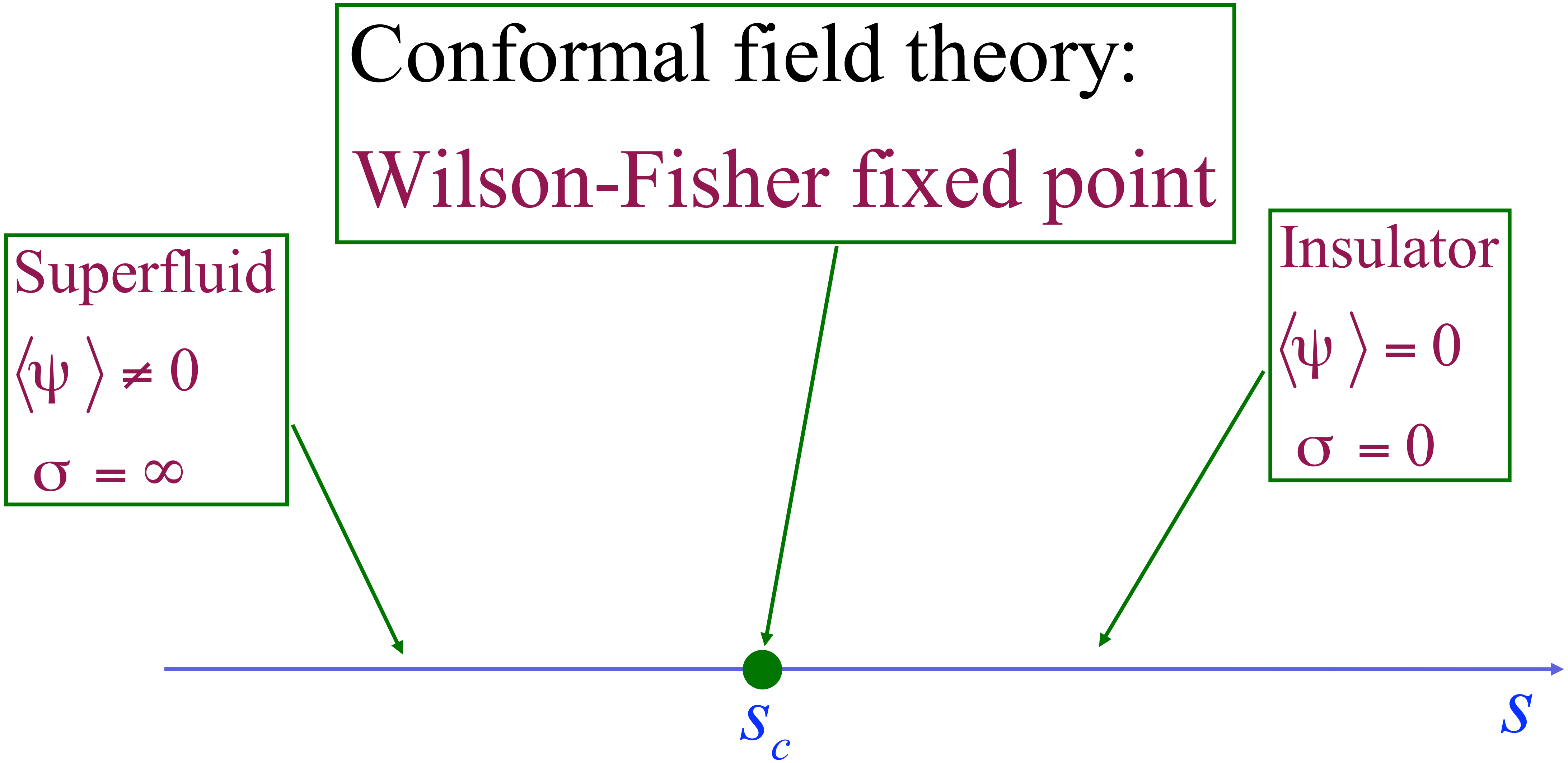
The projectors are defined by

$$P^T_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2} \text{ and } P^L_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - P^T_{\mu\nu} \quad ; \quad p = (k,\omega)$$

while $K^{L,T}(k,\omega)$ are universal functions of ω/T and k/T

Application of Kubo formula shows that

$$\sigma\left(\omega/T\right) = \frac{4e^2}{h} 2\pi K^T(0,\omega) = \frac{4e^2}{h} 2\pi K^L(0,\omega)$$



Using the boson quasiparticle excitations of the insulator $\sim \psi$

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett*. **47**, 1556 (1981)

Conformal field theory:
Wilson-Fisher fixed point

Superfluid

$\langle \psi \rangle \neq 0$

$\langle \varphi \rangle = 0$

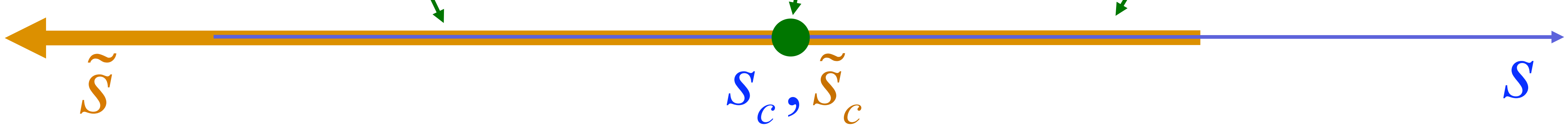
$\sigma = \infty$

Insulator

$\langle \psi \rangle = 0$

$\langle \varphi \rangle \neq 0$

$\sigma = 0$



Using the boson quasiparticle excitations of the insulator $\sim \psi$

$$\mathcal{S} = \int d^3x \left[|\partial_\mu \psi|^2 + s|\psi|^2 + \frac{u}{2}|\psi|^4 \right]$$

<u>is dual to</u>

Using the vortex quasiparticle excitations of the superfluid $\sim \varphi$

$$\mathcal{S}_{\rm dual} = \int d^3x \left[|(\partial_\mu - iA_\mu)\varphi|^2 + \tilde{s}|\varphi|^2 + \frac{\tilde{u}}{2}|\varphi|^4 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]$$

C. Dasgupta and B.I. Halperin, *Phys. Rev. Lett.* **47**, 1556 (1981)

Consequences of duality on CFT correlators of $U(1)$ currents

$$\langle J_\mu(k,\omega) J_\nu(k,\omega)\rangle_{\mathcal{S}} = \sqrt{k^2-\omega^2}\left(P^T_{\mu\nu}K^T(k,\omega) + P^L_{\mu\nu}K^L(k,\omega)\right)$$

$$\left\langle \widetilde{J}_\mu(k,\omega)\widetilde{J}_\nu(k,\omega)\right\rangle_{\mathcal{S}_{\rm dual}} = \sqrt{k^2-\omega^2}\left(P^T_{\mu\nu}\widetilde{K}^T(k,\omega) + P^L_{\mu\nu}\widetilde{K}^L(k,\omega)\right)$$

$$K^L(k,\omega)\widetilde{K}^T(k,\omega) = \frac{1}{4\pi^2}$$

$$K^T(k,\omega)\widetilde{K}^L(k,\omega) = \frac{1}{4\pi^2}$$

Application of Kubo formula shows that

$$\sigma\left(\omega/T\right) = \frac{4e^2}{h} 2\pi K^T(0,\omega) = \frac{4e^2}{h} 2\pi K^L(0,\omega)$$

C. Herzog, P. Kovtun, S. Sachdev, and D.T. Son, hep-th/0701036

Correlations of SO(8) currents of the SYM_3 SCFT at $T > 0$

$$\left\langle J_\mu^a(k,\omega) J_\nu^b(-k,-\omega)\right\rangle = \delta^{ab}\sqrt{k^2-\omega^2}\left(P_{\mu\nu}^T K^T(k,\omega) + P_{\mu\nu}^L K^L(k,\omega)\right)$$

The self-duality of the 4D abelian gauge fields leads to

$$\boxed{K^L(k,\omega)K^T(k,\omega) = \frac{N^3}{18\pi^2}}$$

C. Herzog, P. Kovtun, S. Sachdev, and D.T. Son, hep-th/0701036

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The self-duality of the 4D abelian gauge fields leads to

$$\boxed{K^L(k,\omega)K^T(k,\omega) = \frac{N^3}{18\pi^2}}$$

Analyticity of correlations at $T > 0$ implies

$$K^T(0,\omega) = K^L(0,\omega),$$

and so the conductivity

$$\sigma(\omega/T) = K^T(0,\omega) = K^L(0,\omega) = \sqrt{\frac{N^3}{72\pi^2}}$$

is frequency independent.

C. Herzog, P. Kovtun, S. Sachdev, and D.T. Son, hep-th/0701036

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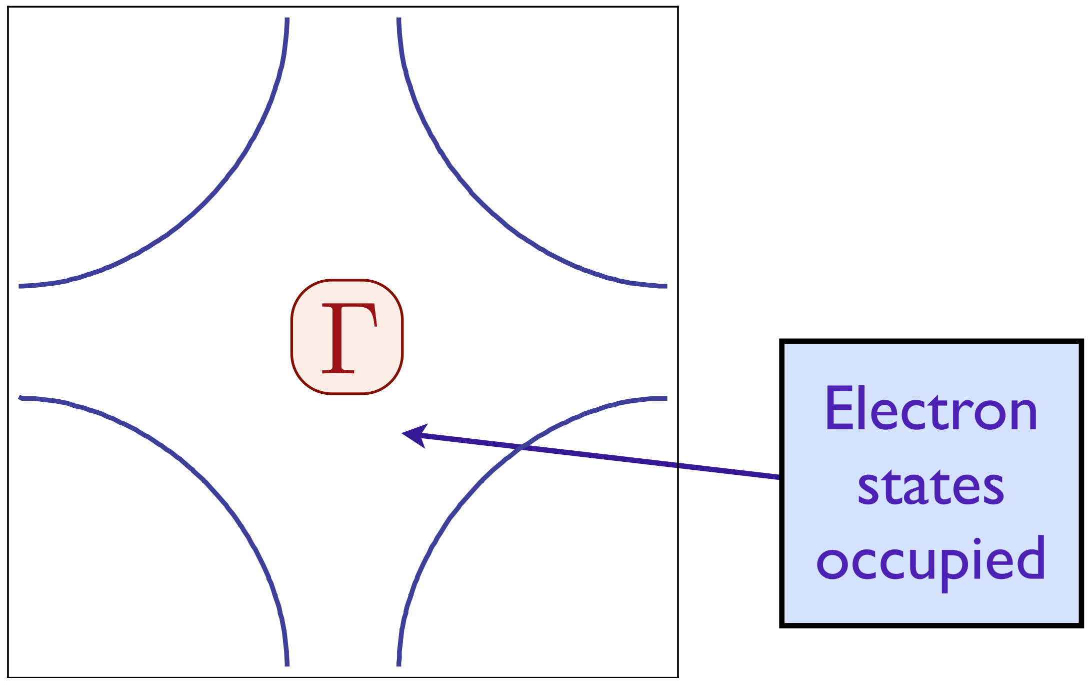
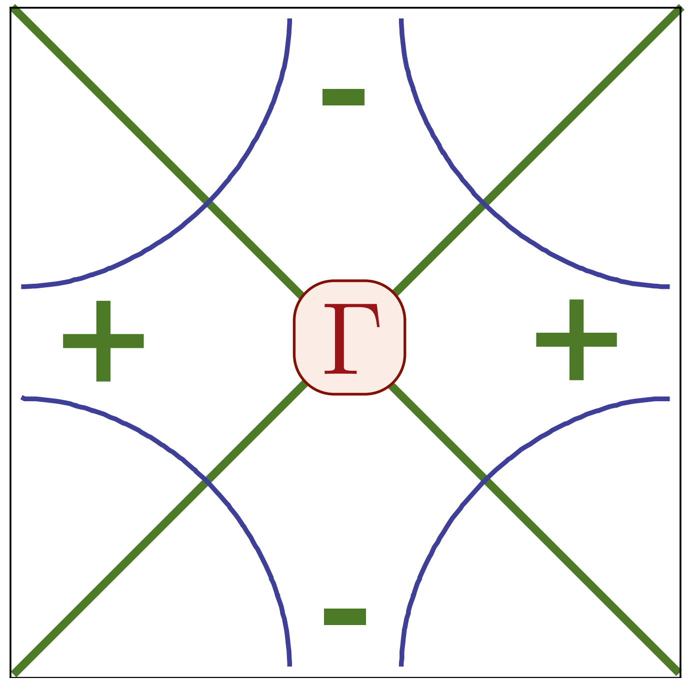
d-wave superconductivity in cuprates



$$H_0 = -\sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \equiv \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha}$$

- Begin with free electrons.

d-wave superconductivity in cuprates

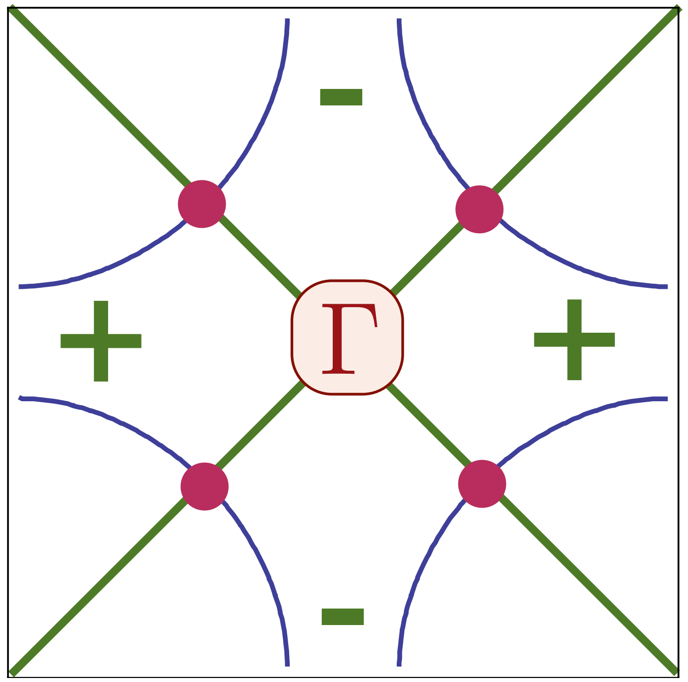


$$H = \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + \text{c.c.} \right)$$

- Begin with free electrons.

- Add d-wave pairing interaction $\Delta_k \sim \cos k_x - \cos k_y$ which vanishes along diagonals

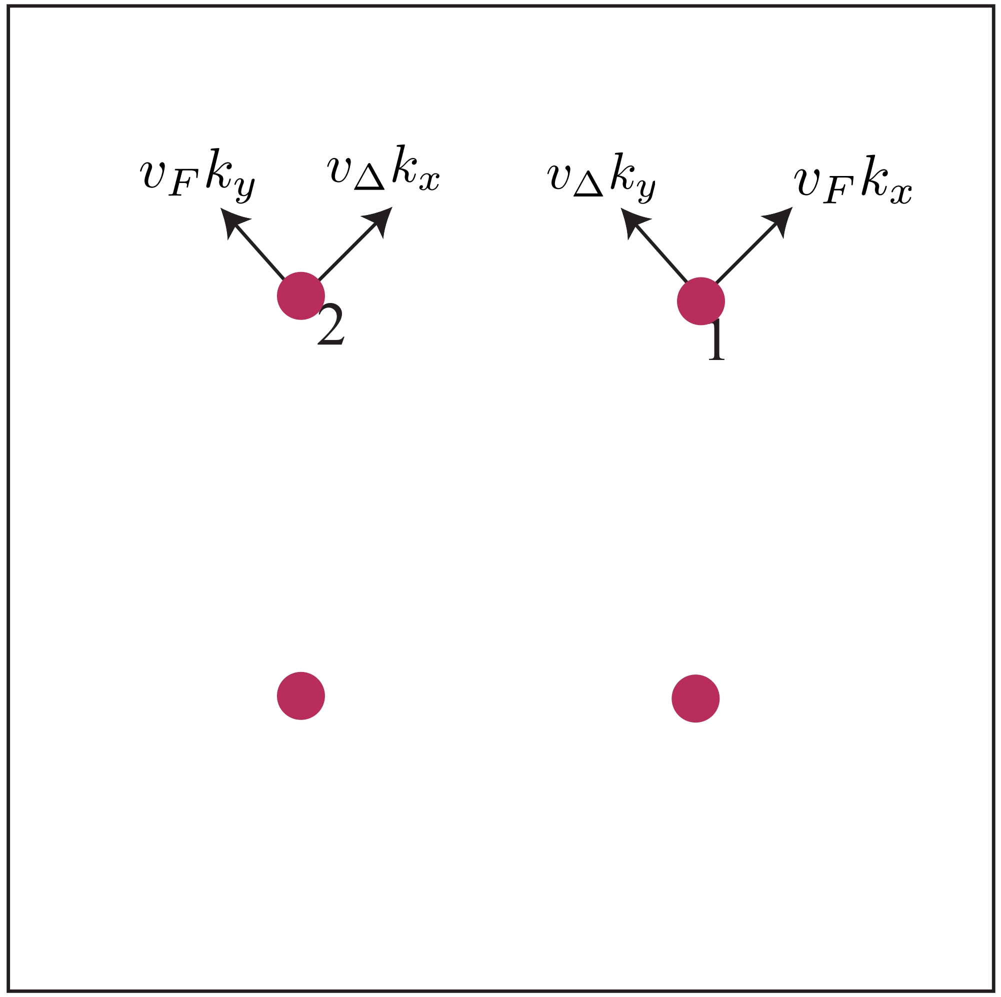
d-wave superconductivity in cuprates



$$H = \sum_{\mathbf{k}} \left(\varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + \text{c.c.} \right)$$

- Begin with free electrons.
- Add d-wave pairing interaction Δ_k which vanishes along diagonals
- Obtain Bogoliubov quasiparticles with dispersion $\sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$

d-wave superconductivity in cuprates



4 two-component Dirac fermions

$$
\begin{aligned}
S_\Psi &= \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{1a}^\dagger \left(-i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x\right) \Psi_{1a} \\
&+ \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \Psi_{2a}^\dagger \left(-i\omega_n + v_F k_y \tau^z + v_\Delta k_x \tau^x\right) \Psi_{2a}.
\end{aligned}
$$

d-wave superconductivity in cuprates

Now consider a discrete spontaneous symmetry breaking, with Ising symmetry, described by a real scalar field ϕ.
Two cases of experimental interest are:

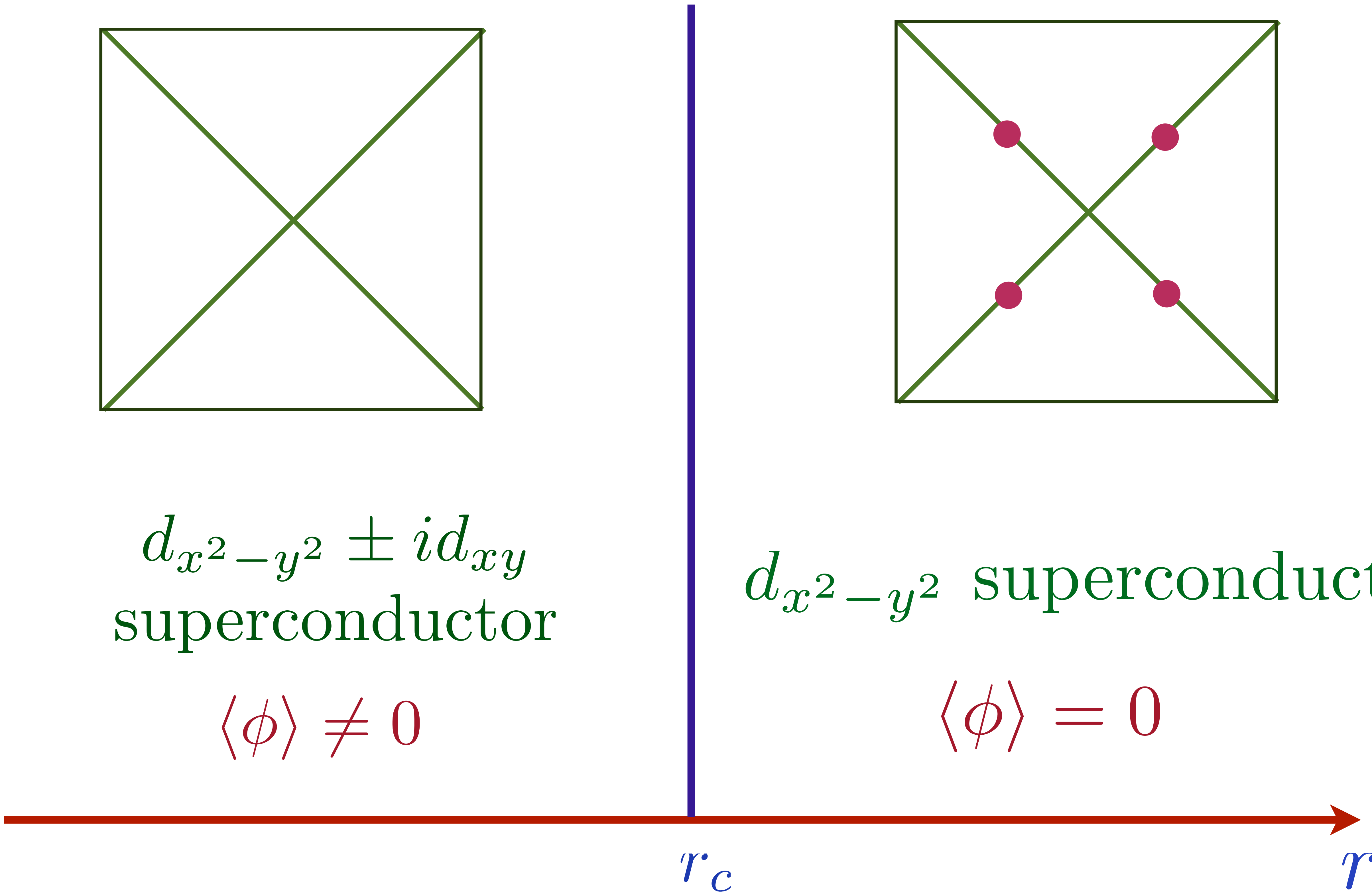
- Time-reversal symmetry breaking: leads to a $d_{x^2-y^2} + id_{xy}$ superconductor, in which the Dirac fermions are massive

- Break 4-fold lattice rotation symmetry to 2-fold lattice rotations: leads to a superconductor with **nematic** order.

