The SYK model

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PHYSICS
HARVARD

What are quasiparticles ?

- **Quasiparticles are additive excitations:**
  The low-lying excitations of the many-body system can be identified as a set $\{n_\alpha\}$ of quasiparticles with energy ε_α

$$E = \sum_\alpha n_\alpha \varepsilon_\alpha + \sum_{\alpha,\beta} F_{\alpha\beta} n_\alpha n_\beta + \ldots$$

In a lattice system of N sites, this parameterizes the energy of $\sim e^{\alpha N}$ states in terms of poly(N) numbers.

What are quasiparticles ?

- Quasiparticles eventually collide with each other. Such collisions eventually leads to thermal equilibration in a chaotic quantum state, but the equilibration takes a long time. In a Fermi liquid, this time diverges as

$$\tau_{\text{eq}} \sim \frac{\hbar E_F}{(k_B T)^2} \quad , \quad \text{as } T \to 0,$$

where E_F is the Fermi energy.

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- This time is much longer than the 'Planckian time' $\hbar/(k_B T)$, which we will find in systems without quasiparticle excitations.

$$\tau_{\rm eq} \gg \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

1. Random matrix quasiparticle model

   q=2, complex SYK

2. Matter without quasiparticles

   q=4, complex SYK

3. The Schwarzian theory

4. Connections to black holes with AdS_2 horizons

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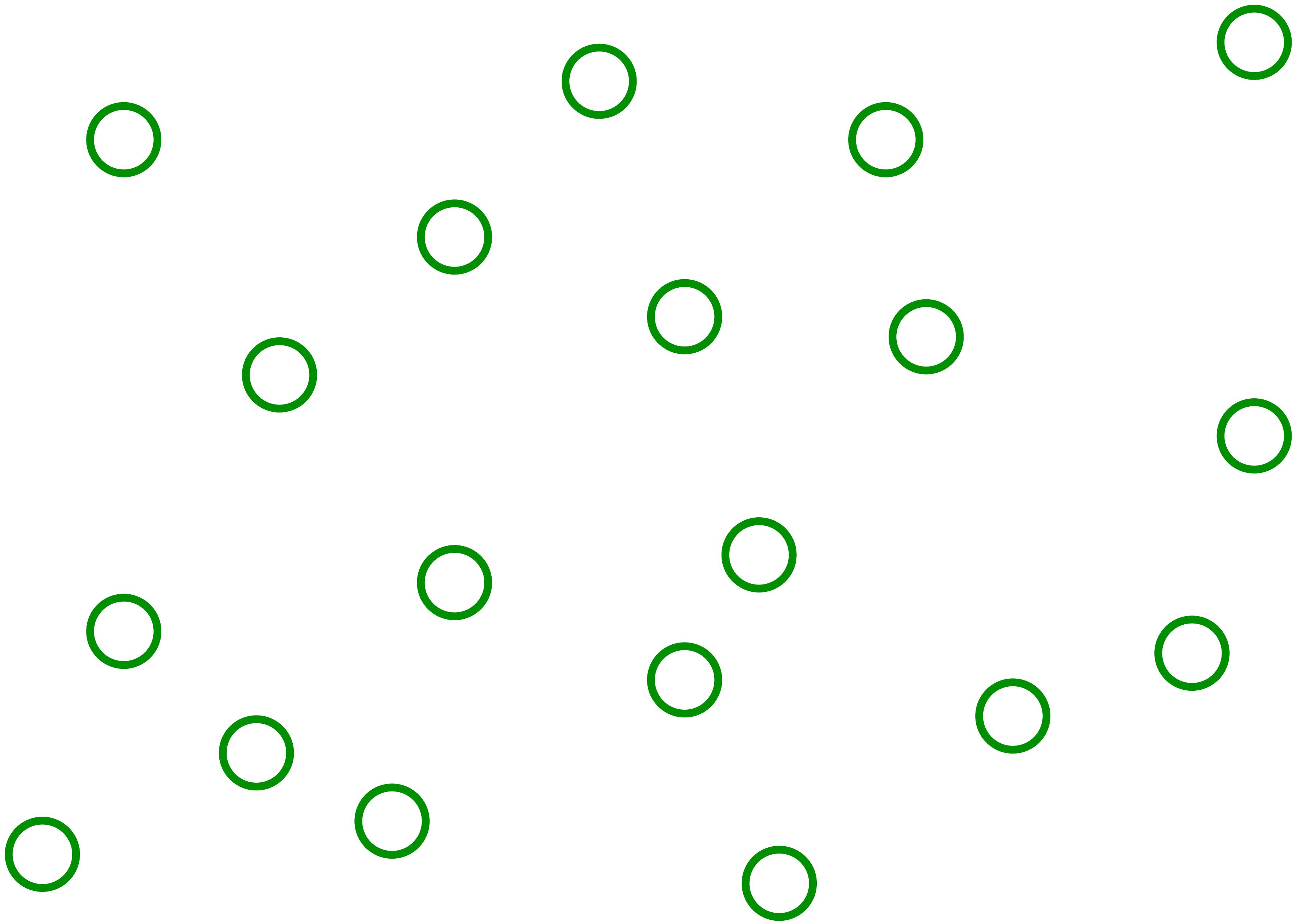
A simple model of a metal with quasiparticles



Pick a set of random positions

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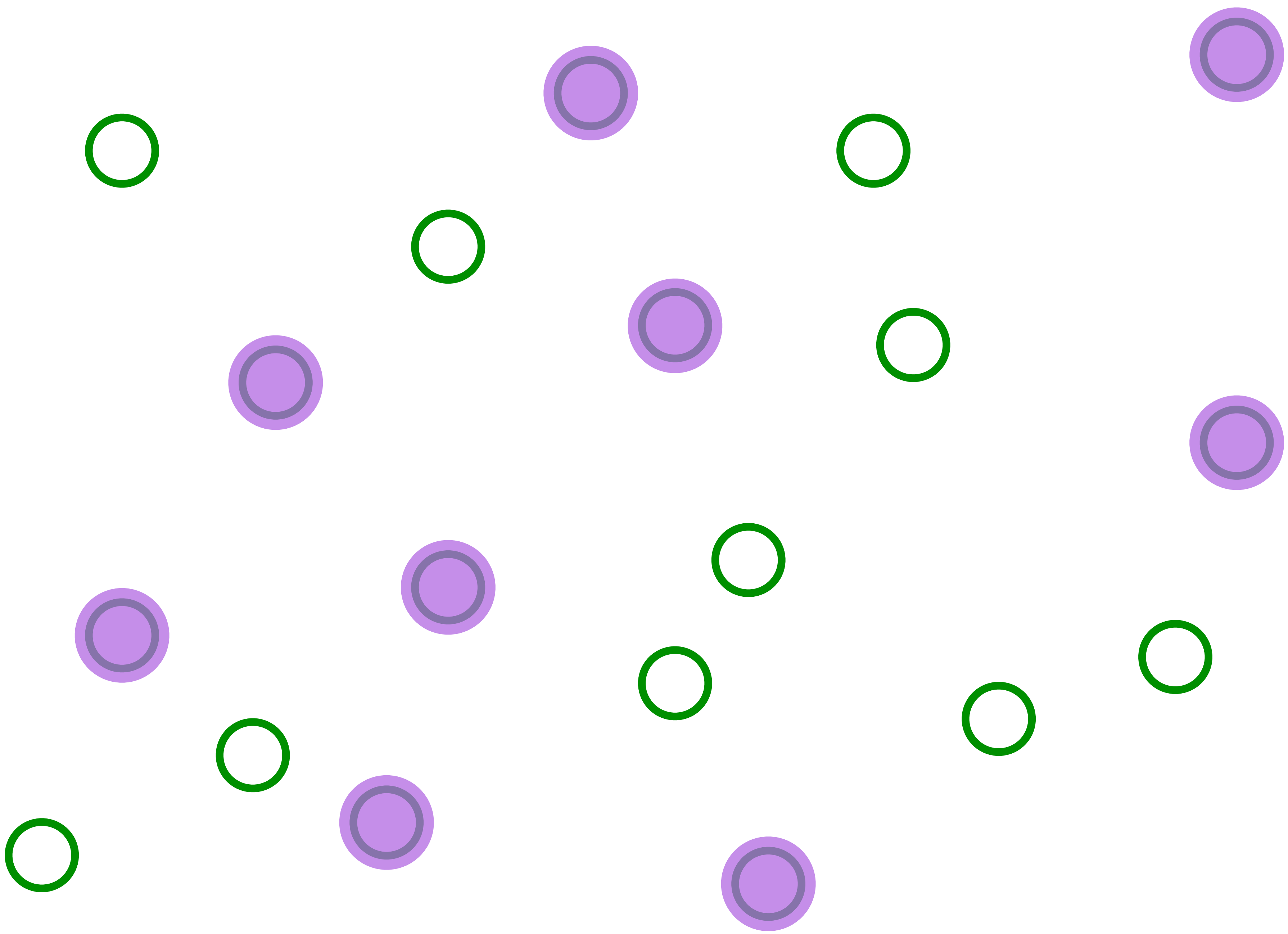

Place electrons randomly on some sites

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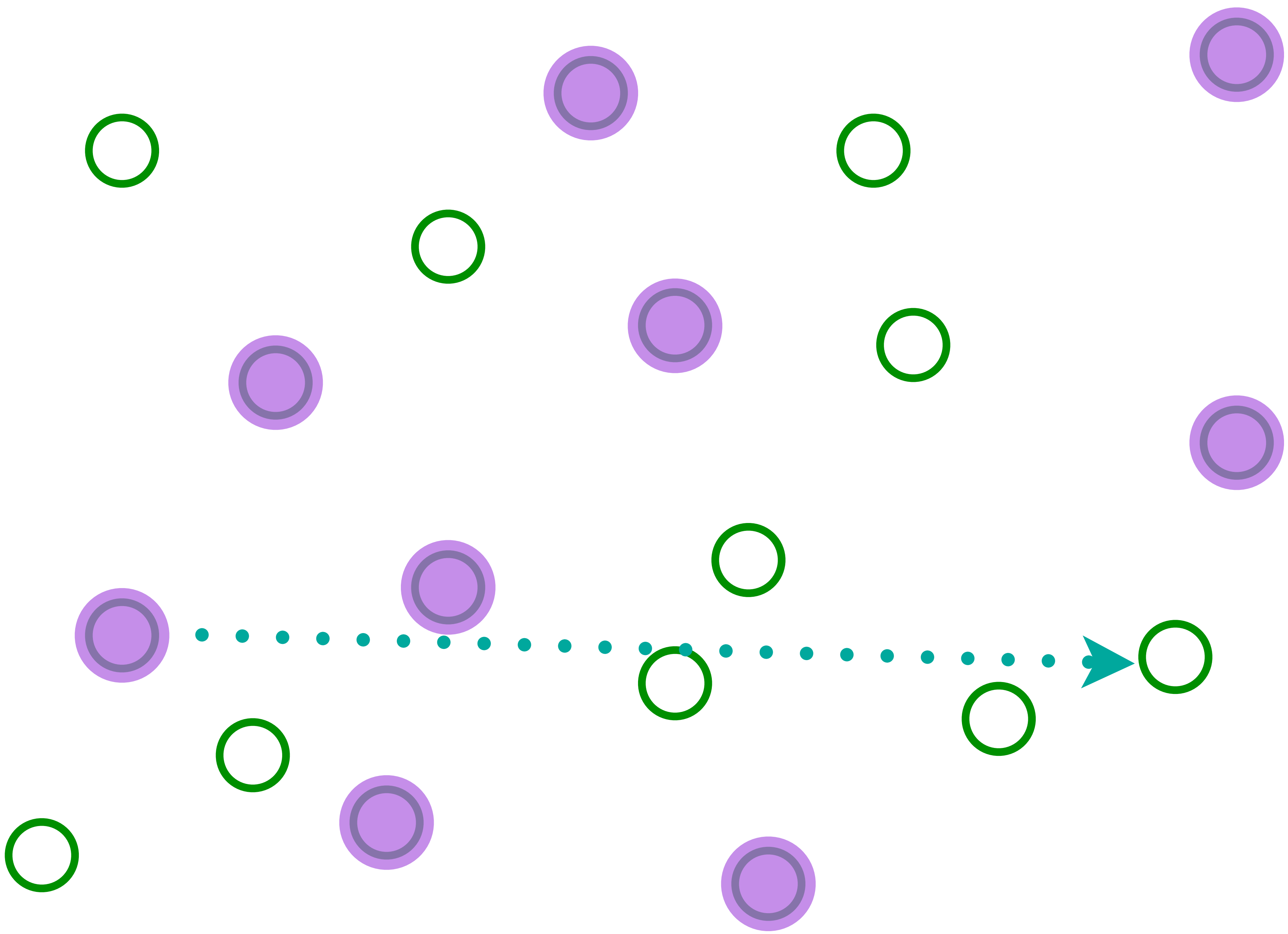

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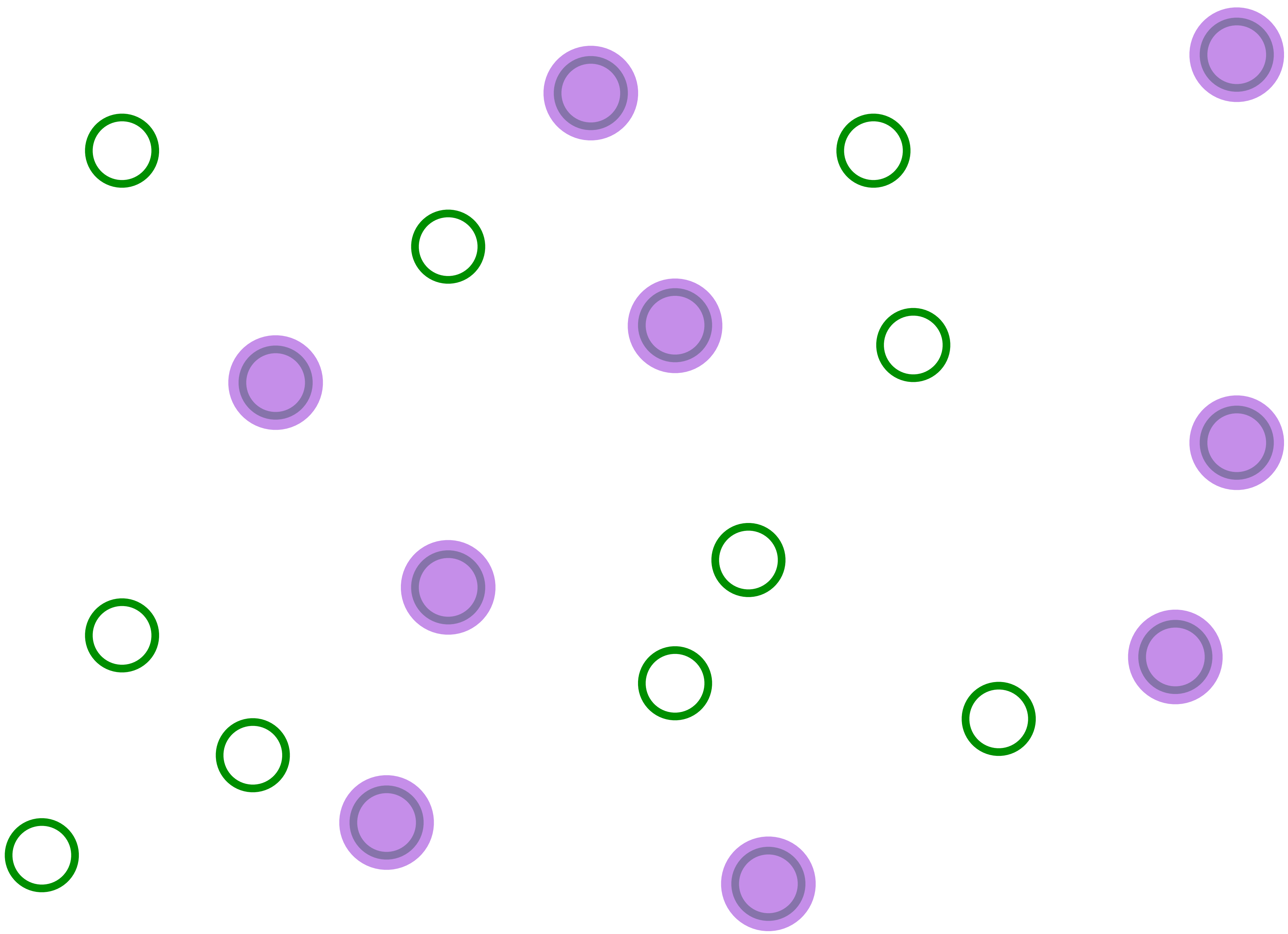