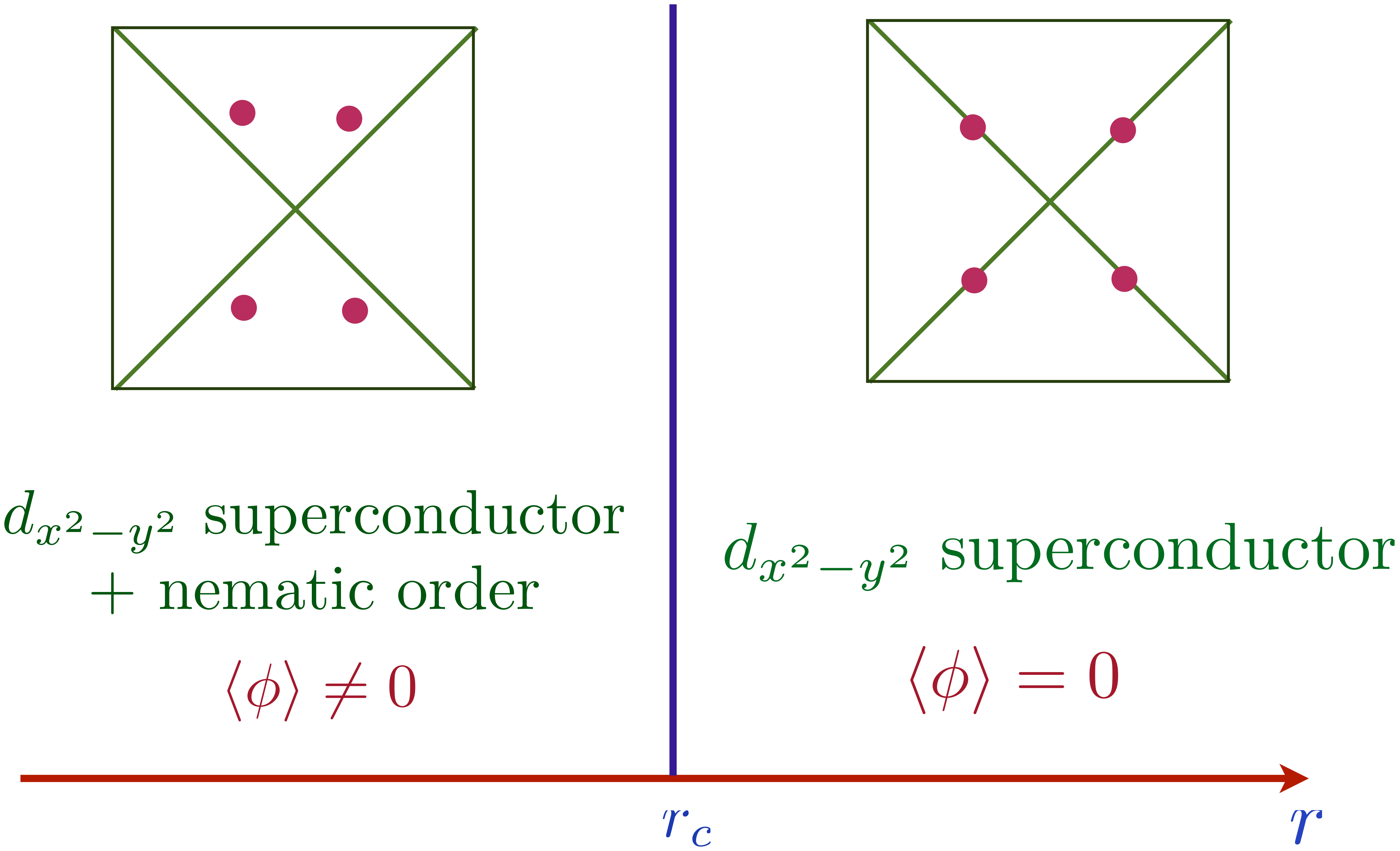
We can write down the usual ϕ^4 theory for the scalar field:

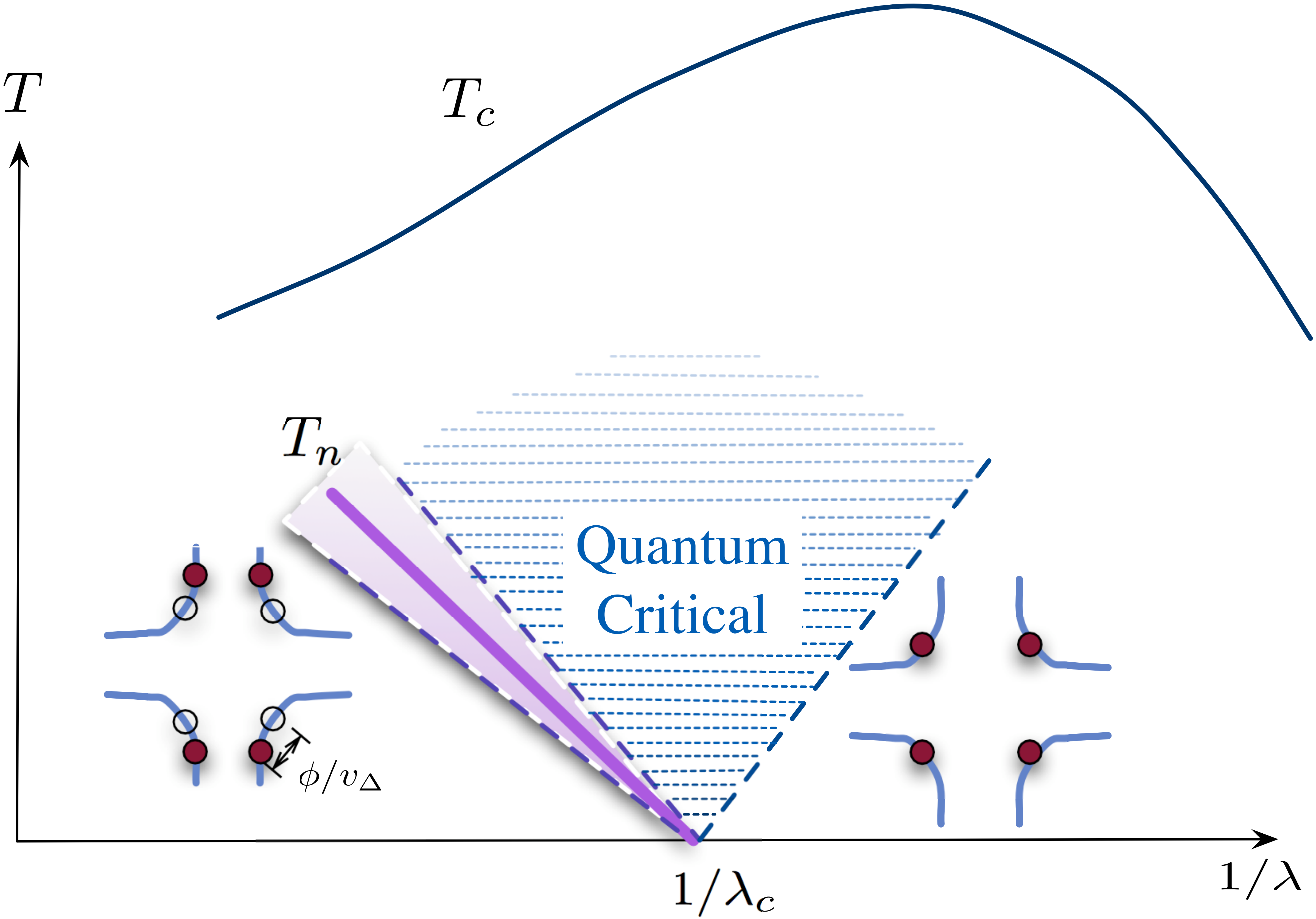
$$S_\phi^0 = \int d^2x d\tau \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{c^2}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 + \frac{u_0}{24}\phi^4 \right]$$

Time-reversal symmetry breaking



Lattice rotation symmetry breaking





M. Vojta, Y. Zhang, and S. Sachdev, Phys. Rev. Lett. **85**, 4940 (2000)
E.-A. Kim, M. J. Lawler, P. Oreto, S. Sachdev, E. Fradkin, S.A. Kivelson, Phys. Rev. B **77**, 184514 (2008).

Ising order and Dirac fermions couple via a "Yukawa" term.

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \left(\Psi^\dagger_{1a} \tau^x \Psi_{1a} + \Psi^\dagger_{2a} \tau^x \Psi_{2a} \right)\right],$$

Nematic ordering

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \left(\Psi^\dagger_{1a} \tau^y \Psi_{1a} + \Psi^\dagger_{2a} \tau^y \Psi_{2a} \right)\right]$$

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Nematic ordering

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \left(\Psi_{1a}^\dagger \tau^y \Psi_{1a} + \Psi_{2a}^\dagger \tau^y \Psi_{2a} \right) \right]$$

Time reversal symmetry breaking

For the latter case *only*, with $v_F = v_\Delta = c$, theory reduces to relativistic Gross-Neveu model

M. Vojta, Y. Zhang, and S. Sachdev, Physical Review Letters **85**, 4940 (2000)

Expansion in number of fermion spin components N_f

Integrating out the fermions yields an effective action for the scalar order parameter

$$S_\phi = \frac{N_f}{v_\Delta v_F}\Gamma\left[\lambda_0\phi(x,\tau);\frac{v_\Delta}{v_F}\right] + \frac{N_f}{2}\int d^2x d\tau \Big(r\phi^2(x,\tau)\Big) + \text{irrelevant terms}$$

where Γ is a non-local and non-analytic functional of ϕ.

The theory has only 2 couplings constants: r and v_Δ/v_F.

Y. Huh and S. Sachdev, Physical Review B **78**, 064512 (2008).

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where Γ is a non-local and non-analytic functional of ϕ.

There is a systematic expansion in powers of $1/N_f$ for renormalization group equations and all critical properties.

Y. Huh and S. Sachdev, Physical Review B **78**, 064512 (2008).

1. Quantum-critical transport

   Collisionless-to-hydrodynamic crossover of CFT3s

2. Exact solution from AdS/CFT

   Constraints from duality relations

3. Quantum criticality of Dirac fermions

   "Vector" 1/N expansion

4. Quantum criticality of Fermi surfaces

   The genus expansion

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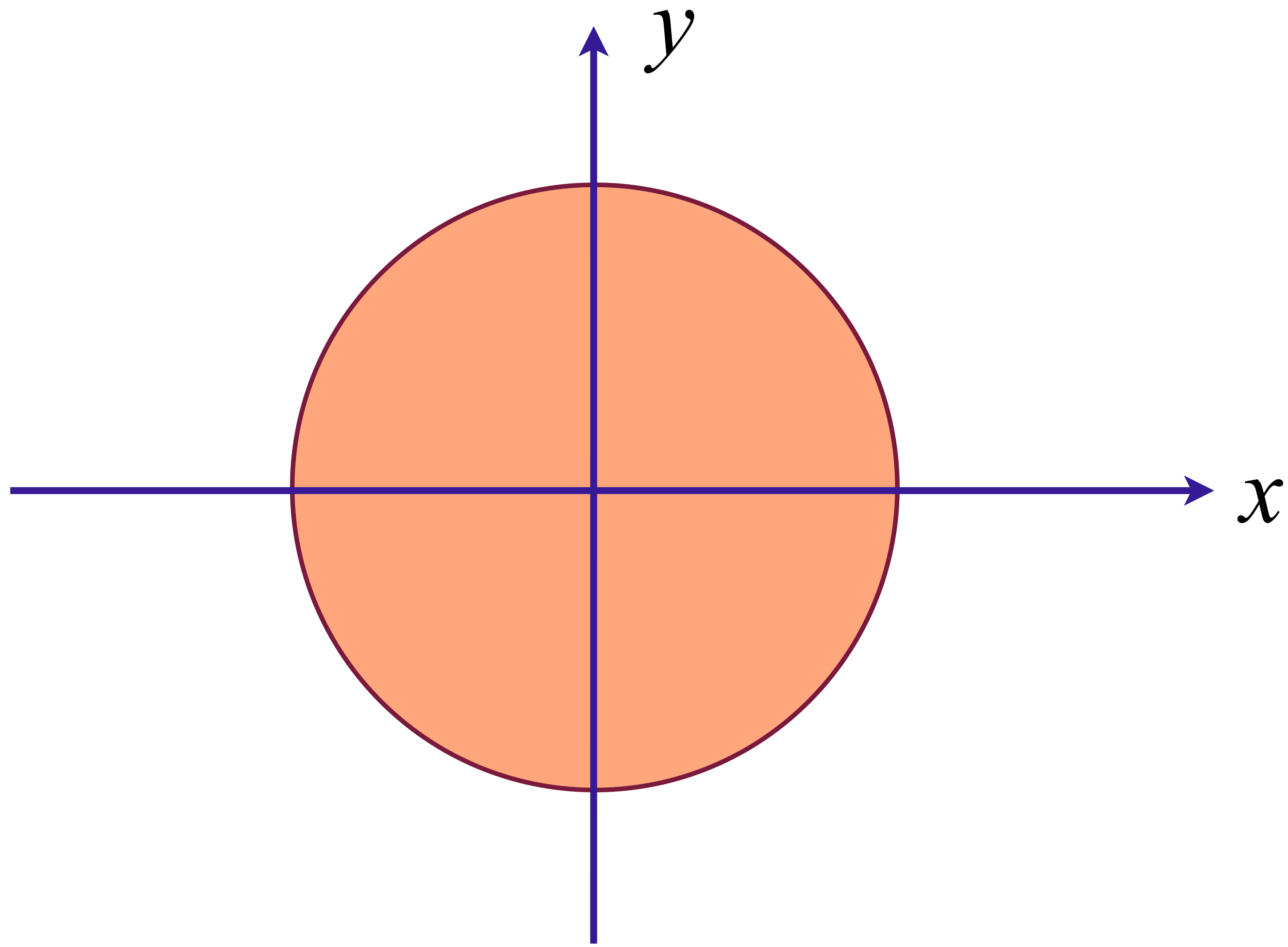
The genus expansion

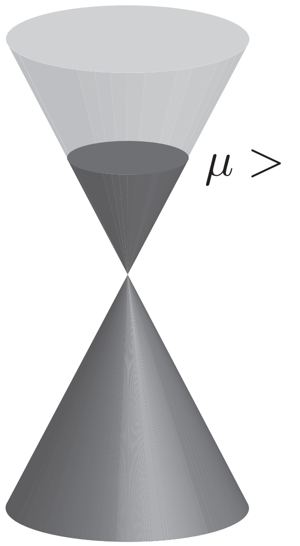
Quantum criticality of Pomeranchuk instability



Fermi surface with full square lattice symmetry

Electron Green's function in Fermi liquid (T=0)

$$G(k,\omega) = \frac{Z}{\omega - v_F(k-k_F) - i\omega^2 \mathcal{F}\left(\frac{k-k_F}{\omega}\right)} + \ldots$$



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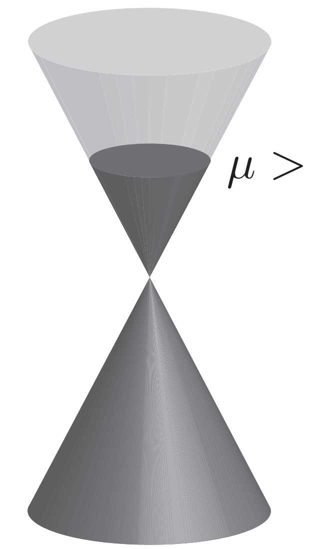
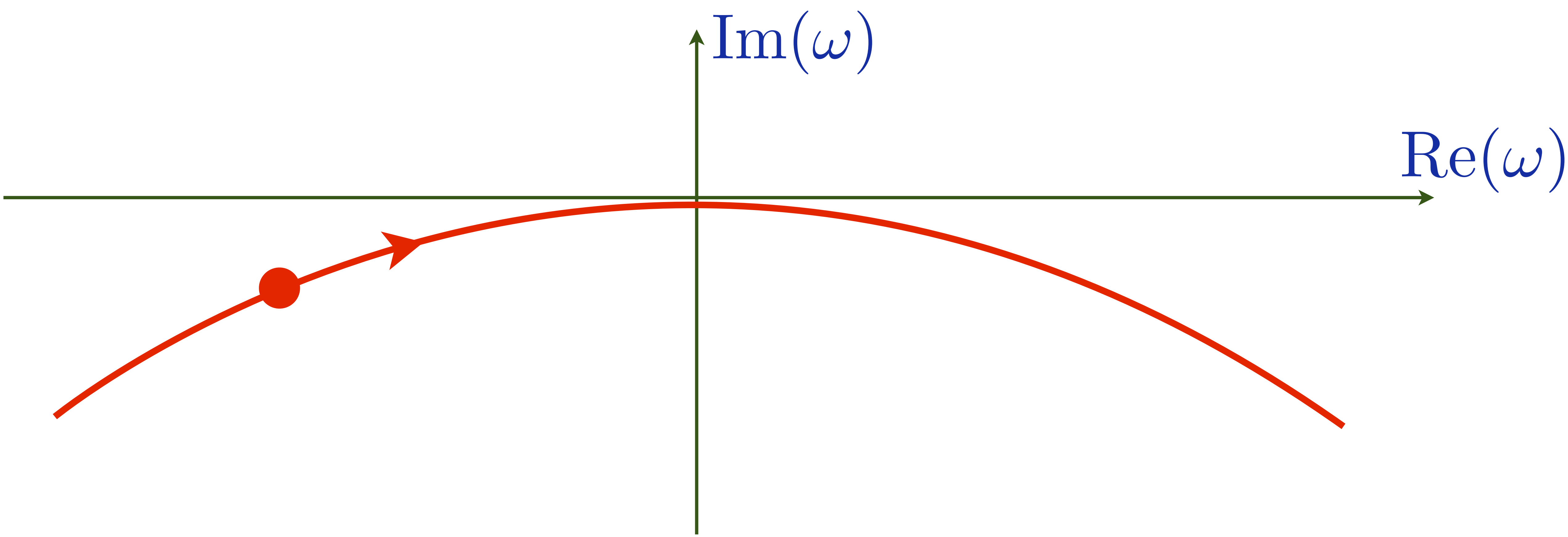


$$G(k,\omega) = \frac{Z}{\omega - v_F(k-k_F) - i\omega^2 \mathcal{F}\left(\frac{k-k_F}{\omega}\right)} + \ldots$$