Electrons move one-by-one randomly

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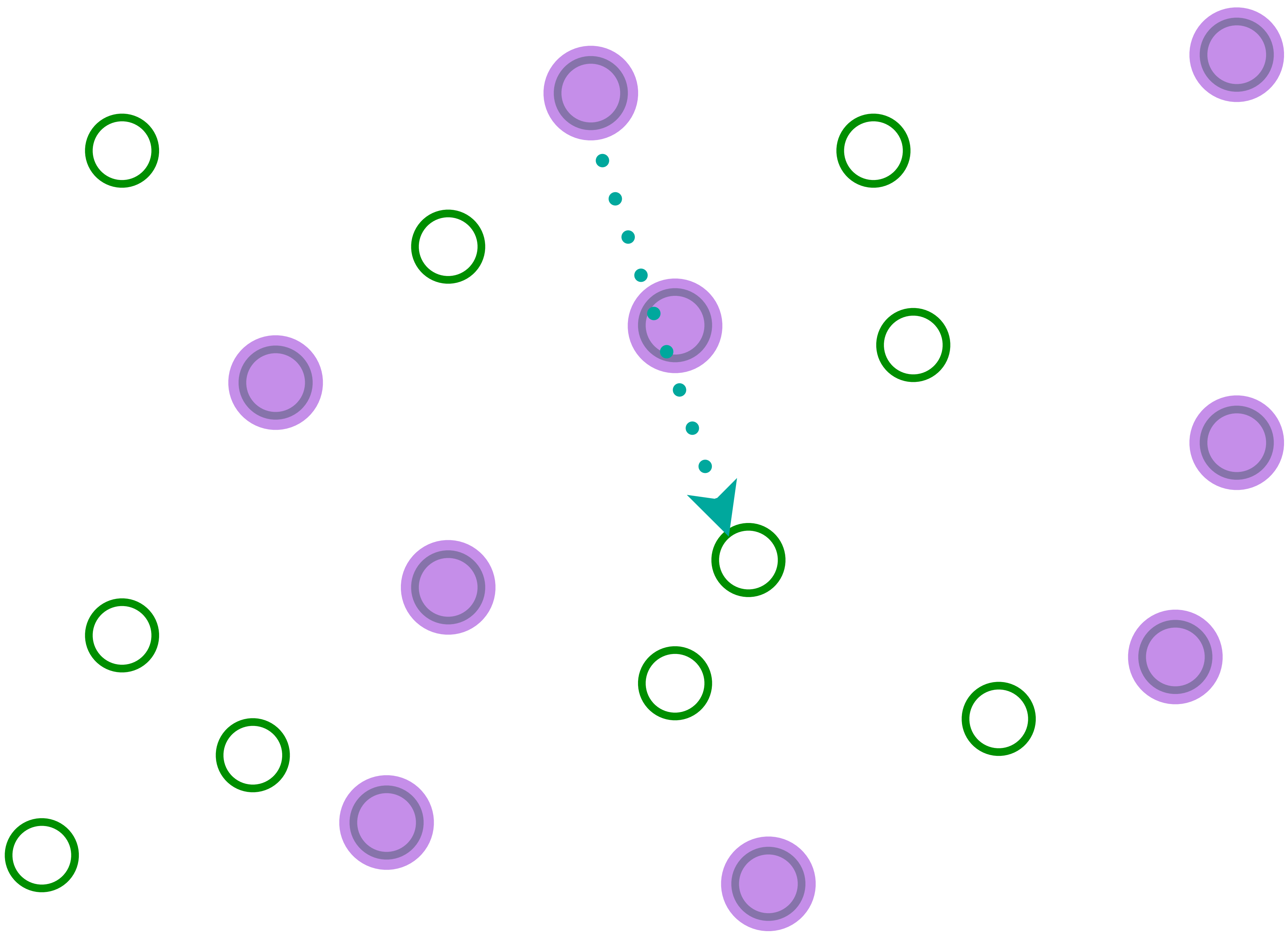

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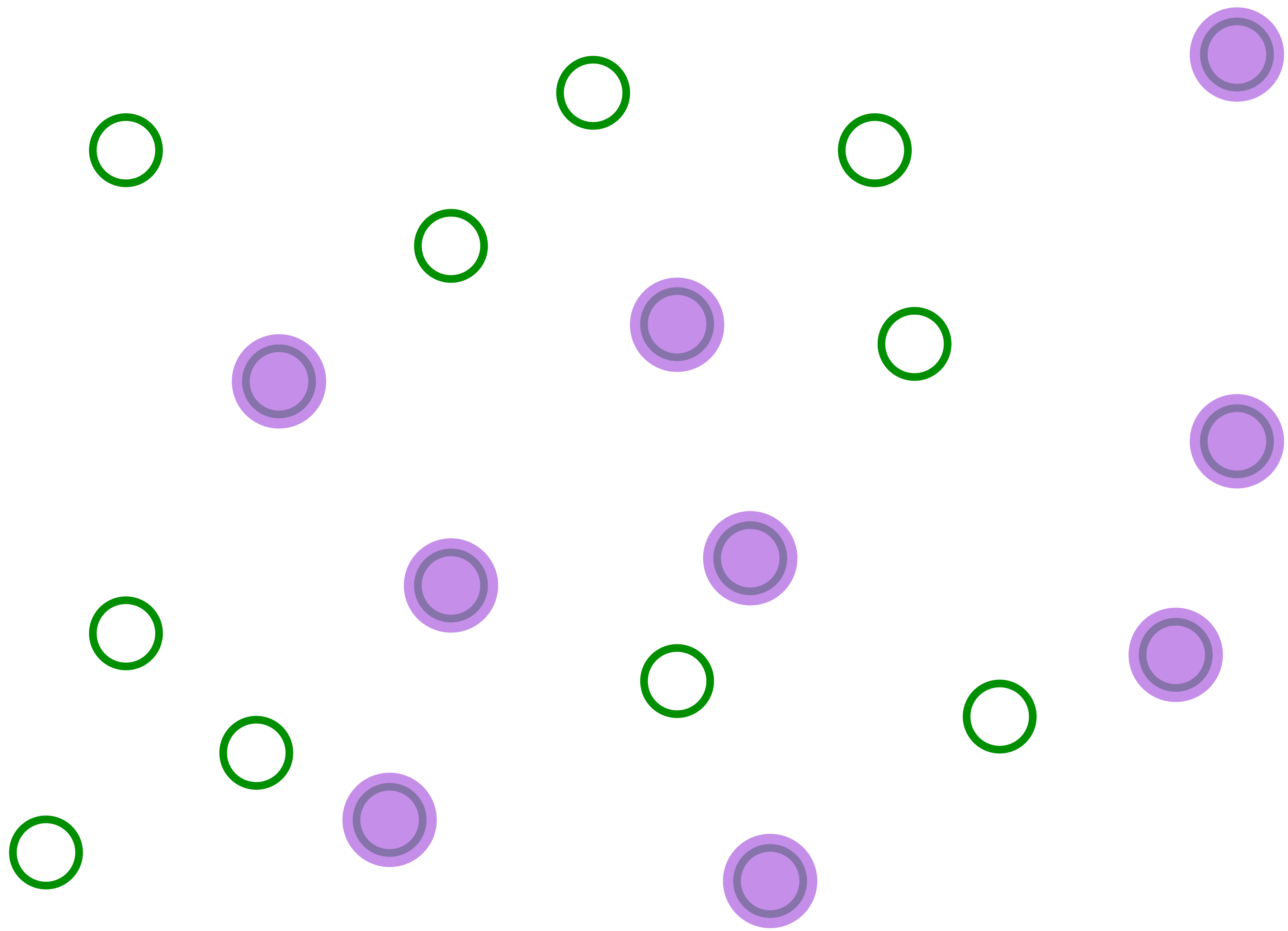