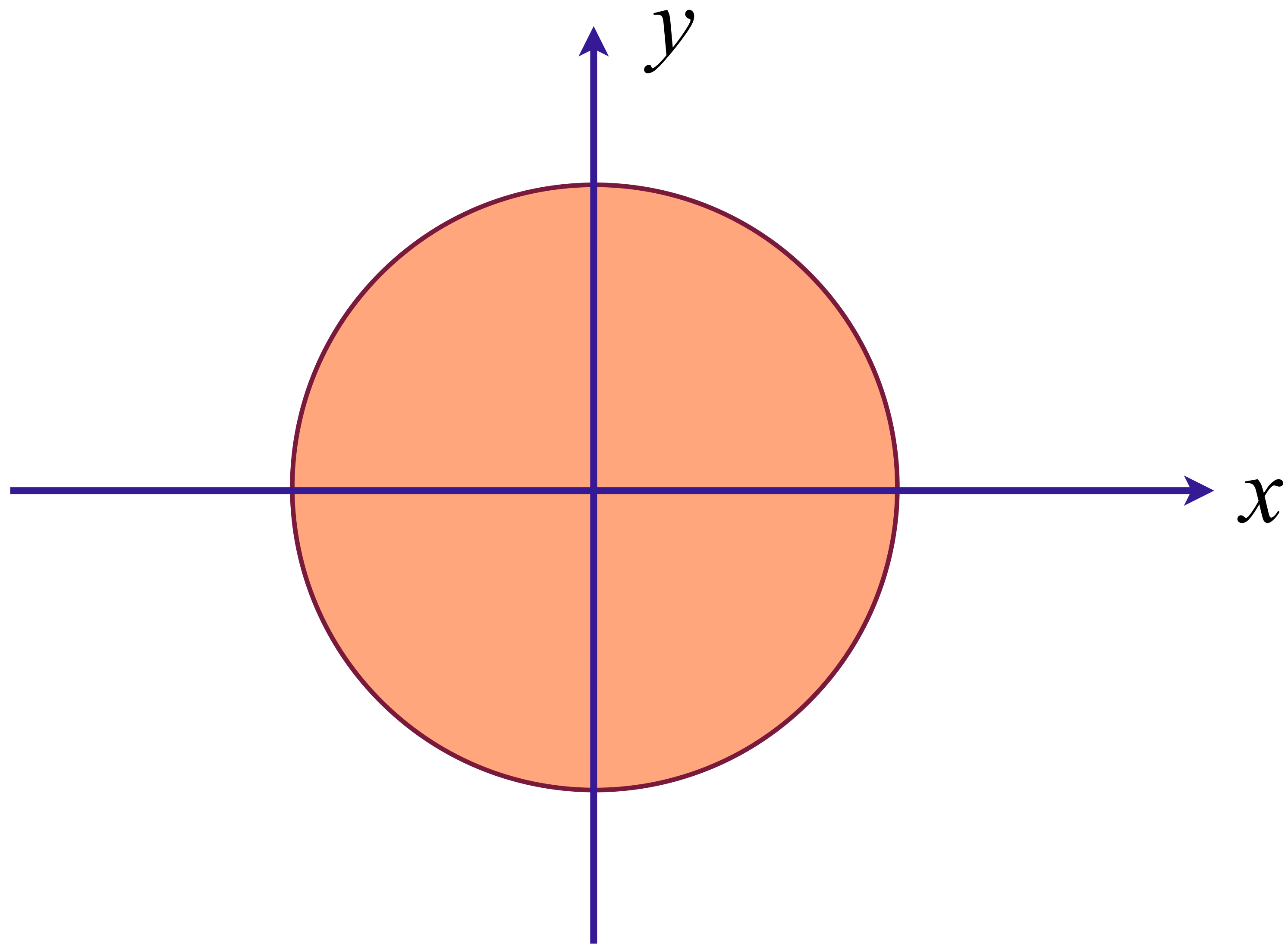
Green's function has a pole in the LHP at

$$\omega = v_F(k-k_F) - i\alpha(k-k_F)^2 + \ldots$$

Pole is at $\omega = 0$ precisely at $k = k_F$ *i.e.* on a sphere of radius k_F in momentum space. This is the *Fermi surface.*

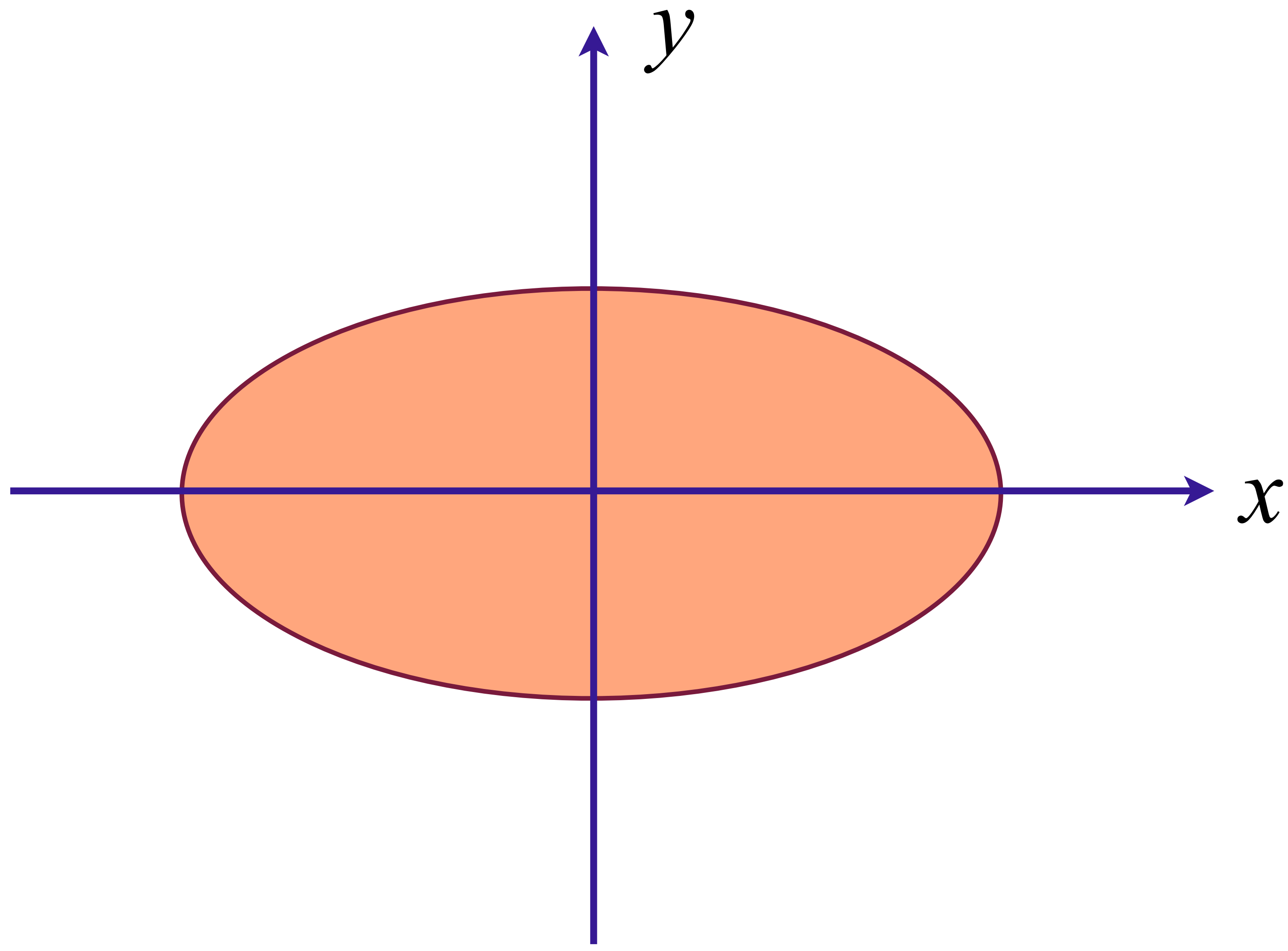


Quantum criticality of Pomeranchuk instability



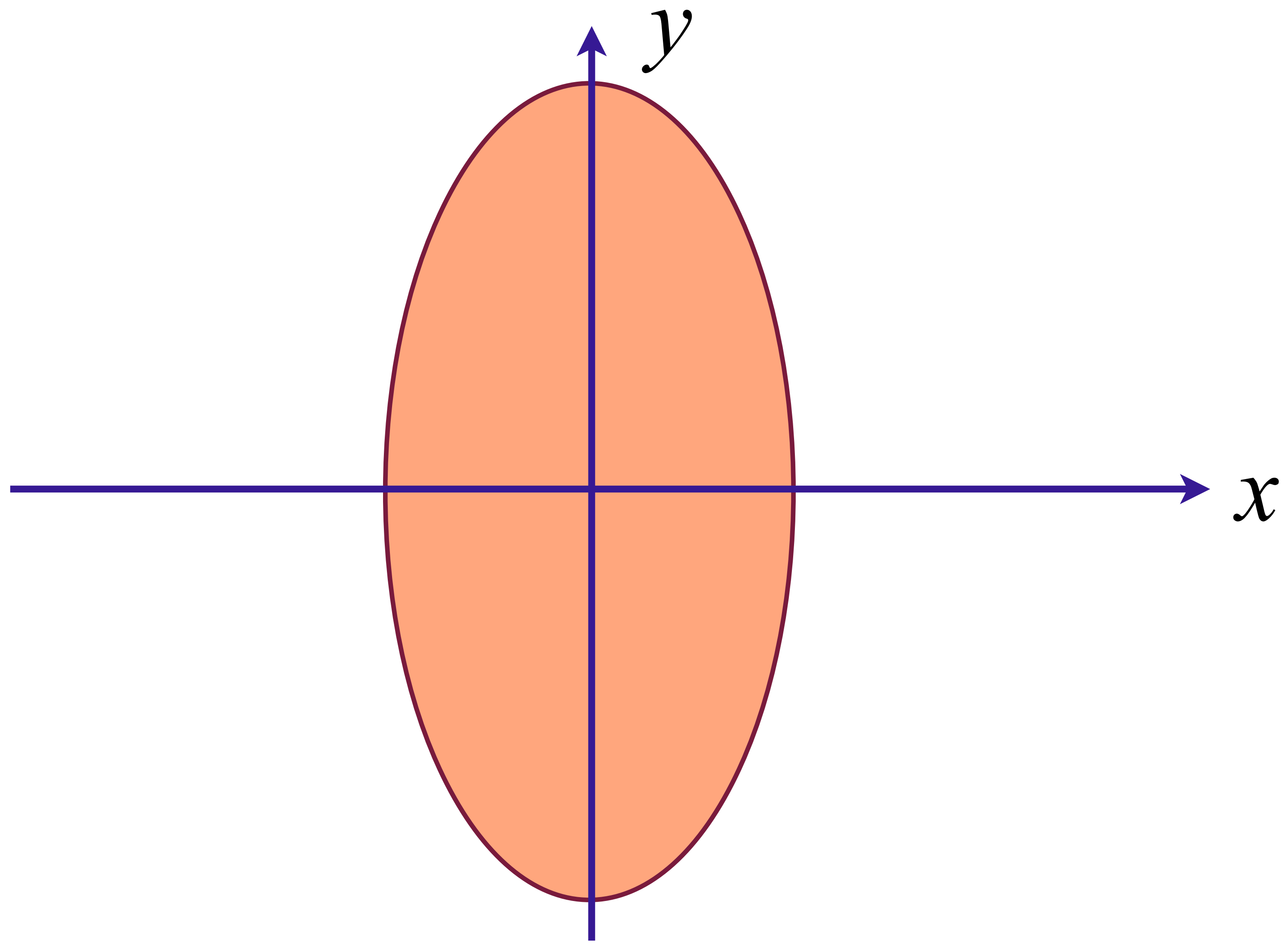
Fermi surface with full square lattice symmetry

Quantum criticality of Pomeranchuk instability



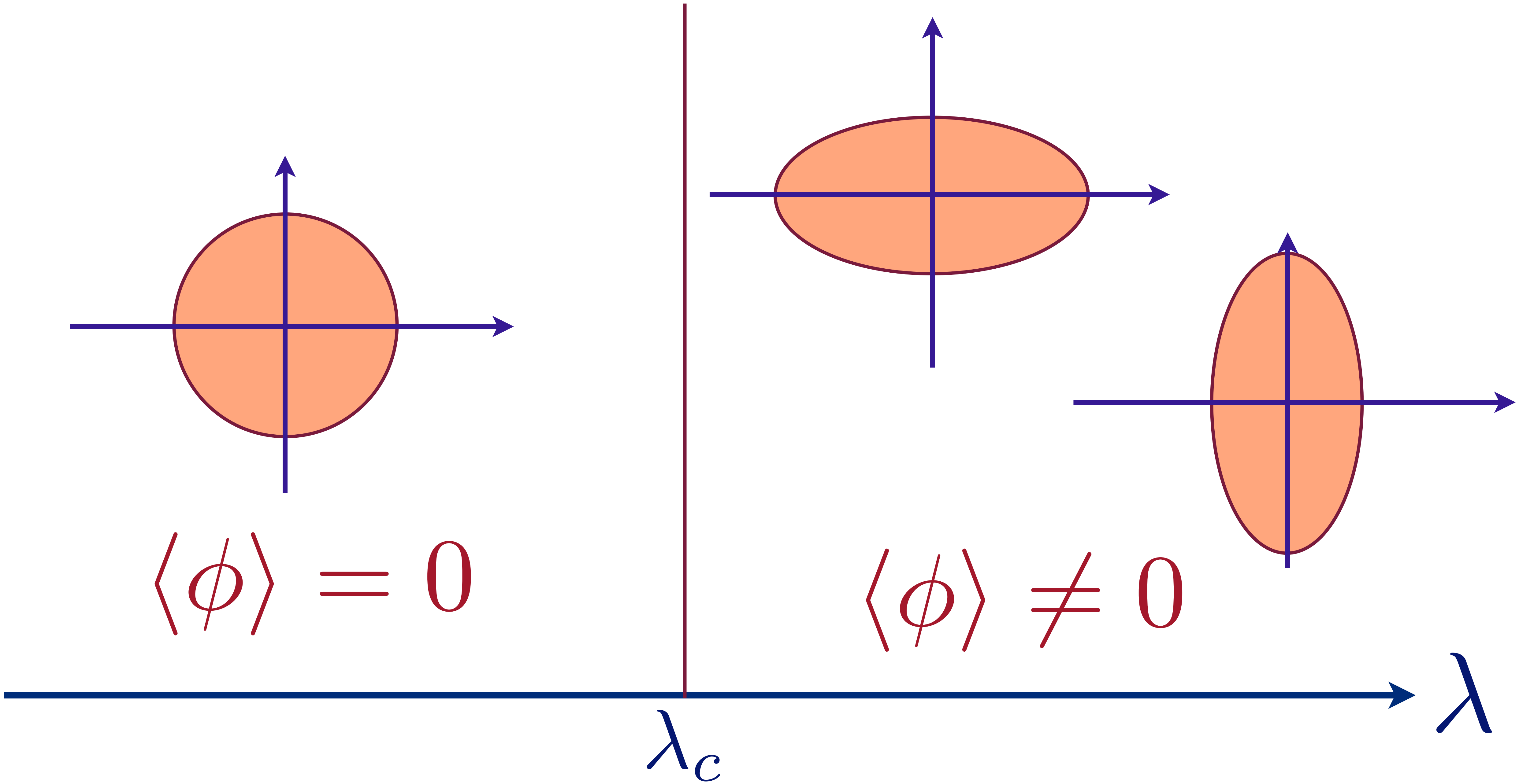
Spontaneous elongation along x direction:
Ising order parameter $\phi > 0$.

Quantum criticality of Pomeranchuk instability



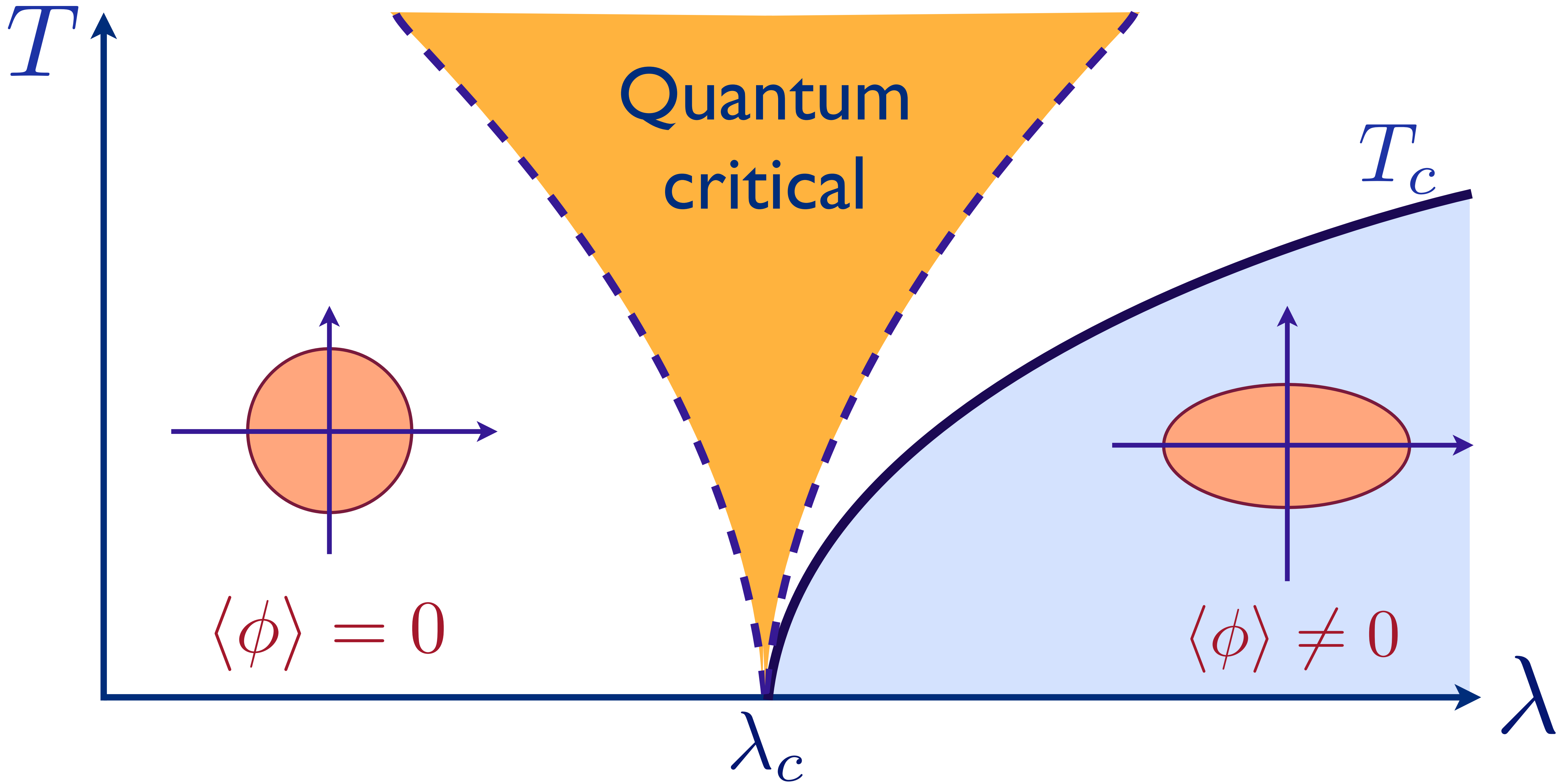
Spontaneous elongation along y direction:
Ising order parameter $\phi < 0$.

Quantum criticality of Pomeranchuk instability



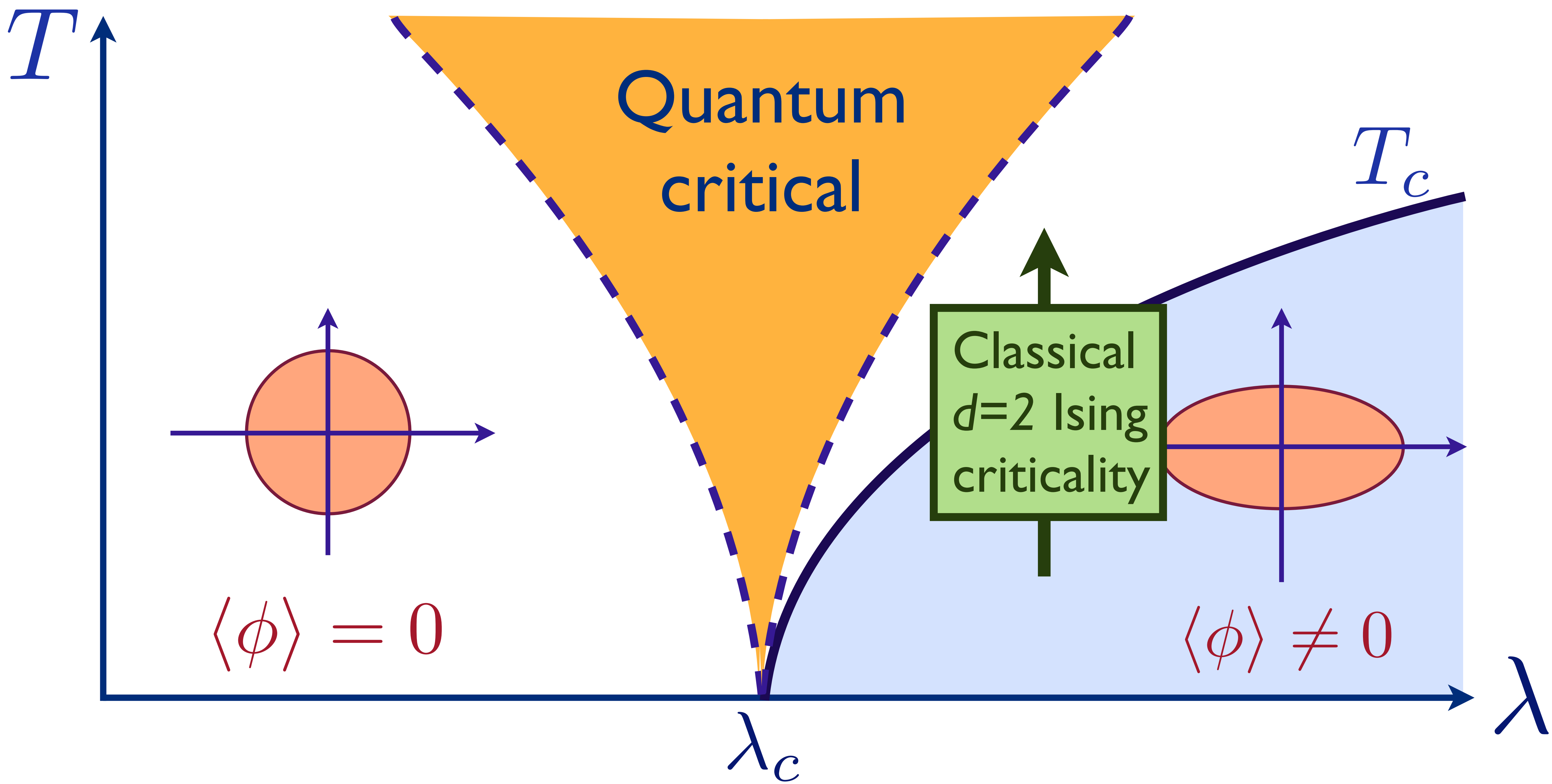
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Pomeranchuk instability



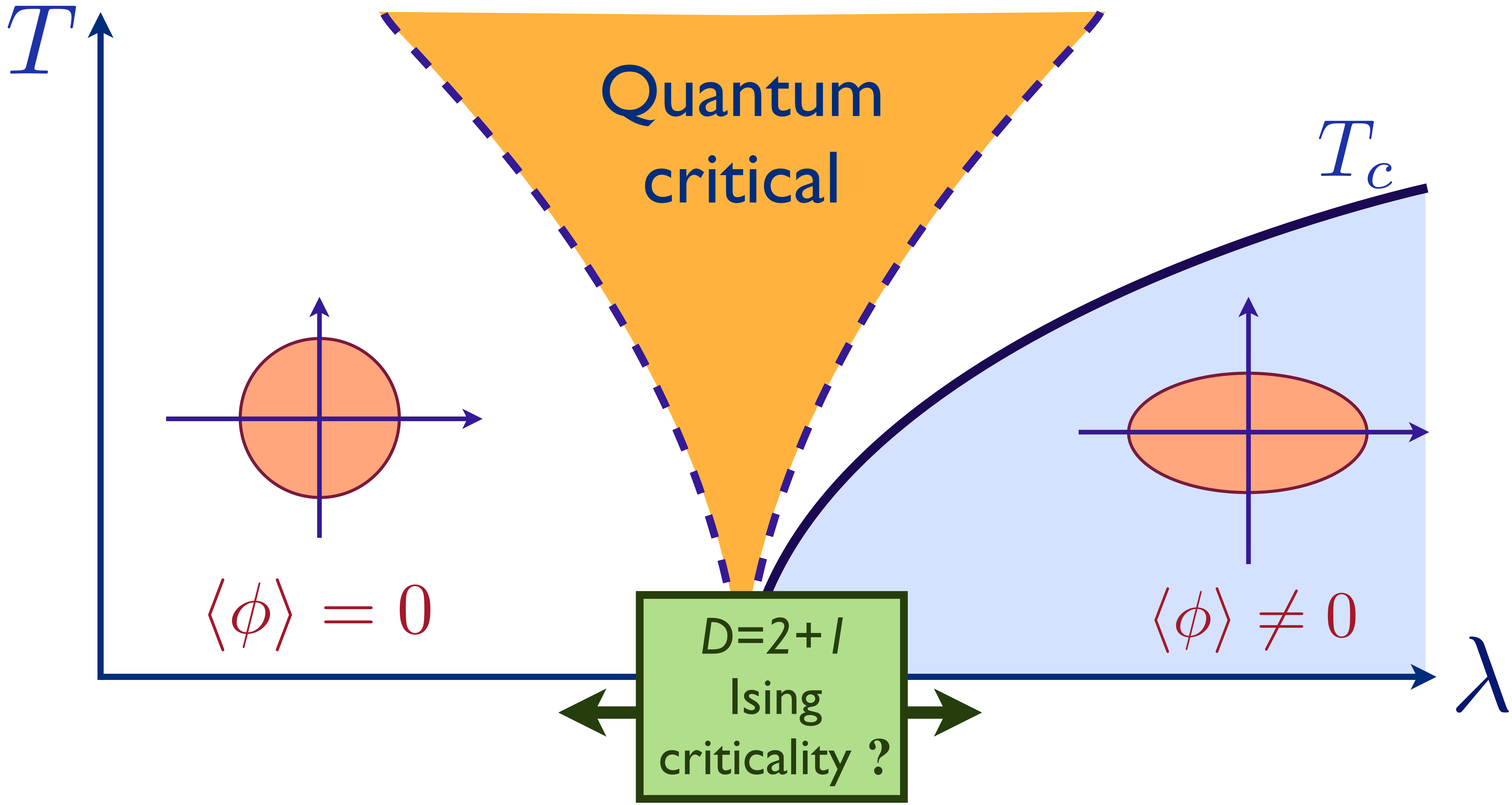
Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability



Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability



Phase diagram as a function of T and λ

Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

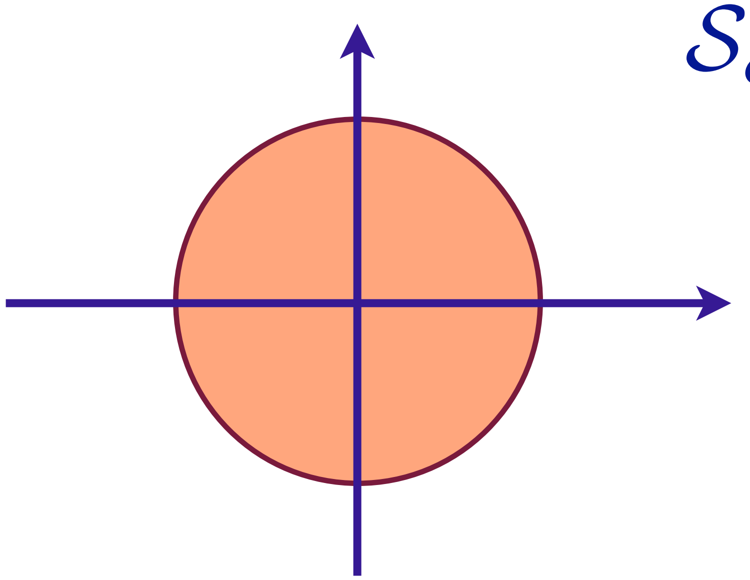
$$\mathcal{S}_\phi = \int d^2 r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$

Quantum criticality of Pomeranchuk instability

Effective action for Ising order parameter

$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right]$$

Effective action for electrons:



$$\mathcal{S}_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[\sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right]$$

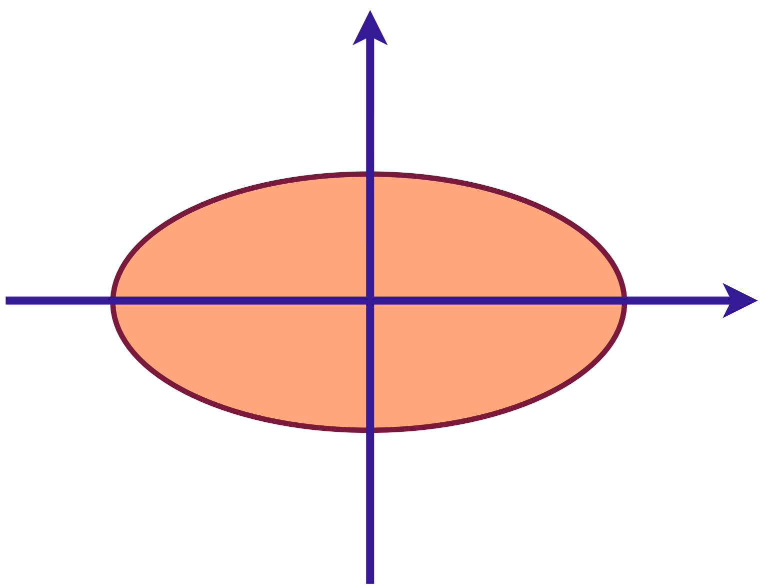
$$\equiv \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

Quantum criticality of Pomeranchuk instability

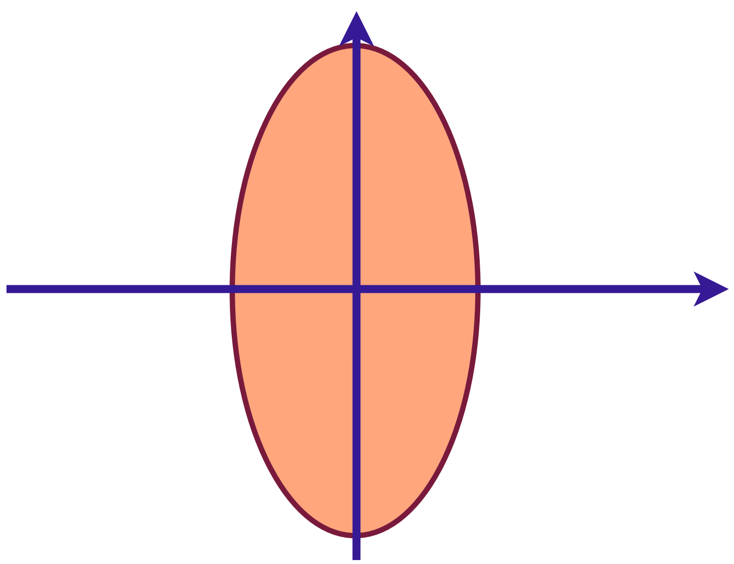
Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \, \phi \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} (\cos k_x - \cos k_y) c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha}$$

for spatially independent ϕ



$\langle \phi \rangle > 0$



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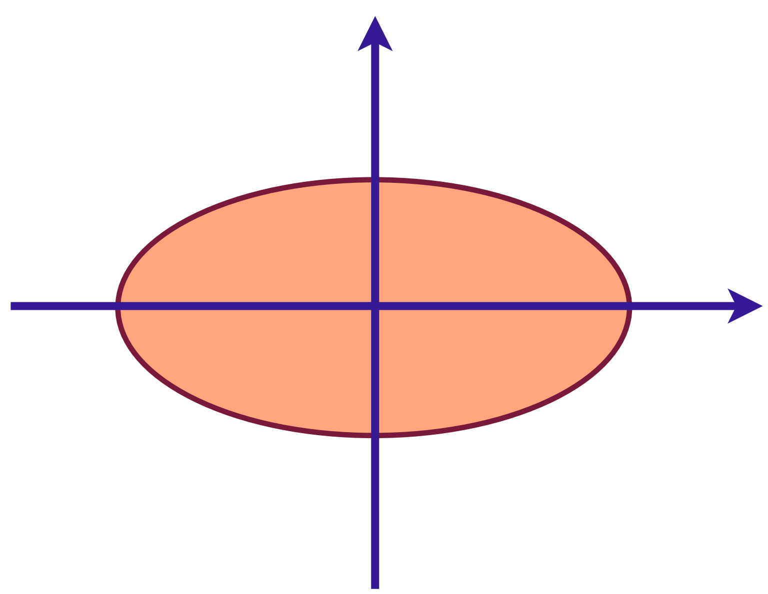
Quantum criticality of Pomeranchuk instability

Coupling between Ising order and electrons

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{q}} \left(\cos k_x - \cos k_y\right) c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\alpha} c_{\mathbf{k}-\mathbf{q}/2,\alpha}$$

for spatially dependent ϕ



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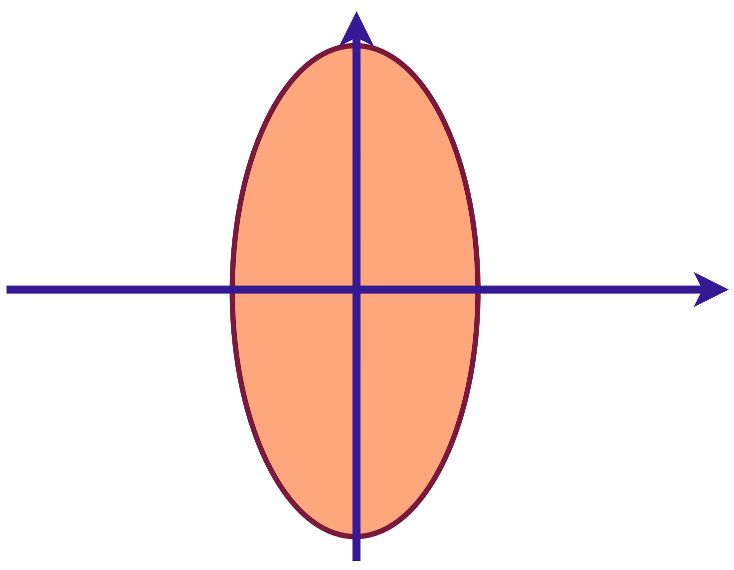

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Quantum criticality of Pomeranchuk instability

$$\mathcal{S}_\phi = \int d^2r d\tau \left[(\partial_\tau \phi)^2 + c^2 (\nabla \phi)^2 + (\lambda - \lambda_c) \phi^2 + u \phi^4 \right]$$

$$\mathcal{S}_c = \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k}} \int d\tau c_{\mathbf{k}\alpha}^\dagger \left(\partial_\tau + \varepsilon_{\mathbf{k}} \right) c_{\mathbf{k}\alpha}$$

$$\mathcal{S}_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{q}} \left(\cos k_x - \cos k_y \right) c_{\mathbf{k}+\mathbf{q}/2,\alpha}^\dagger c_{\mathbf{k}-\mathbf{q}/2,\alpha}$$

Quantum critical field theory

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}c_{i\alpha} \exp\left(-\mathcal{S}_\phi - \mathcal{S}_c - \mathcal{S}_{\phi c} \right)$$

Quantum criticality of Pomeranchuk instability

Hertz theory

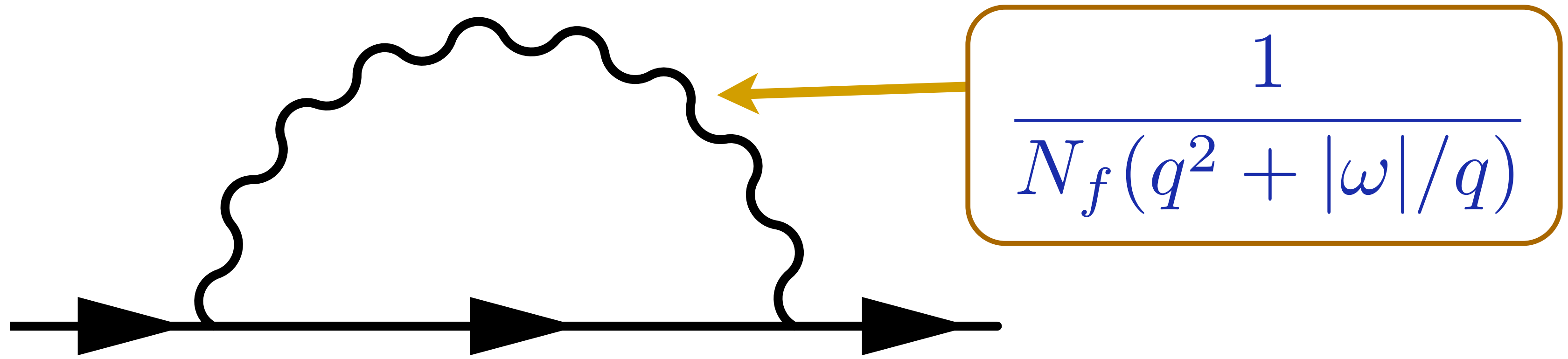
Integrate out c_α fermions and obtain non-local corrections to ϕ action

$$\delta\mathcal{S}_\phi \sim N_f \gamma^2 \int \frac{d^2q}{4\pi^2} \int \frac{d\omega}{2\pi} |\phi(\mathbf{q},\omega)|^2 \left[\frac{|\omega|}{q} + q^2\right] + \dots$$

This leads to a critical point with dynamic critical exponent $z = 3$ and quantum criticality controlled by the Gaussian fixed point.