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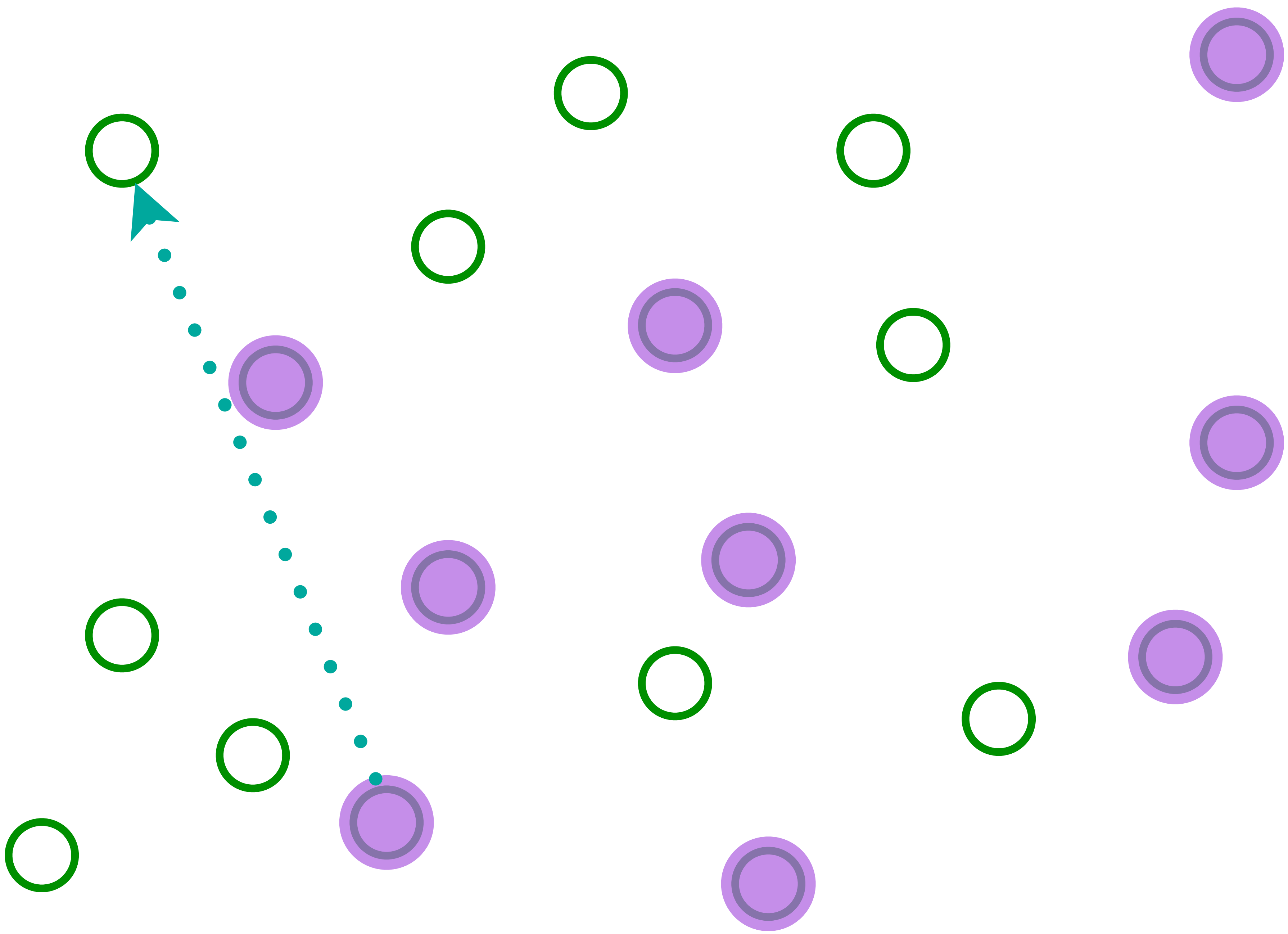