Quantum criticality of Pomeranchuk instability

Hertz theory



Self energy of c_α fermions to order $1/N_f$

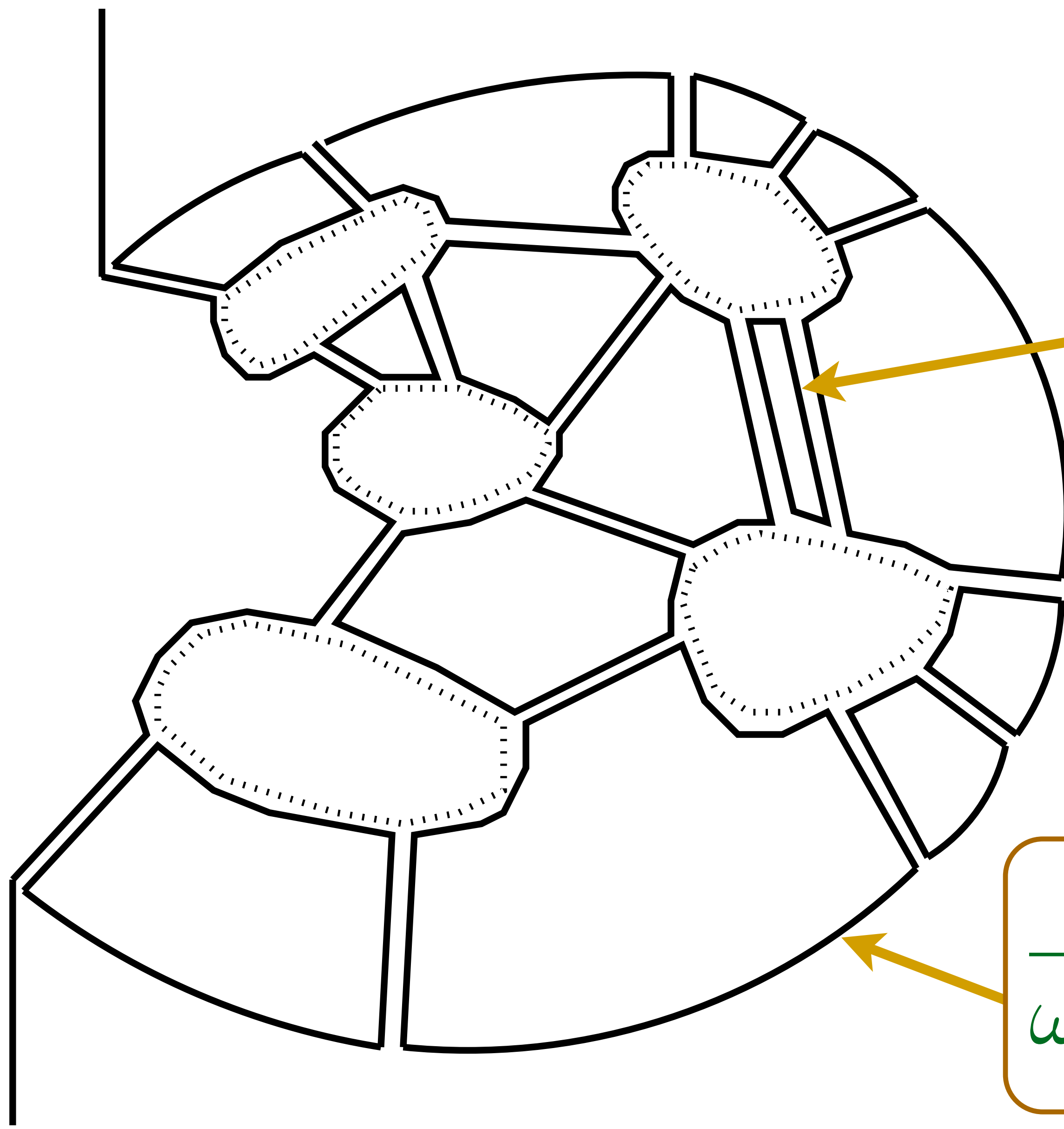
$$\Sigma_c(k,\omega) \sim \frac{i}{N_f}\omega^{2/3}$$

This leads to the Green's function

$$G(k,\omega) \approx \frac{1}{\omega - v_F(k-k_F) - \frac{i}{N_f}\omega^{2/3}}$$

Note that the order $1/N_f$ term is more singular in the infrared than the bare term; this leads to problems in the bare $1/N_f$ expansion in terms that are dominated by low frequency fermions.

Quantum criticality of Pomeranchuk instability



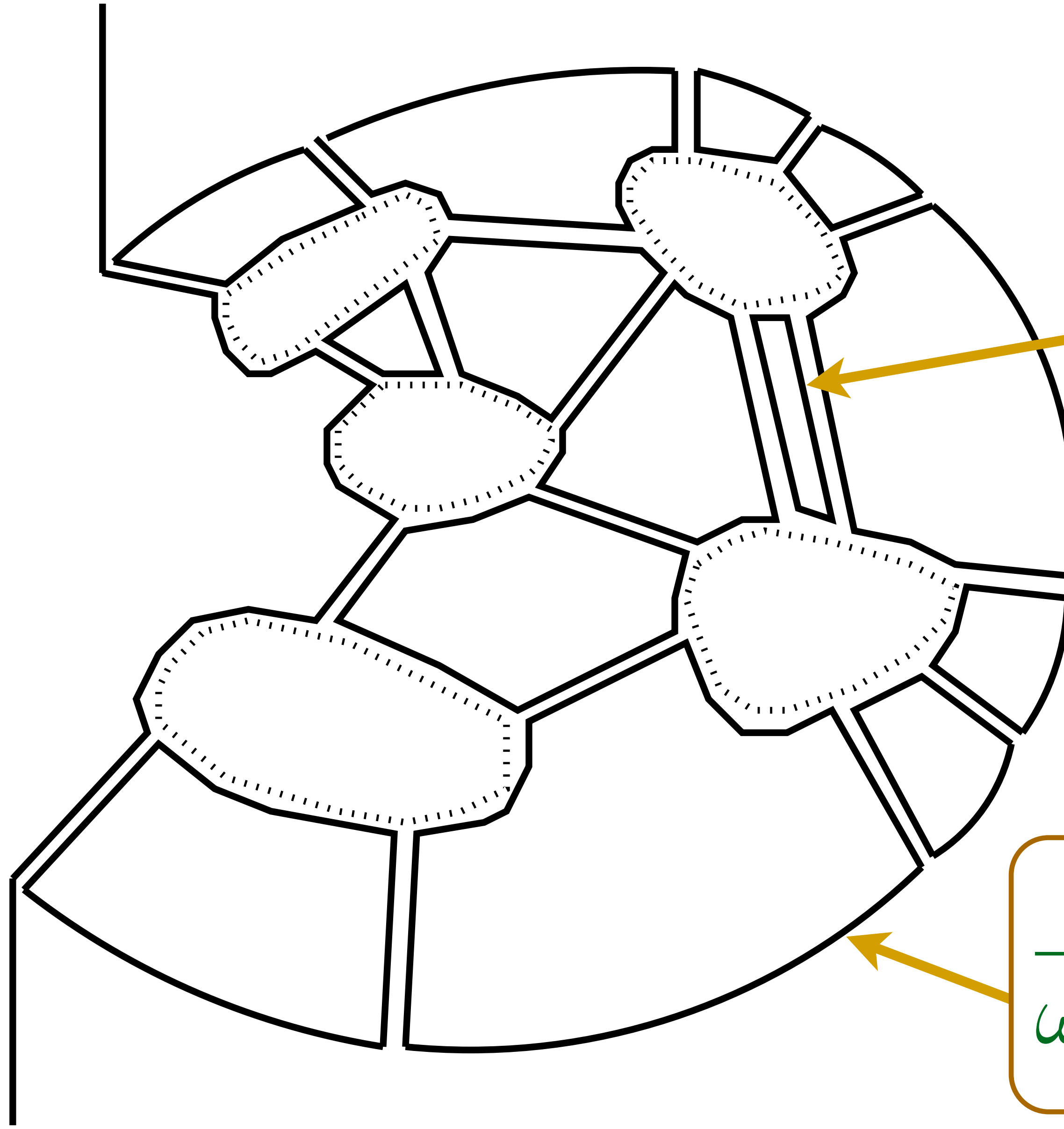
$$\frac{1}{N_f(q^2 + |\omega|/q)}$$

$$\frac{1}{\omega - v_F(k - k_F) - \frac{i}{N_f}\omega^{2/3}}$$

The infrared singularities of fermion particle-hole pairs are most severe on planar graphs: these all contribute at leading order in $1/N_f$.

Sung-Sik Lee, *Physical Review* B **80**, 165102 (2009)

Quantum criticality of Pomeranchuk instability

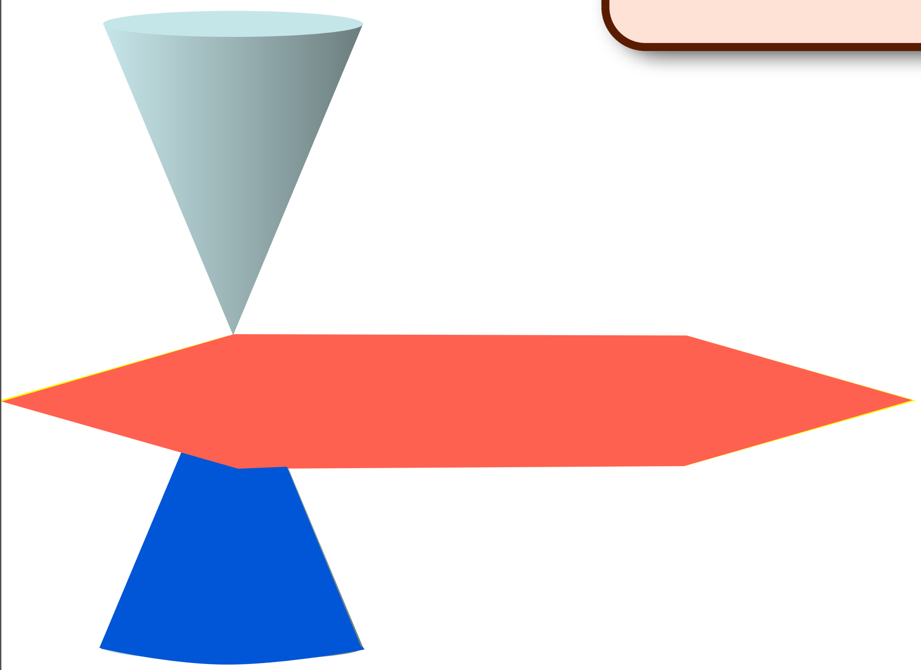


$$\frac{1}{N_f(q^2 + |\omega|/q)}$$

$$\frac{1}{\omega - v_F(k - k_F) - \frac{i}{N_f}\omega^{2/3}}$$

A string theory for the Fermi surface ?

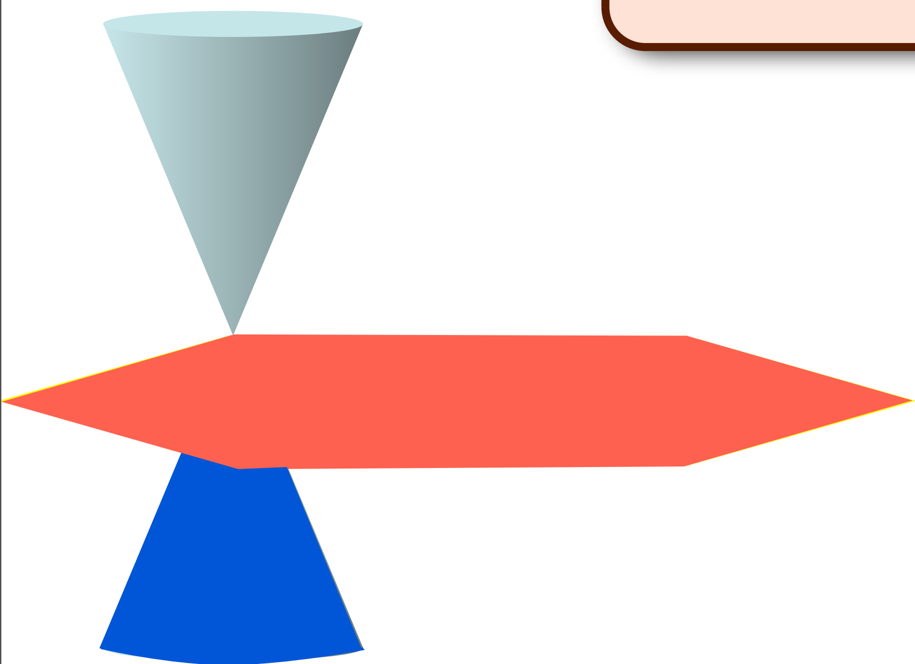
Conformal field theory
in 2+1 dimensions at $T = 0$



$\Updownarrow$

Einstein gravity
on AdS_4

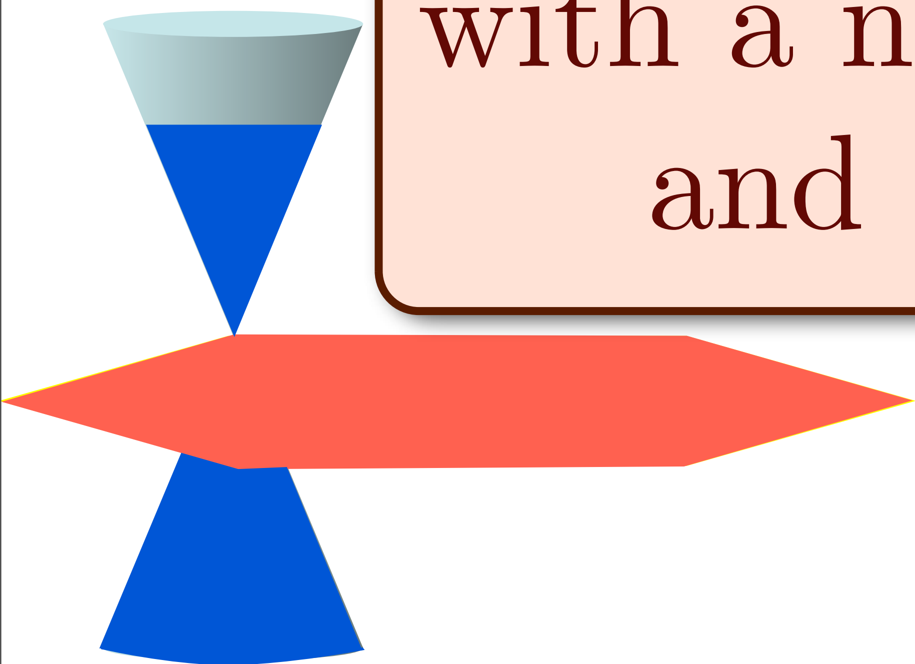
Conformal field theory
in 2+1 dimensions at $T > 0$



$\Updownarrow$

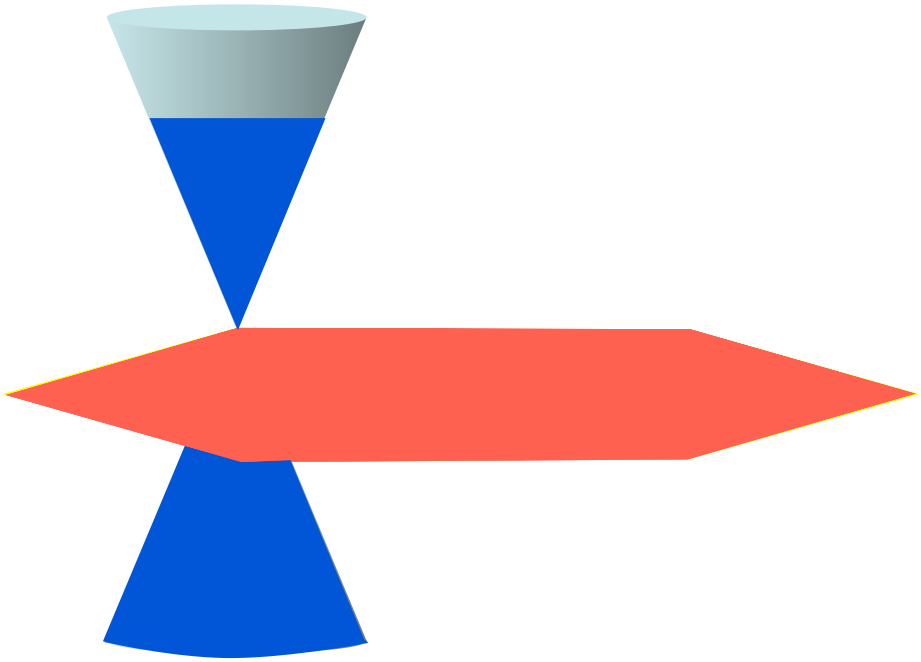
Einstein gravity on AdS_4
with a Schwarzschild
black hole

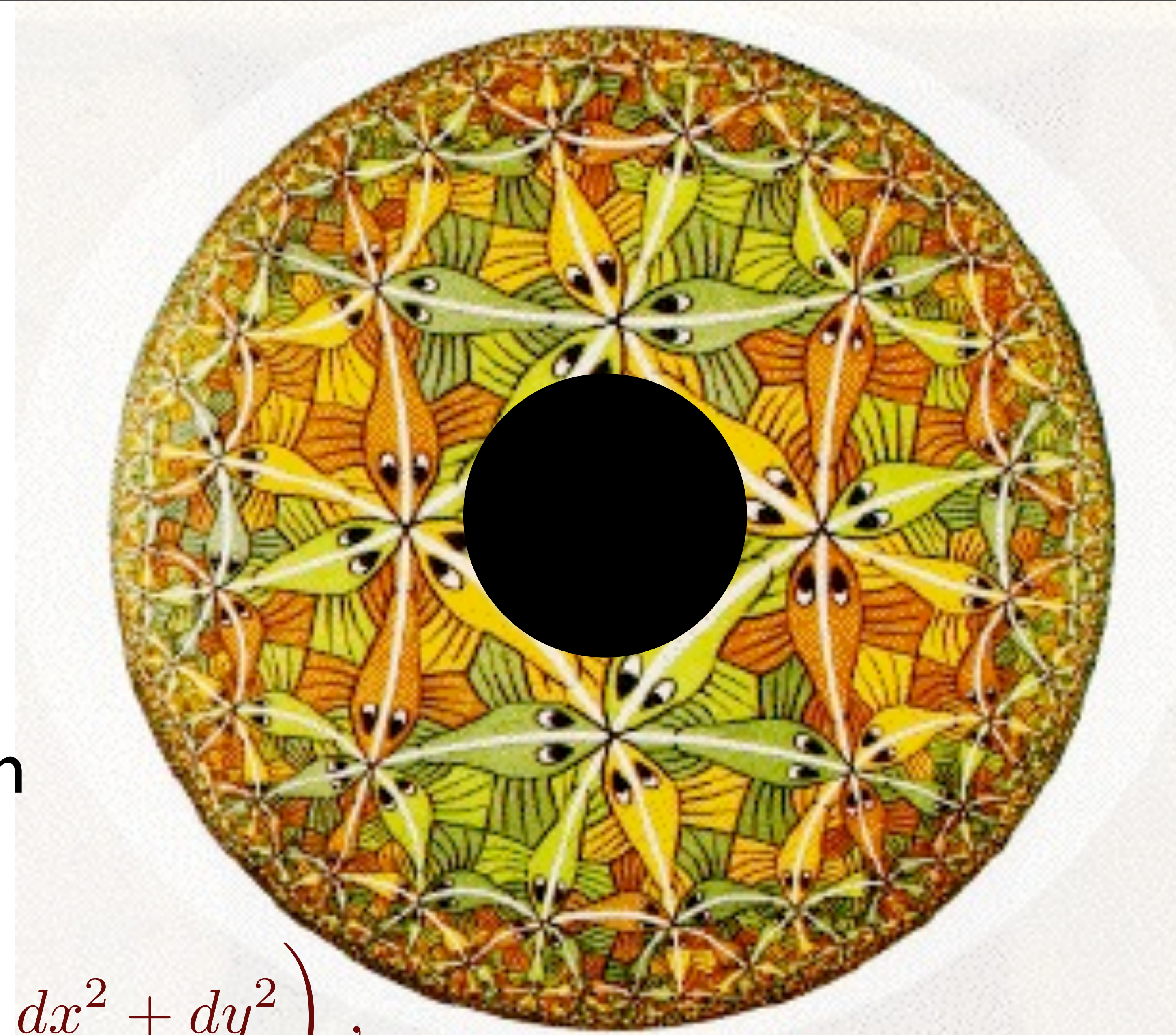
Conformal field theory
in 2+1 dimensions at $T > 0$,
with a non-zero chemical potential, μ
and applied magnetic field, B



$\Updownarrow$

Einstein gravity on AdS_4
with a Reissner-Nordstrom
black hole carrying electric
and magnetic charges





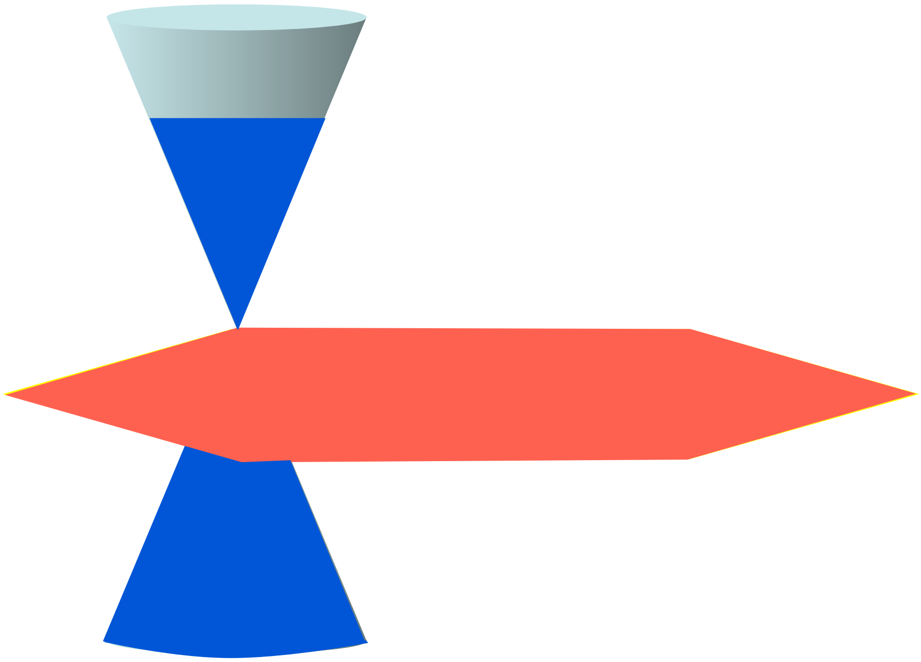
AdS_4-Reissner-Nordstrom black hole

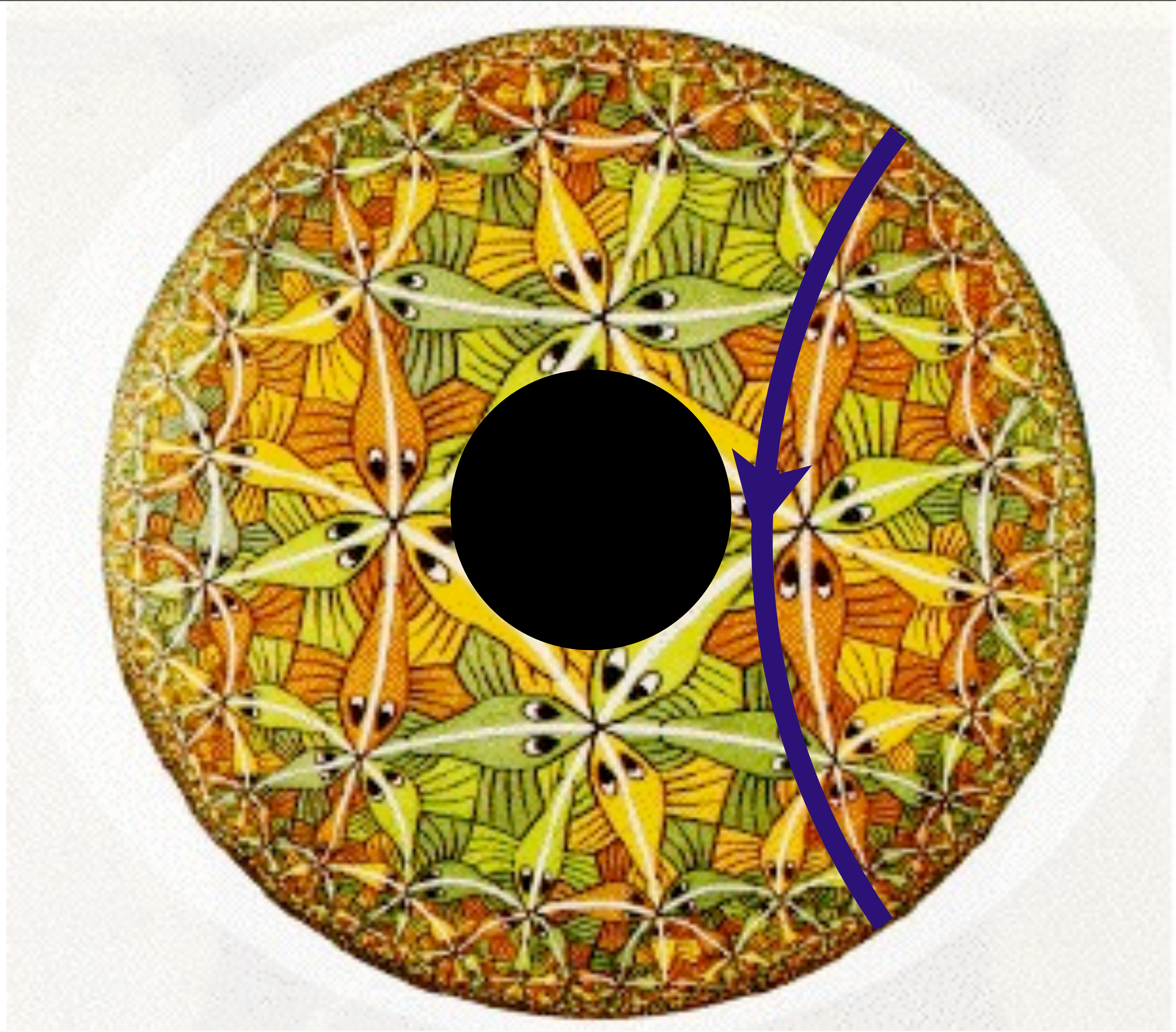
$$ds^2 = \frac{L^2}{r^2}\left(f(r)d\tau^2 + \frac{dr^2}{f(r)} + dx^2 + dy^2\right),$$

$$f(r) = 1 - \left(1 + \frac{(r_+^2\mu^2 + r_+^4 B^2)}{\gamma^2}\right)\left(\frac{r}{r_+}\right)^3 + \frac{(r_+^2\mu^2 + r_+^4 B^2)}{\gamma^2}\left(\frac{r}{r_+}\right)^4,$$

$$A = i\mu\left[1 - \frac{r}{r_+}\right]d\tau + Bx\,dy\,.$$

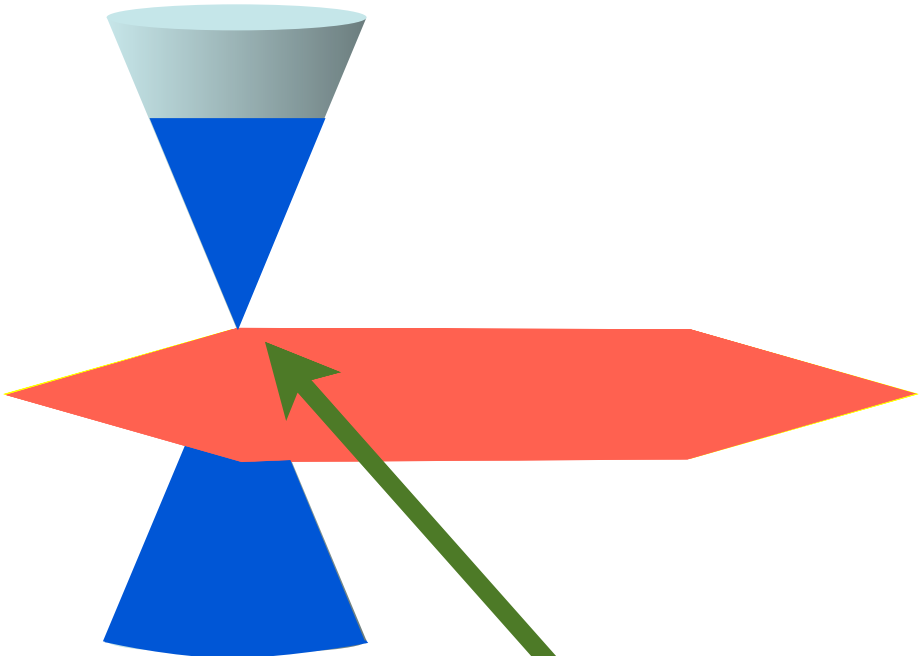
$$T = \frac{1}{4\pi r_+}\left(3 - \frac{r_+^2\mu^2}{\gamma^2} - \frac{r_+^4 B^2}{\gamma^2}\right).$$

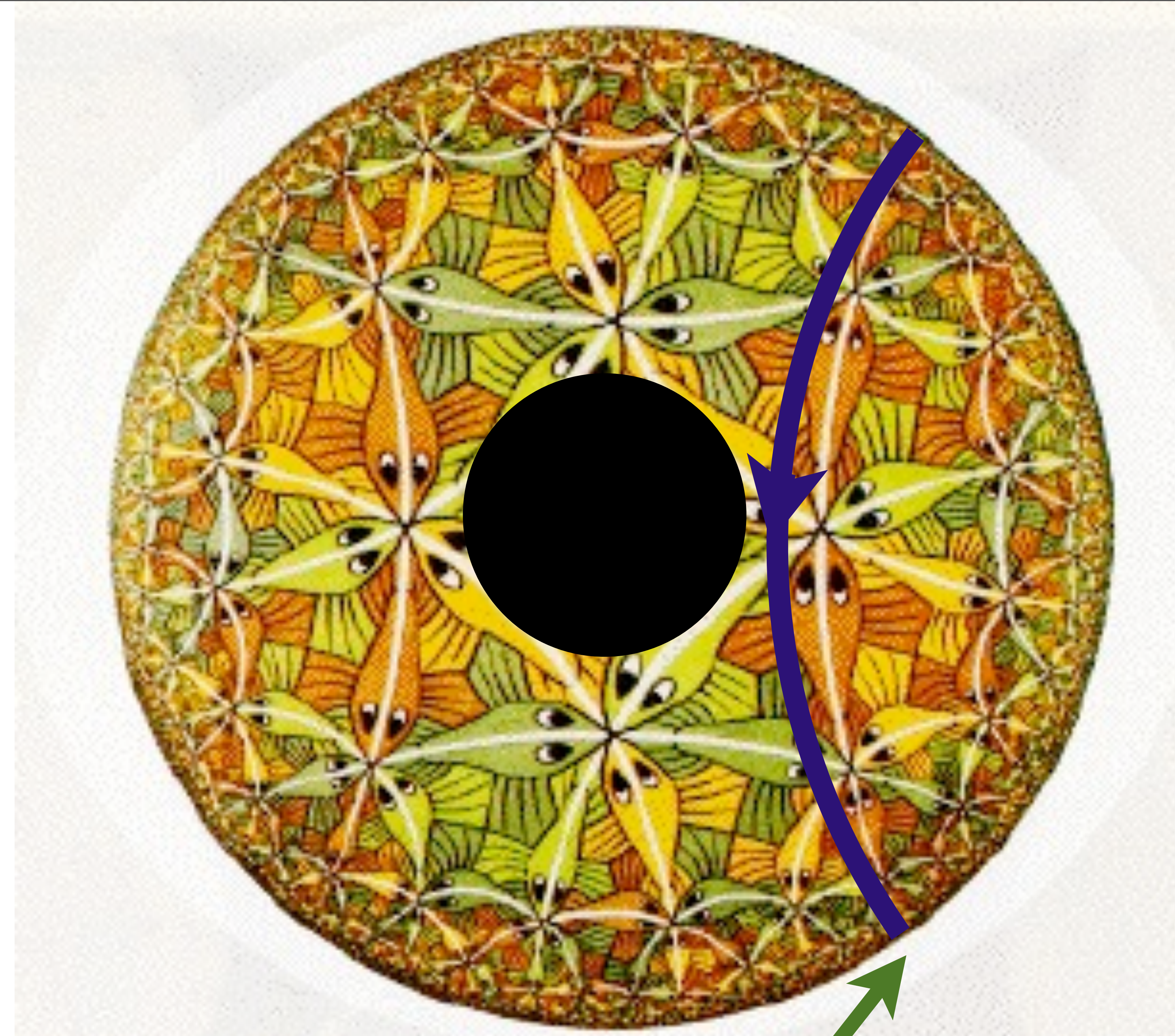




Examine free energy and Green's function of a probe particle

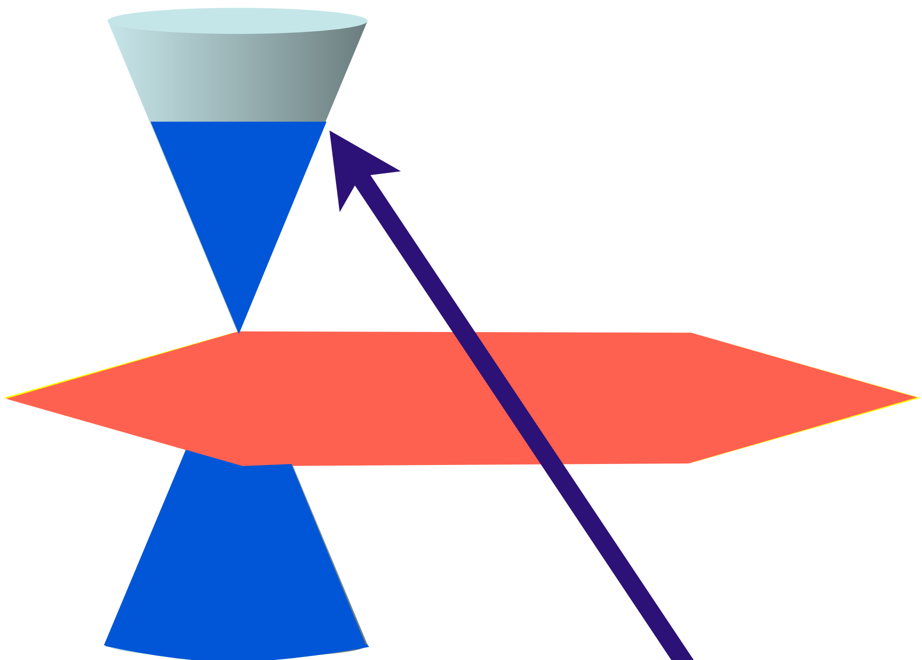
T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694
F. Denef, S. Hartnoll, and S. Sachdev, arXiv:0908.1788

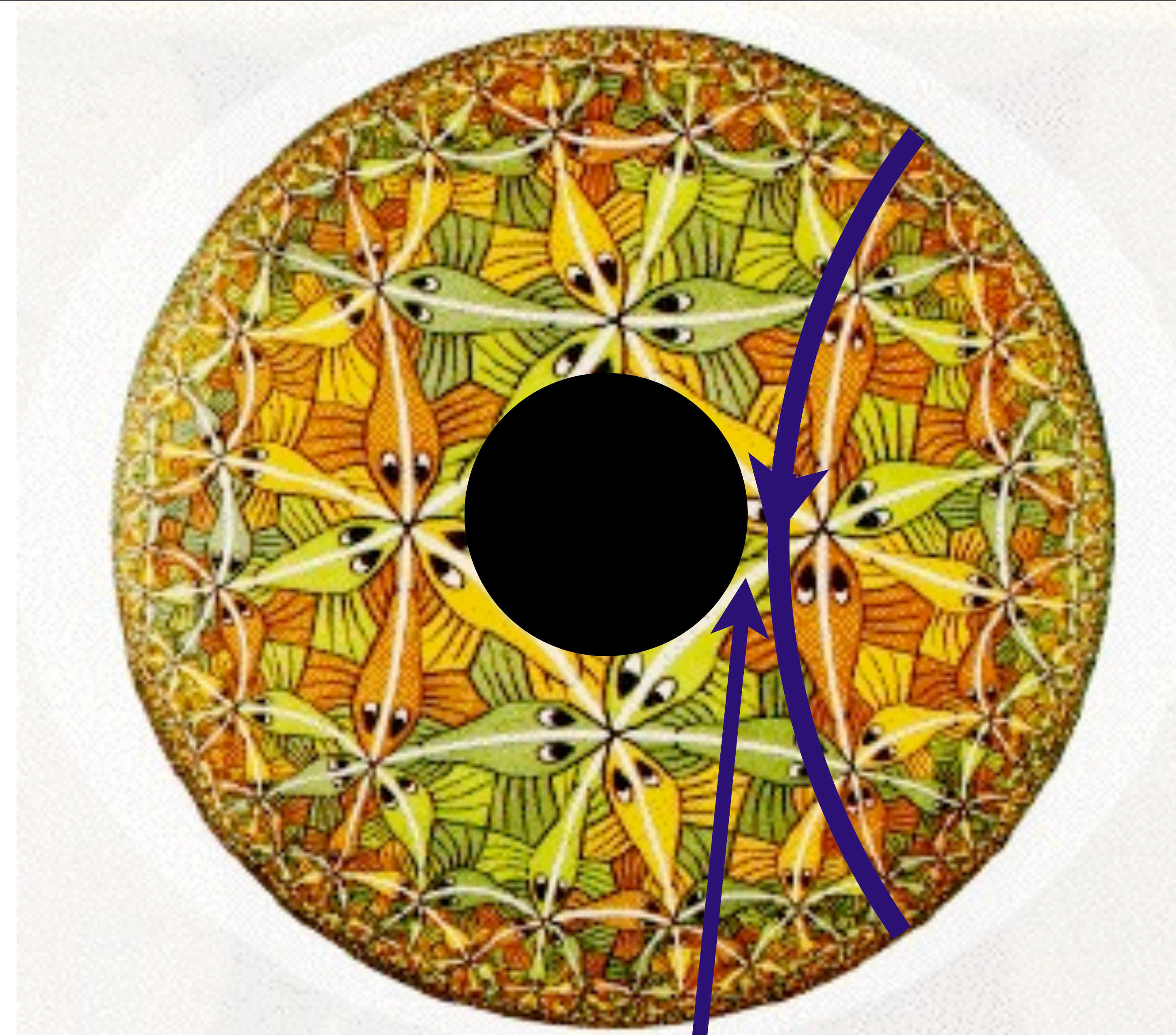




Short time behavior depends upon conformal AdS_4 geometry near boundary

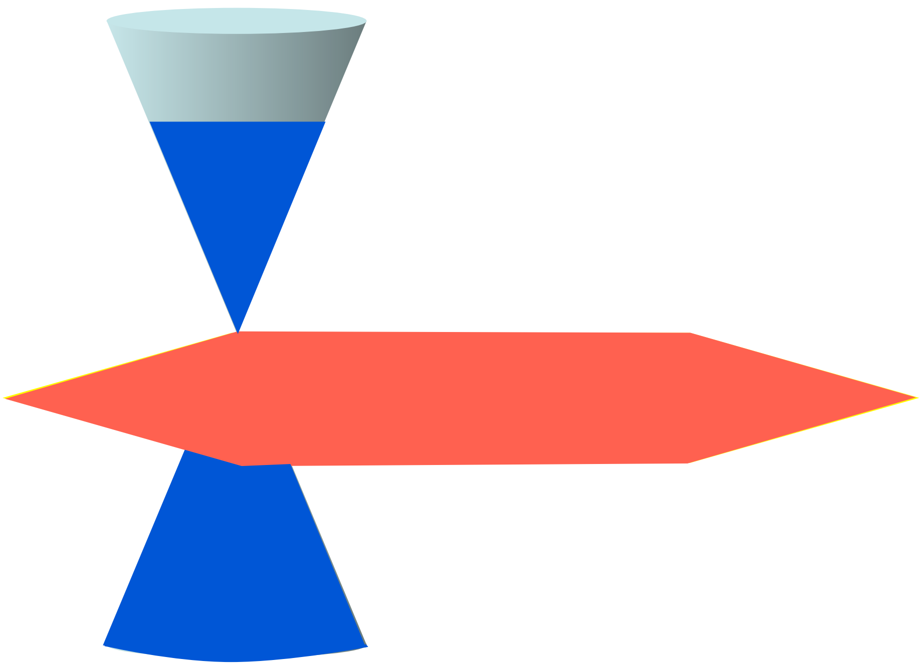
T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694
F. Denef, S. Hartnoll, and S. Sachdev, arXiv:0908.1788