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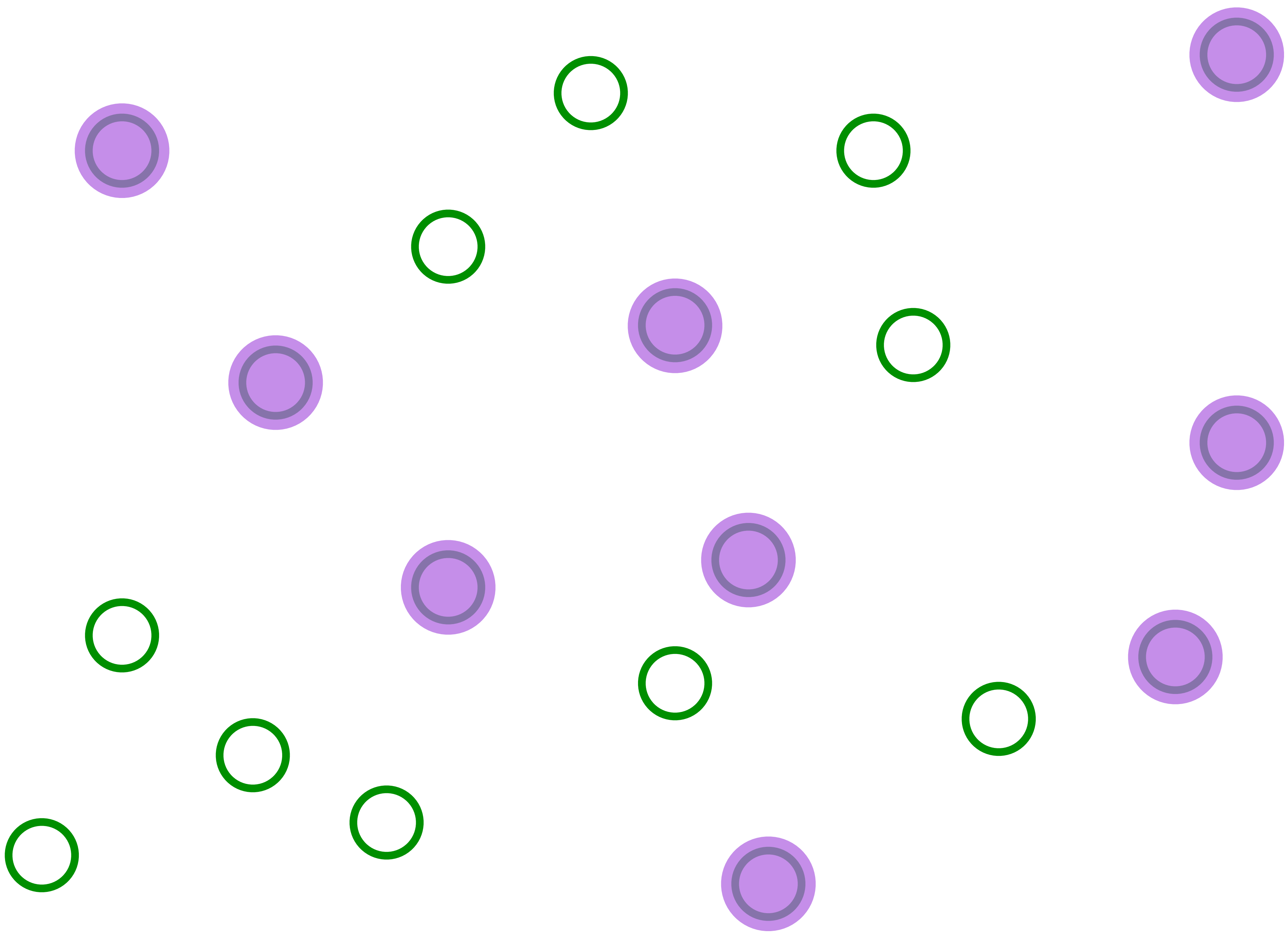

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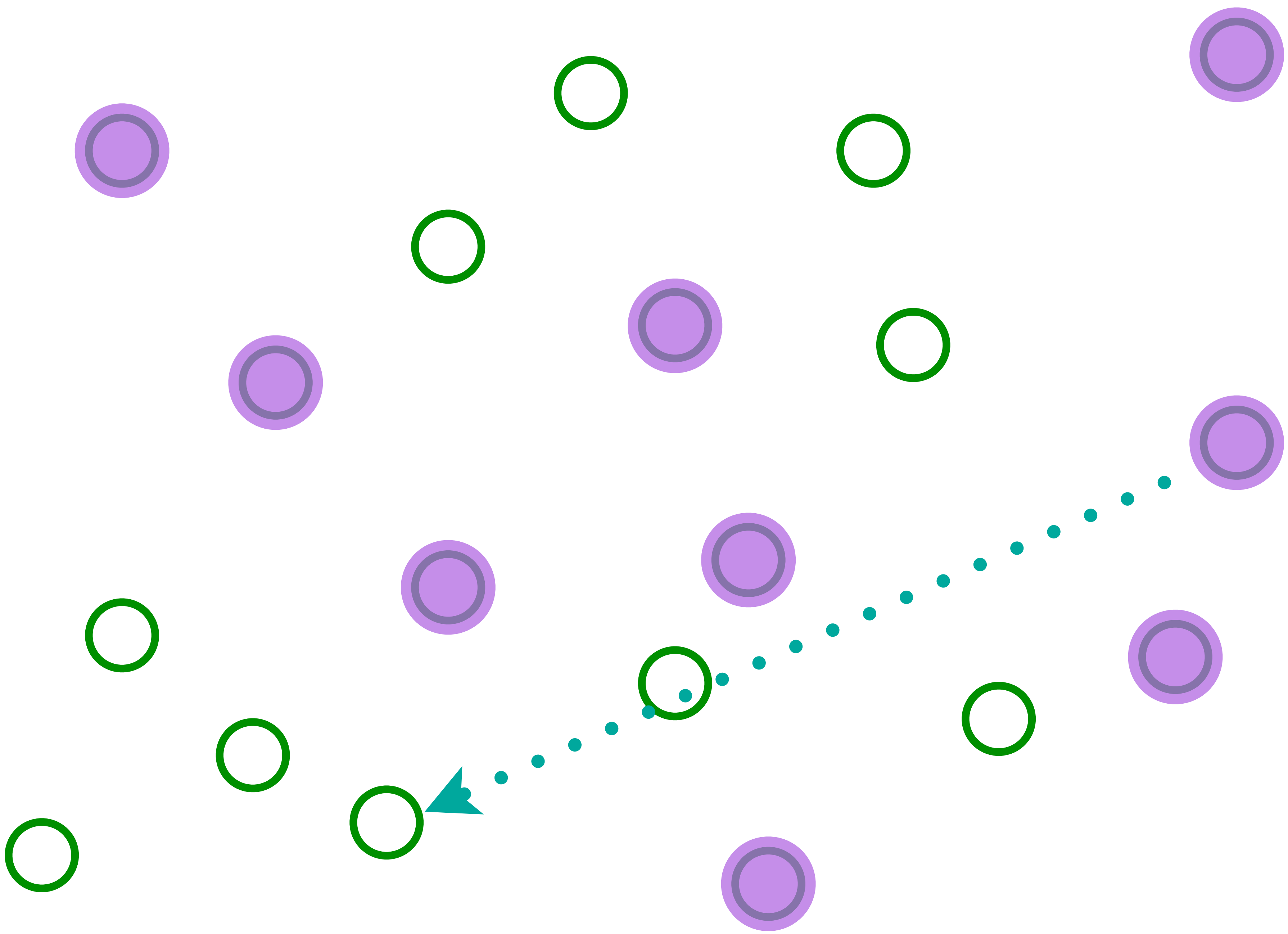