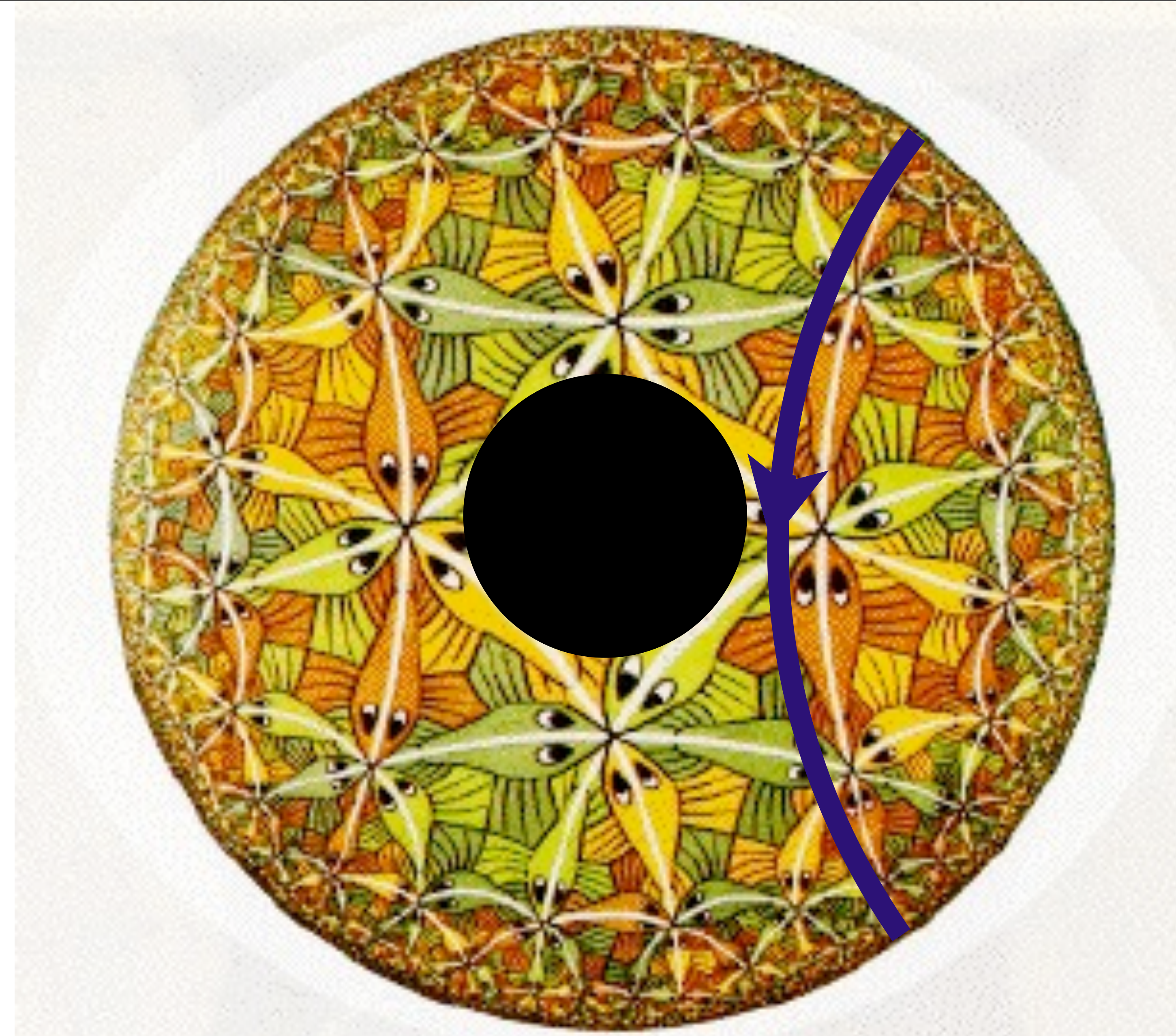




Long time behavior depends upon near-horizon geometry of black hole

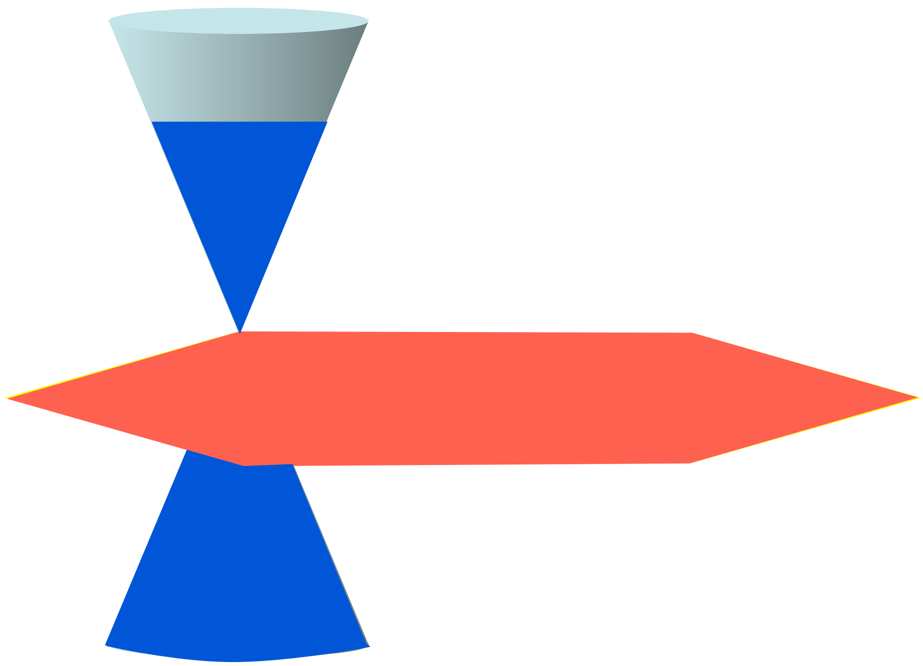
T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694
F. Denef, S. Hartnoll, and S. Sachdev, arXiv:0908.1788

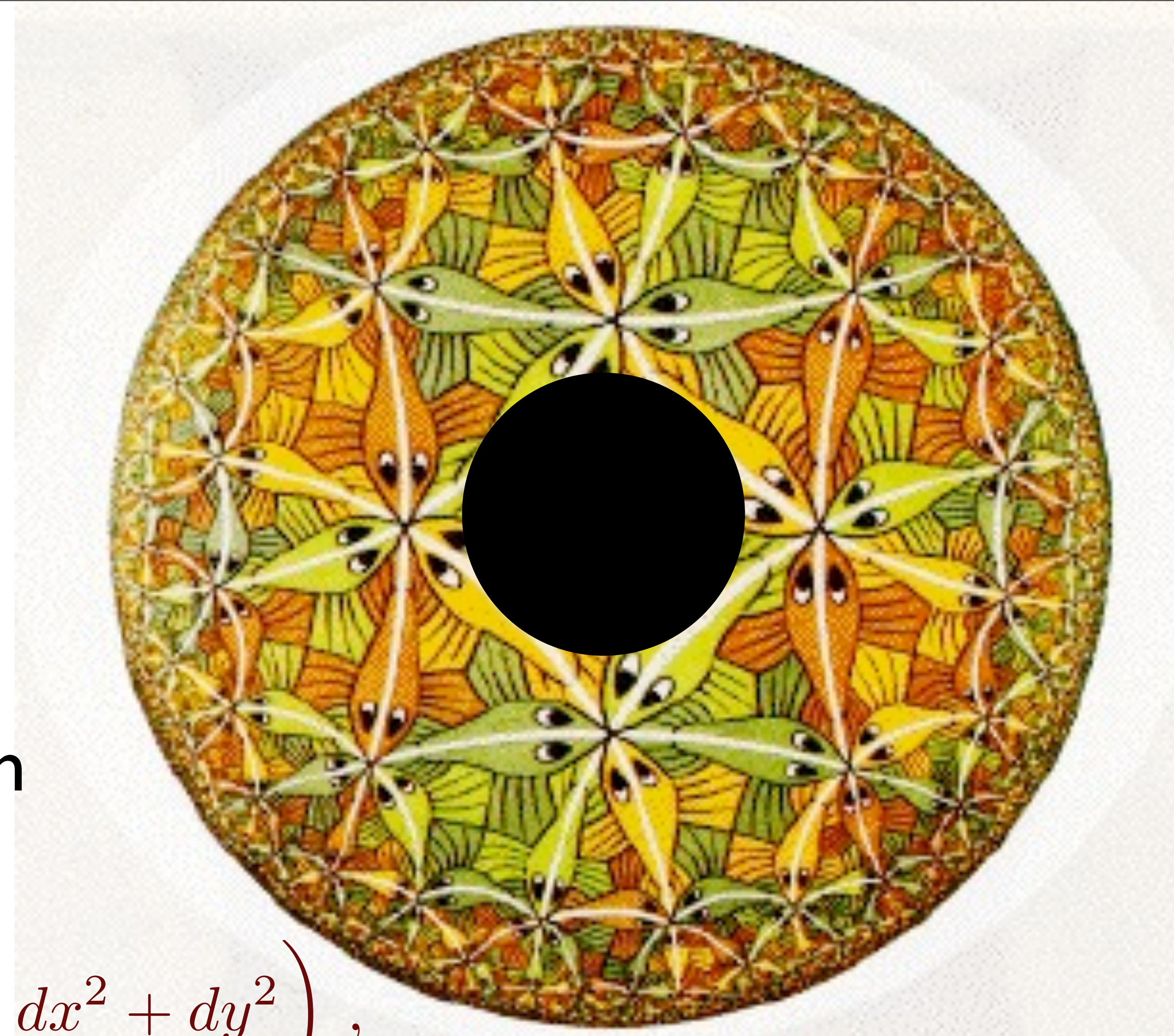




Radial direction of gravity theory is measure of energy scale in CFT

T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694
F. Denef, S. Hartnoll, and S. Sachdev, arXiv:0908.1788





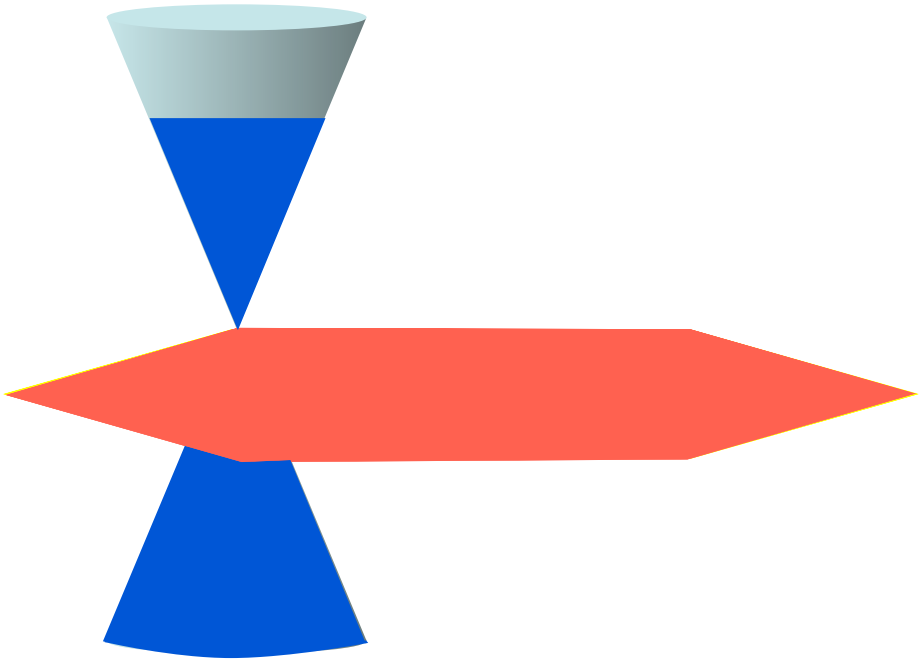
AdS$_4$-Reissner-Nordstrom black hole

$$ds^2 = \frac{L^2}{r^2}\left(f(r)d\tau^2 + \frac{dr^2}{f(r)} + dx^2 + dy^2\right),$$

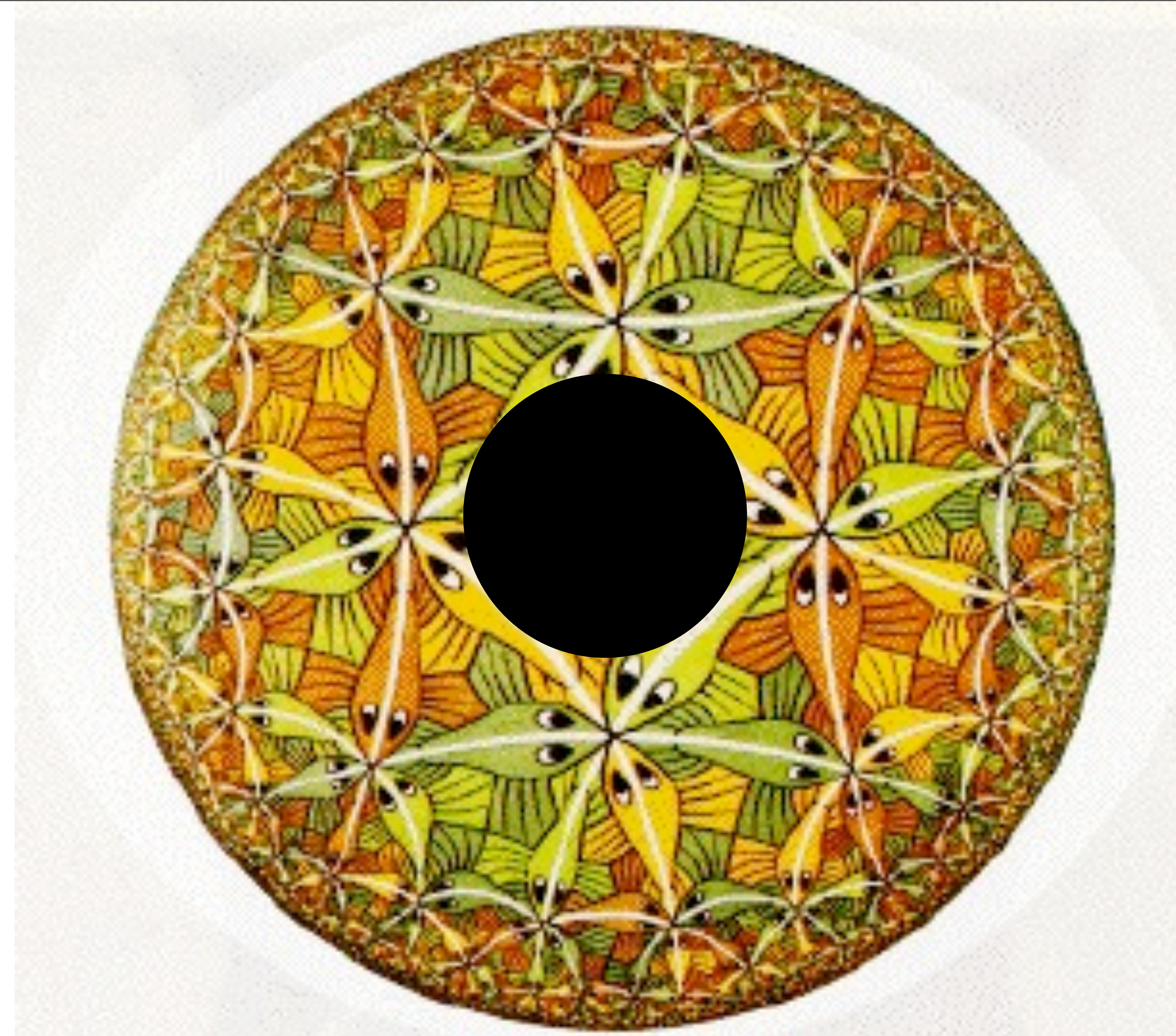
$$f(r) = 1 - \left(1 + \frac{(r_+^2\mu^2 + r_+^4 B^2)}{\gamma^2}\right)\left(\frac{r}{r_+}\right)^3 + \frac{(r_+^2\mu^2 + r_+^4 B^2)}{\gamma^2}\left(\frac{r}{r_+}\right)^4,$$

$$A = i\mu\left[1 - \frac{r}{r_+}\right]d\tau + Bx\,dy\,.$$

$$T = \frac{1}{4\pi r_+}\left(3 - \frac{r_+^2\mu^2}{\gamma^2} - \frac{r_+^4 B^2}{\gamma^2}\right).$$

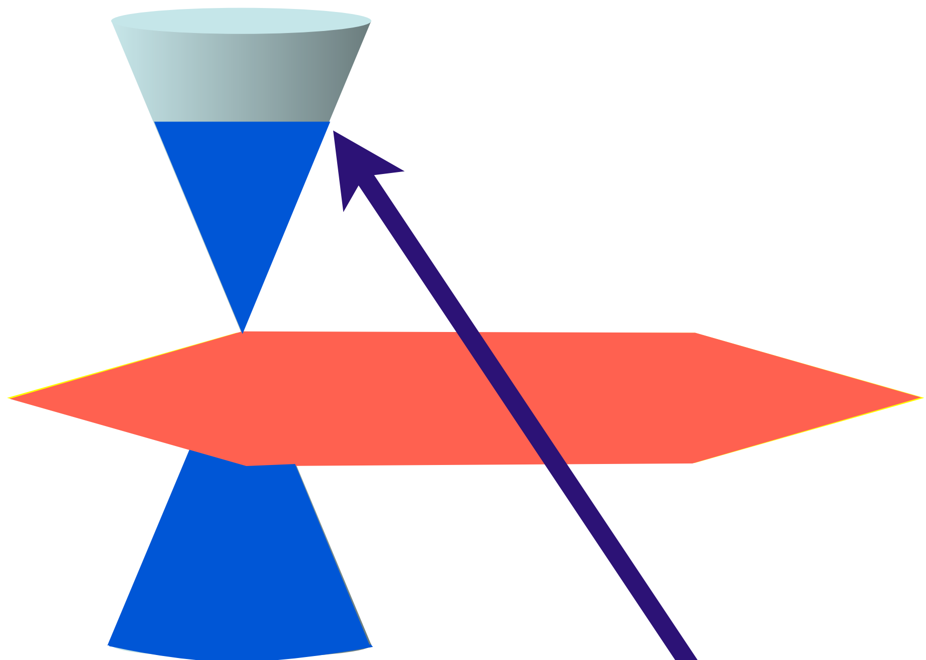


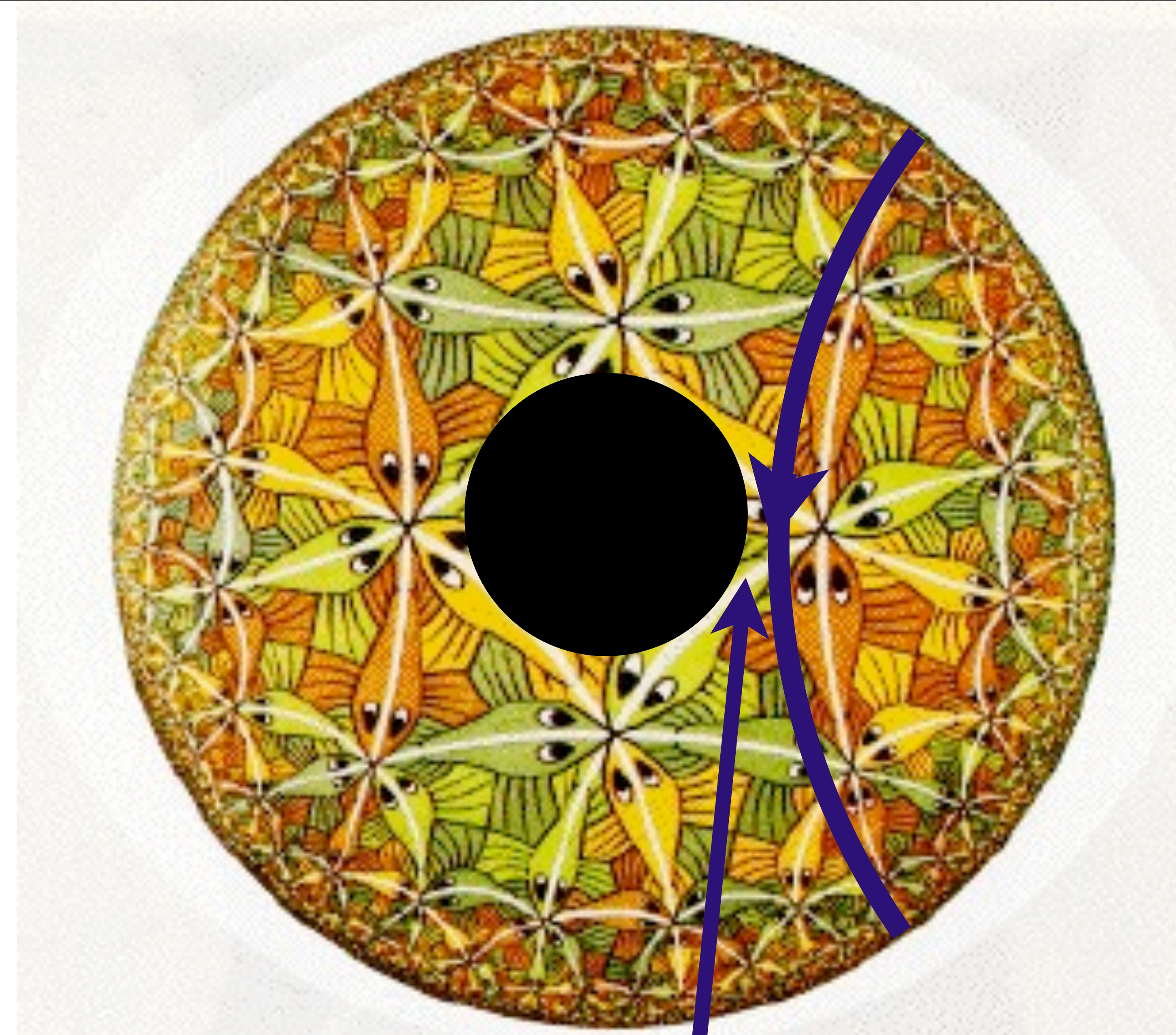
$AdS_2 \times R^2$ near-horizon geometry



$$r - r_+ \sim \frac{1}{\zeta}$$

$$ds^2 = \frac{R^2}{\zeta^2}\left(-d\tau^2 + d\zeta^2\right) + \frac{r_+^2}{R^2}\left(dx^2 + dy^2\right)$$

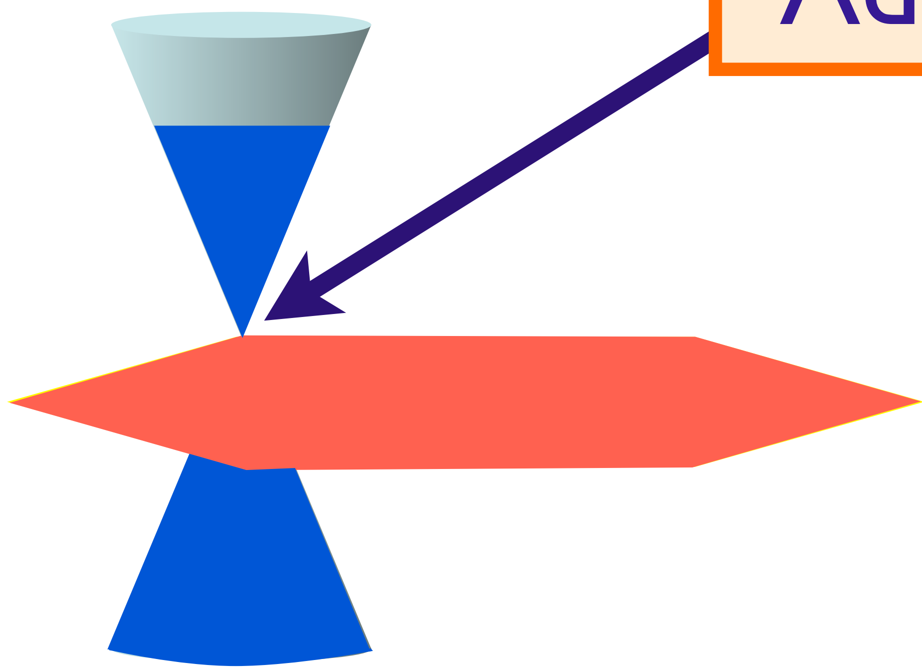


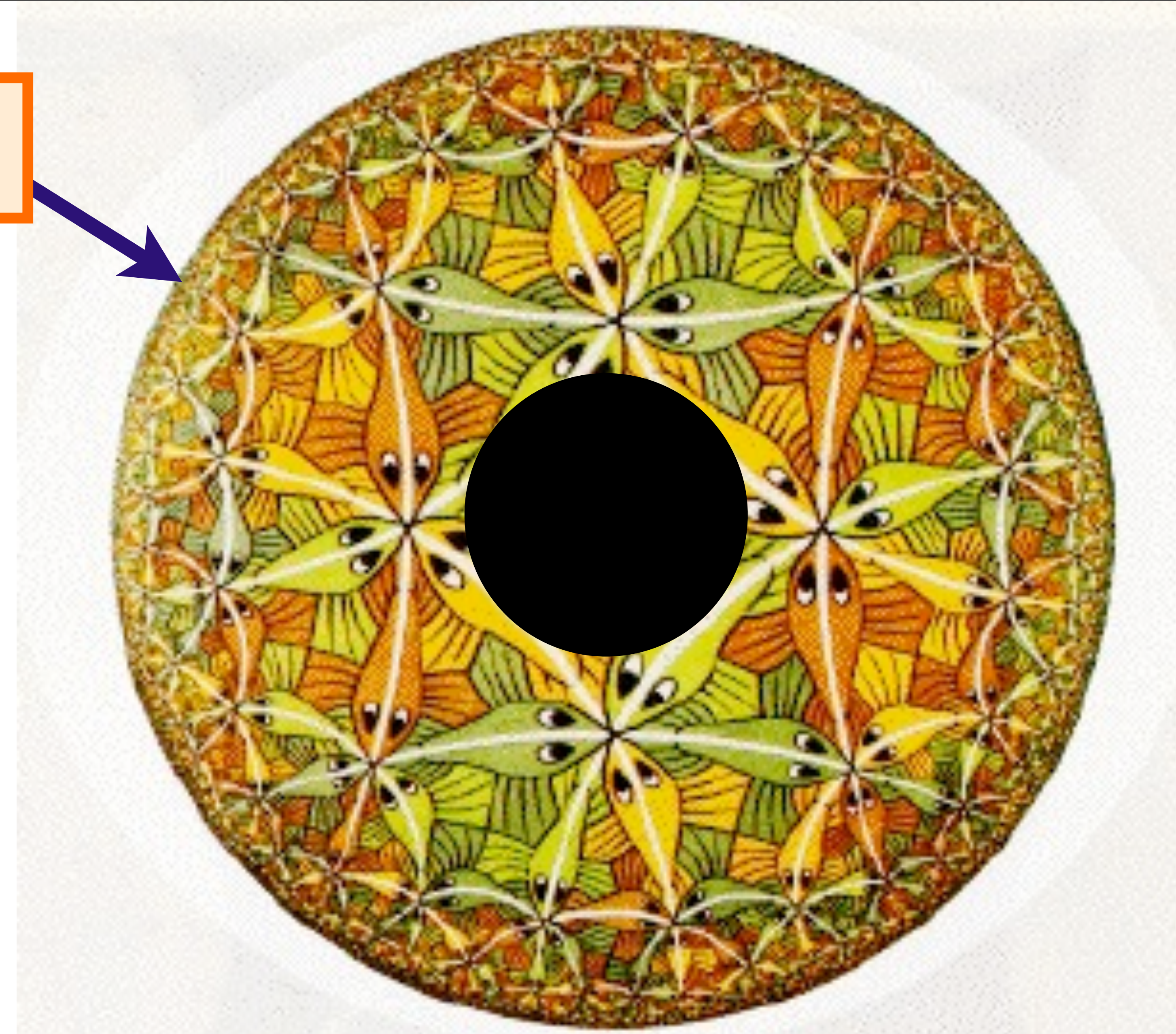


Infrared physics of Fermi surface is linked to the near horizon AdS_2 geometry of Reissner-Nordstrom black hole

T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694

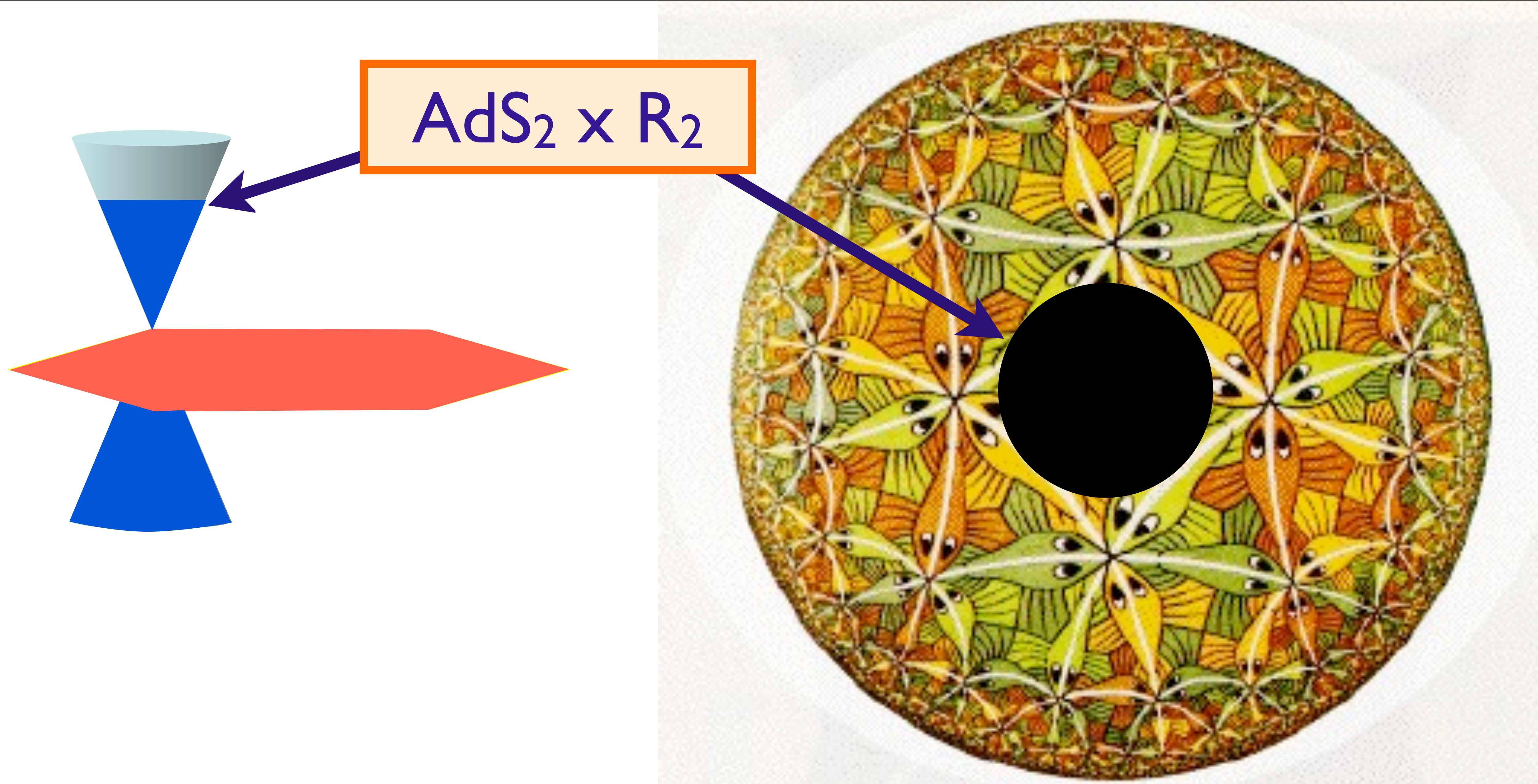
AdS$_4$





Geometric interpretation of RG flow

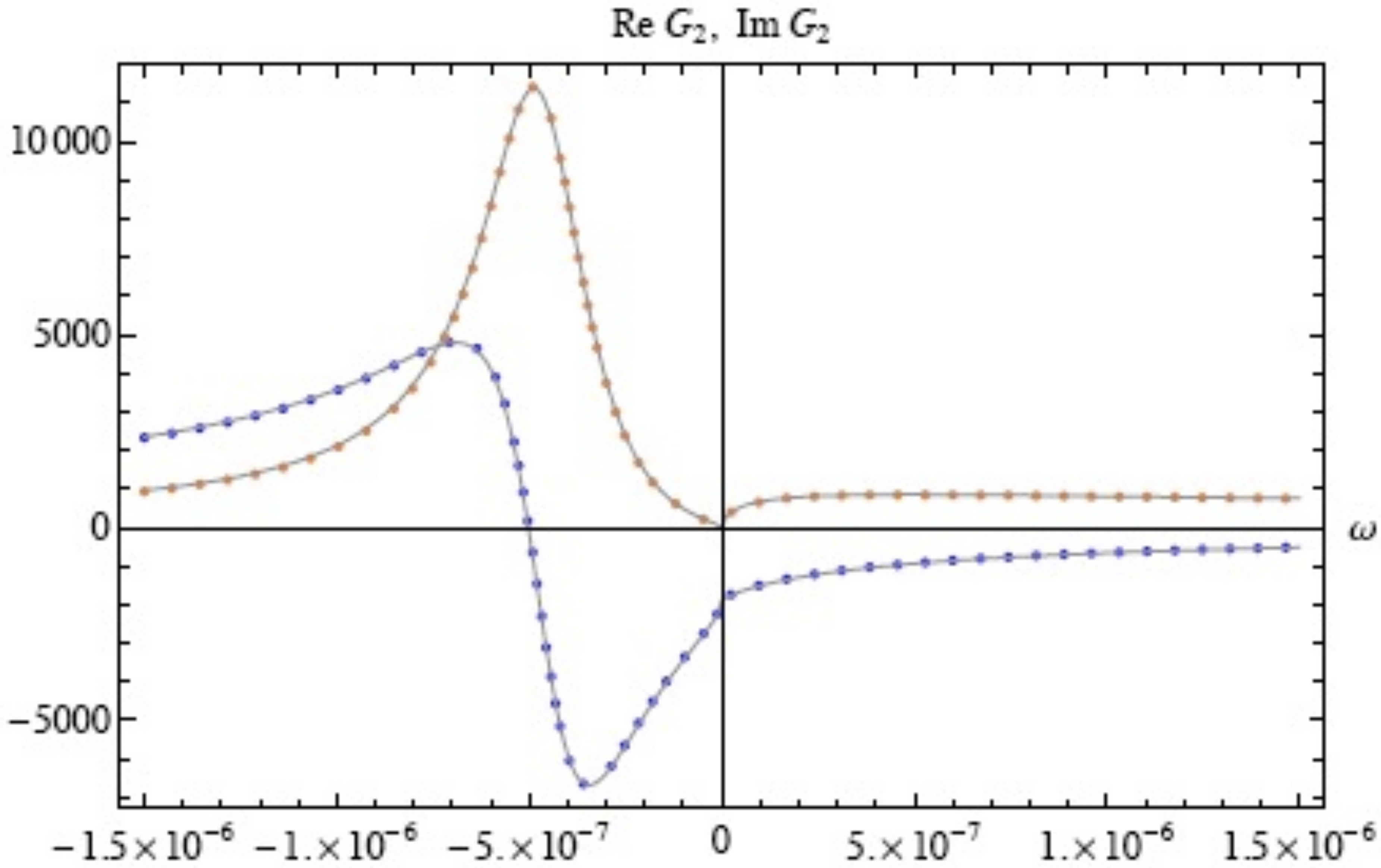
T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694



Geometric interpretation of RG flow

T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694

Green's function of a fermion

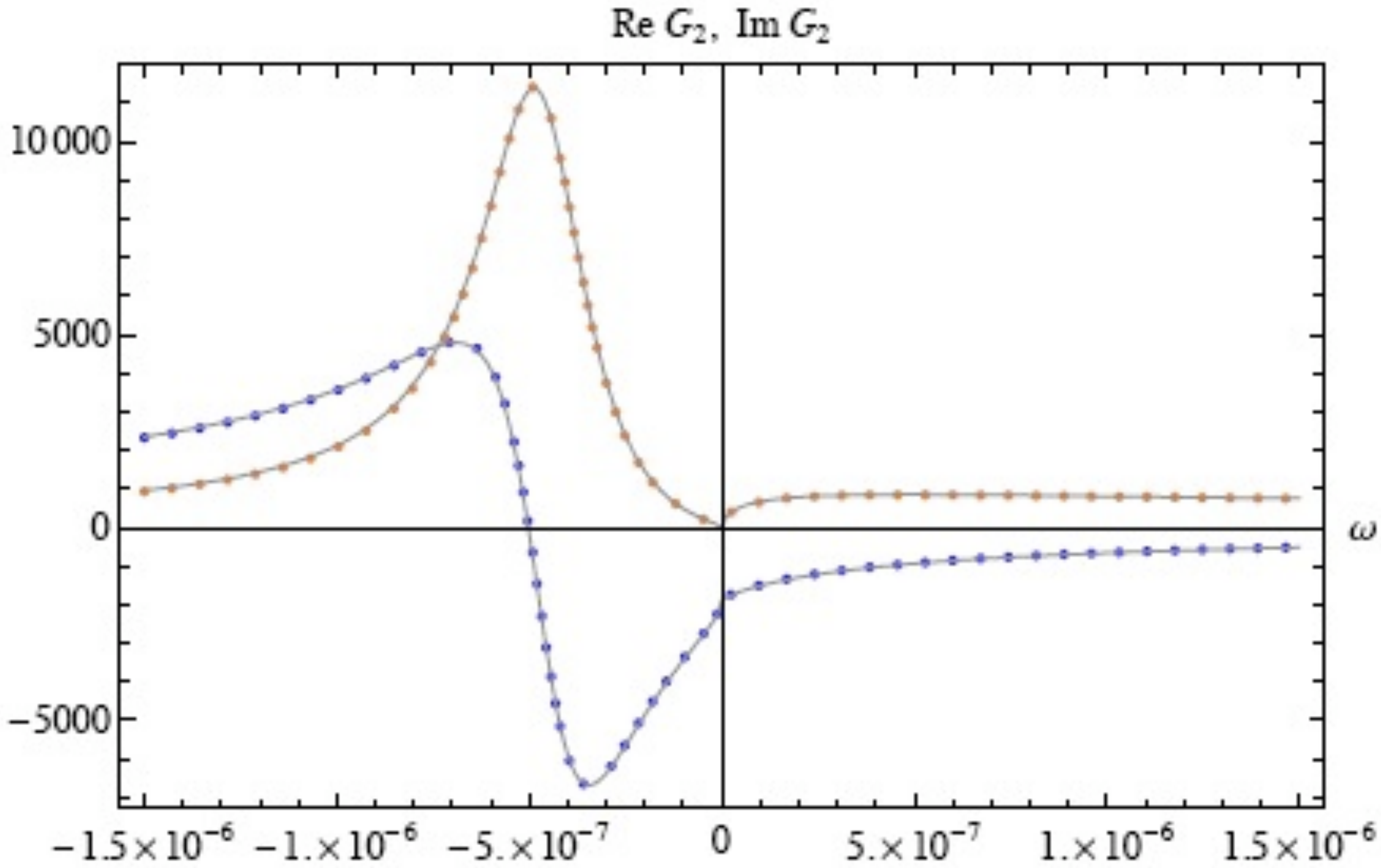


T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694

$$G(k,\omega) \approx \frac{1}{\omega - v_F(k-k_F) - i\omega^{\theta(k)}}$$

See also M. Cubrovic, J Zaanen, and K. Schalm, arXiv:0904.1993

Green's function of a fermion



T. Faulkner, H. Liu, J. McGreevy, and D. Vegh, arXiv:0907.2694

$$G(k,\omega) \approx \frac{1}{\omega - v_F(k-k_F) - i\omega^{\theta(k)}}$$

Similar to non-Fermi liquid theories of Fermi surfaces coupled to gauge fields, and at quantum critical points

Free energy from gravity theory

The free energy is expressed as a sum over the "quasinormal frequencies", z_ℓ, of the black hole. Here ℓ represents any set of quantum numbers:

$$\mathcal{F}_{\text{boson}} = -T \sum_{\ell} \ln \left(\frac{|z_\ell|}{2\pi T} \left| \Gamma \left(\frac{i z_\ell}{2\pi T} \right) \right|^2 \right)$$

$$\mathcal{F}_{\text{fermion}} = T \sum_{\ell} \ln \left(\left| \Gamma \left(\frac{i z_\ell}{2\pi T} + \frac{1}{2} \right) \right|^2 \right)$$

Application of this formula shows that the fermions exhibit the dHvA quantum oscillations with expected period ($2\pi/$(Fermi surface ares)) in $1/B$, but with an amplitude corrected from the Fermi liquid formula of Lifshitz-Kosevich.

F. Denef, S. Hartnoll, and S. Sachdev, arXiv:0908.1788

Conclusions

General theory of finite temperature dynamics and transport near quantum critical points, with applications to antiferromagnets, graphene, and superconductors

Conclusions

The AdS/CFT offers promise in providing a new understanding of strongly interacting quantum matter at non-zero density