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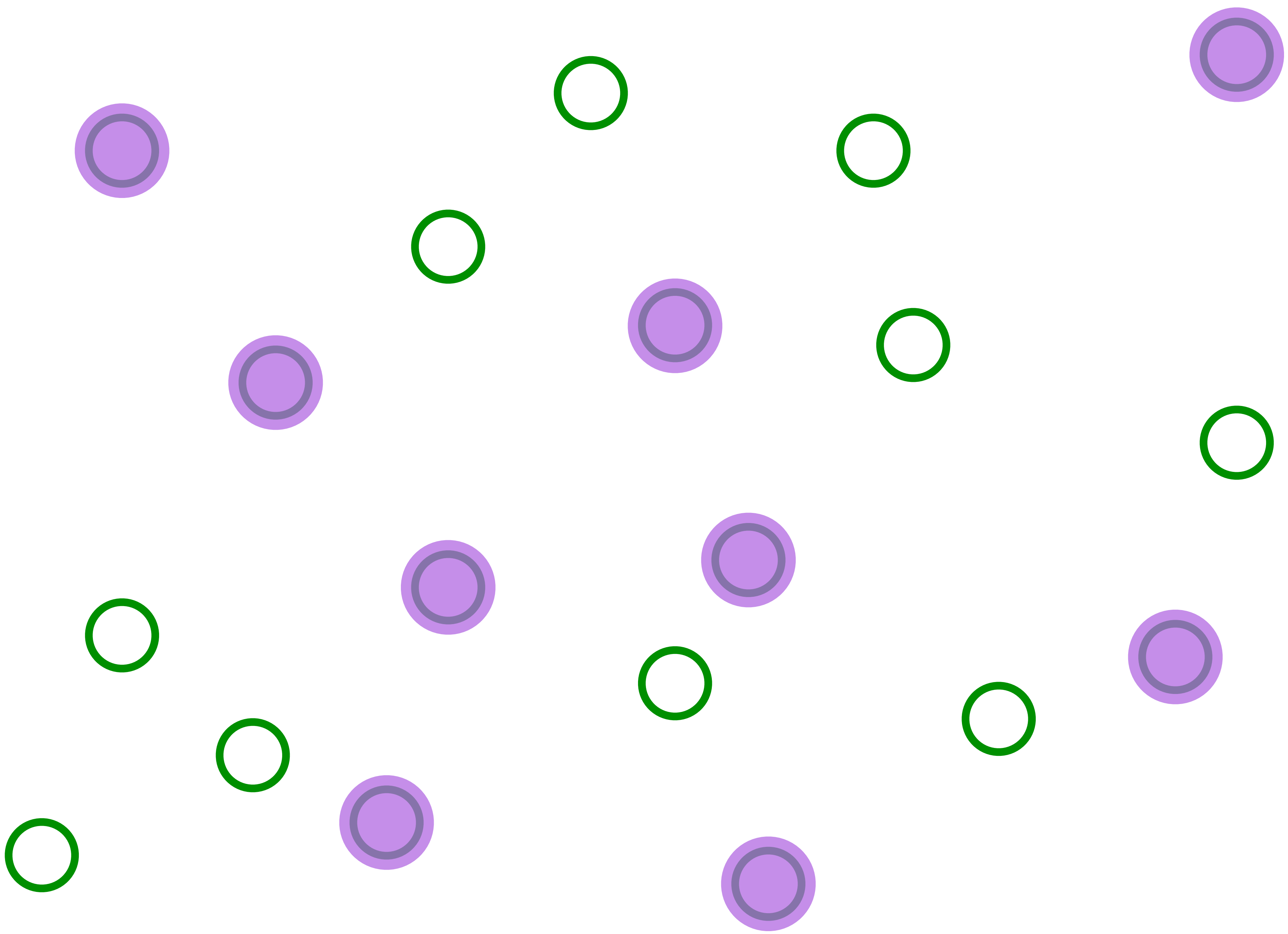


Electrons move one-by-one randomly

A simple model of a metal with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

Fermions occupying the eigenstates of a *N* x *N* random matrix

A simple model of a metal with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

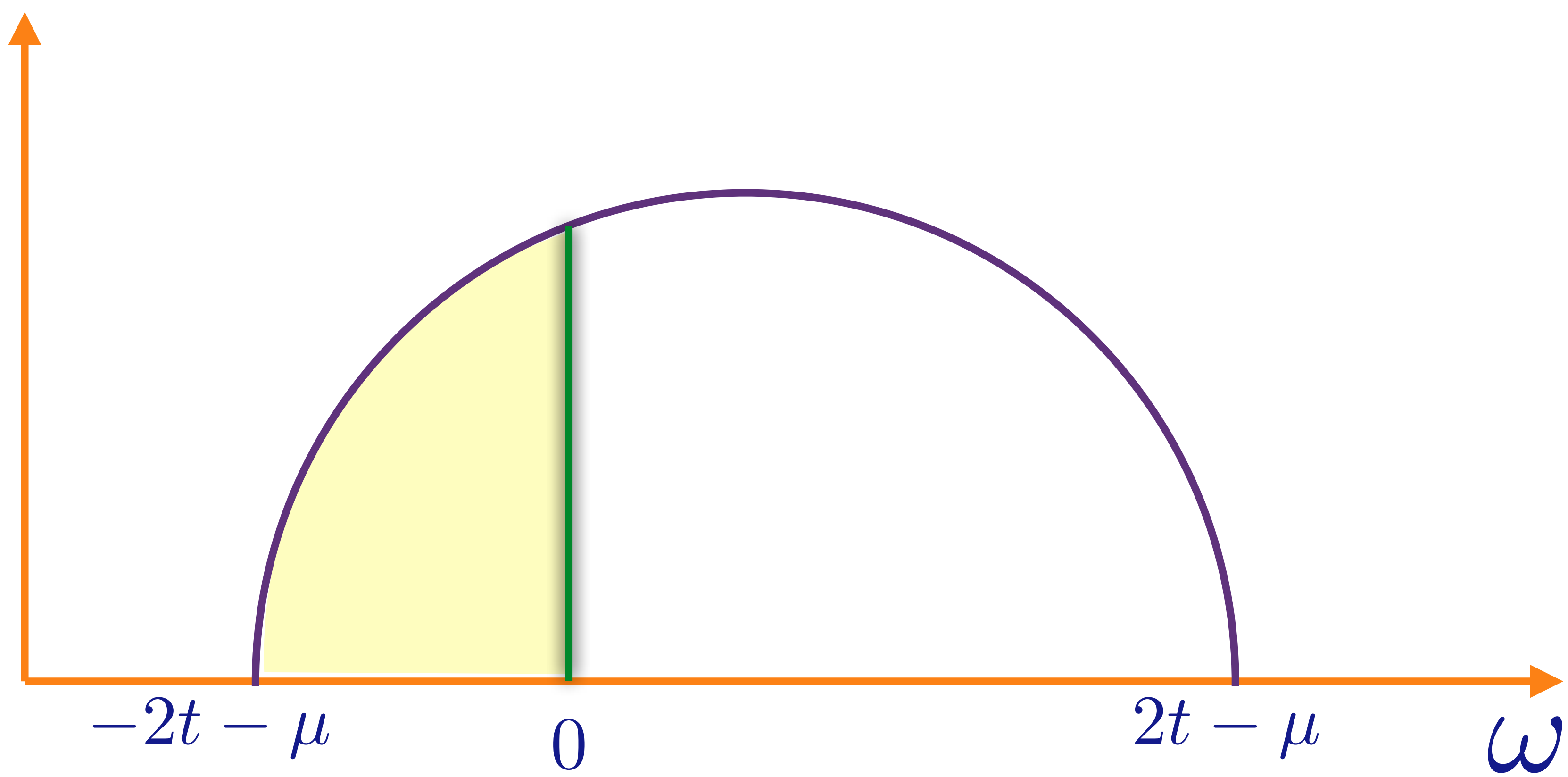
$$G(\tau) \equiv -T_\tau \left\langle c_i(\tau) c_i^\dagger(0) \right\rangle$$

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$

$$G(\tau = 0^-) = \mathcal{Q}.$$

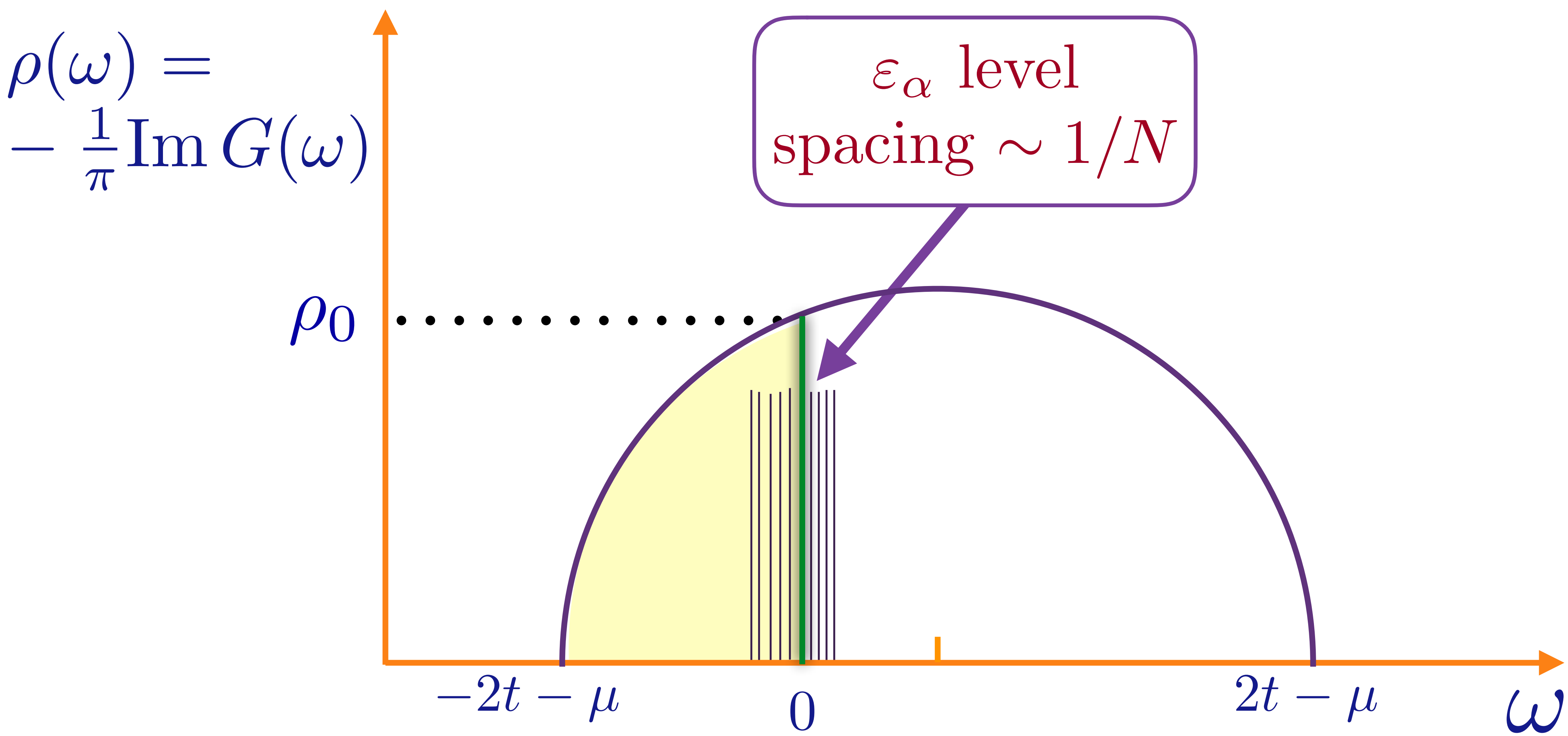
$G(\omega)$ can be determined by solving a quadratic equation.

$$\rho(\omega) = -\frac{1}{\pi} \mathrm{Im}\, G(\omega)$$



A simple model of a metal with quasiparticles

Let ε_α be the eigenvalues of the matrix $t_{ij}/\sqrt{N}$. The fermions will occupy the lowest NQ eigenvalues, upto the Fermi energy E_F. The single-particle density of states is
$\rho(\omega) = (1/N)\sum_\alpha \delta(\omega - \varepsilon_\alpha)$, and $\rho_0 \equiv \rho(\omega = 0)$.



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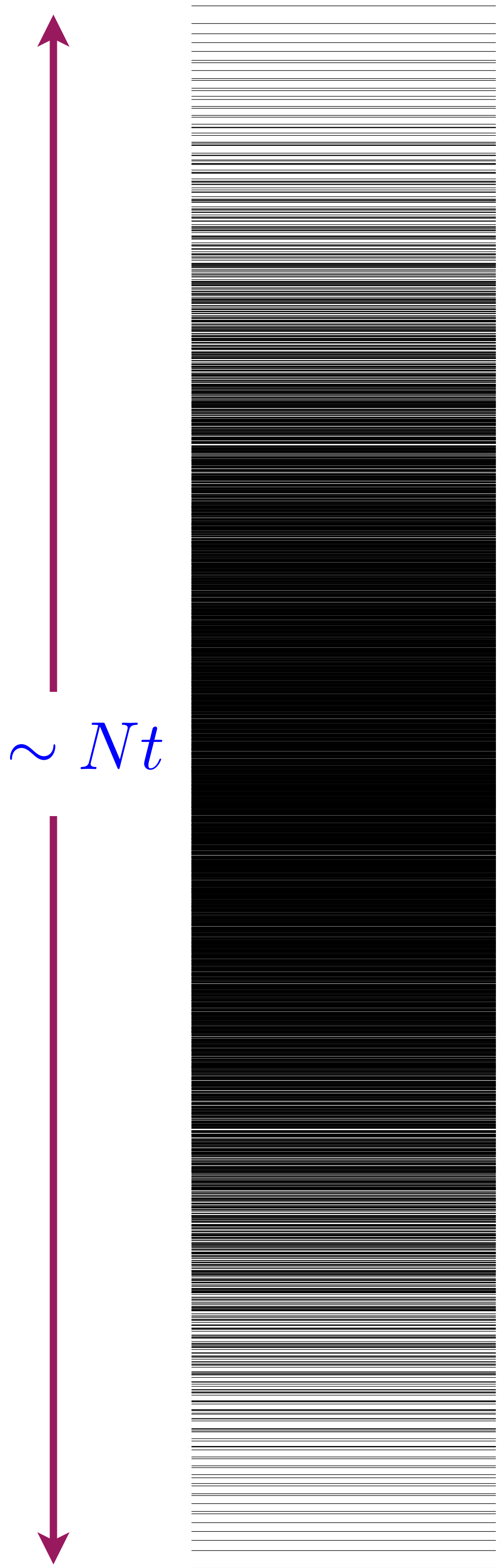

Many-body level spacing $\sim 2^{-N}$

Quasiparticle excitations with spacing $\sim 1/N$

There are 2^N many body levels with energy

$$E = \sum_{\alpha=1}^{N} n_\alpha \varepsilon_\alpha ,$$

where $n_\alpha = 0, 1$. Shown are all values of E for a single cluster of size $N = 12$. The ε_α have a level spacing $\sim 1/N$.

A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N\left(-\frac{\pi^2}{6}\rho_0 T^2 + \mathcal{O}(T^4)\right) + \dots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.

We can also define the *many* body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$\boxed{D(E) \sim \exp\left(\pi\sqrt{\frac{2N\rho_0(E-E_0)}{3}}\right) \quad , \quad E > E_0 \ , \ \frac{1}{N} \ll \rho_0(E-E_0) \ll N}$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

A simple model of a metal with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$. We compute the lifetime of a quasiparticle, τ_α, in an exact eigenstate $\psi_\alpha(i)$ of the free particle Hamitonian with energy ε_α. By Fermi's Golden rule, for ε_α at the Fermi energy

$$\begin{aligned}\frac{1}{\tau_\alpha} &= \pi U^2 \rho_0^2 \int d\varepsilon_\beta d\varepsilon_\gamma d\varepsilon_\delta f(\varepsilon_\beta)(1 - f(\varepsilon_\gamma))(1 - f(\varepsilon_\delta))\delta(\varepsilon_\alpha + \varepsilon_\beta - \varepsilon_\gamma - \varepsilon_\delta) \\ &= \frac{\pi^3 U^2 \rho_0^2}{4} T^2\end{aligned}$$

where ρ_0 is the density of states at the Fermi energy, and $f(\epsilon) = 1/(e^{\epsilon/T} + 1)$ is the Fermi function.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as $\sim T^{-2}$ at the Fermi level.

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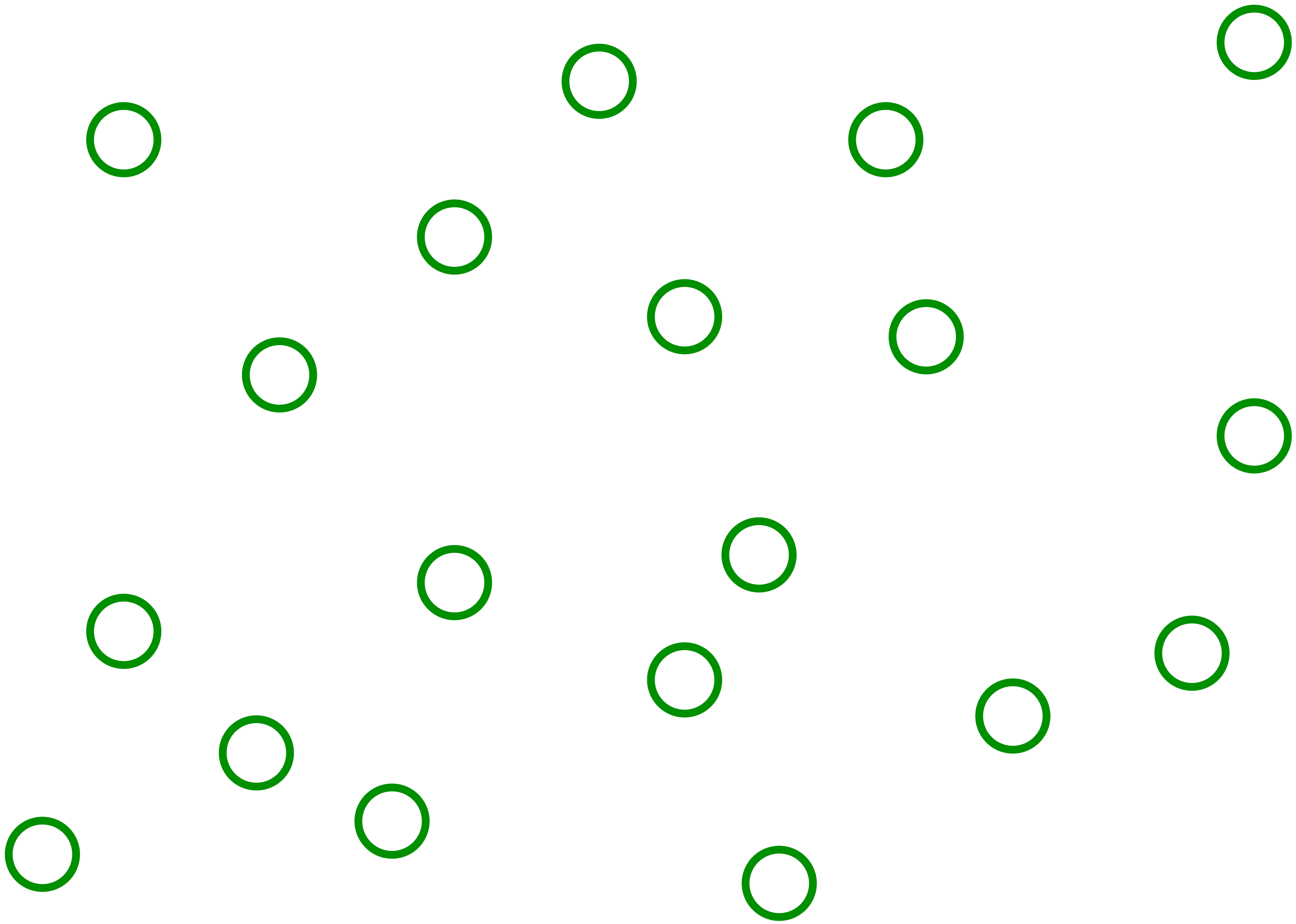
The Sachdev-Ye-Kitaev (SYK) model



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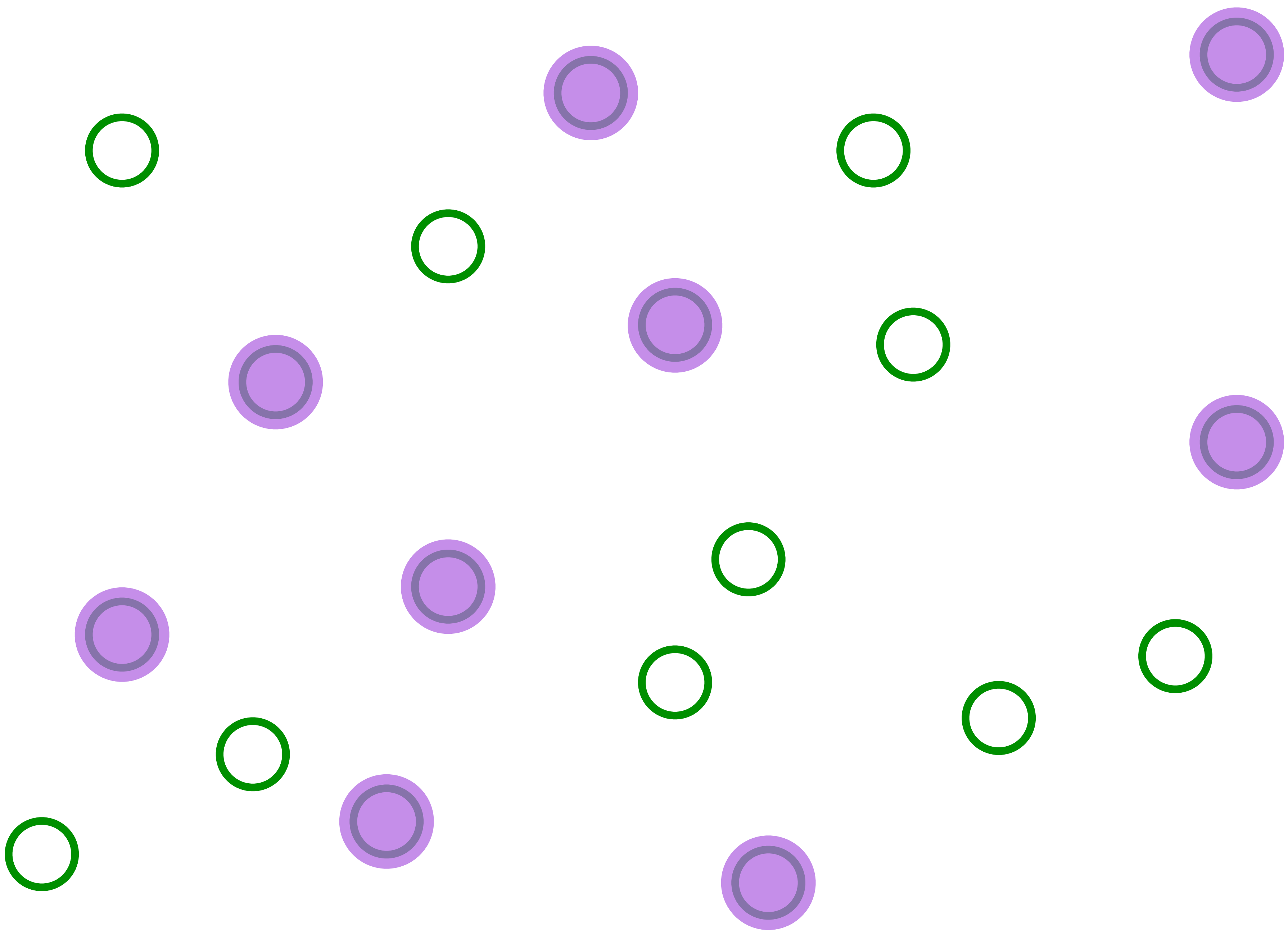
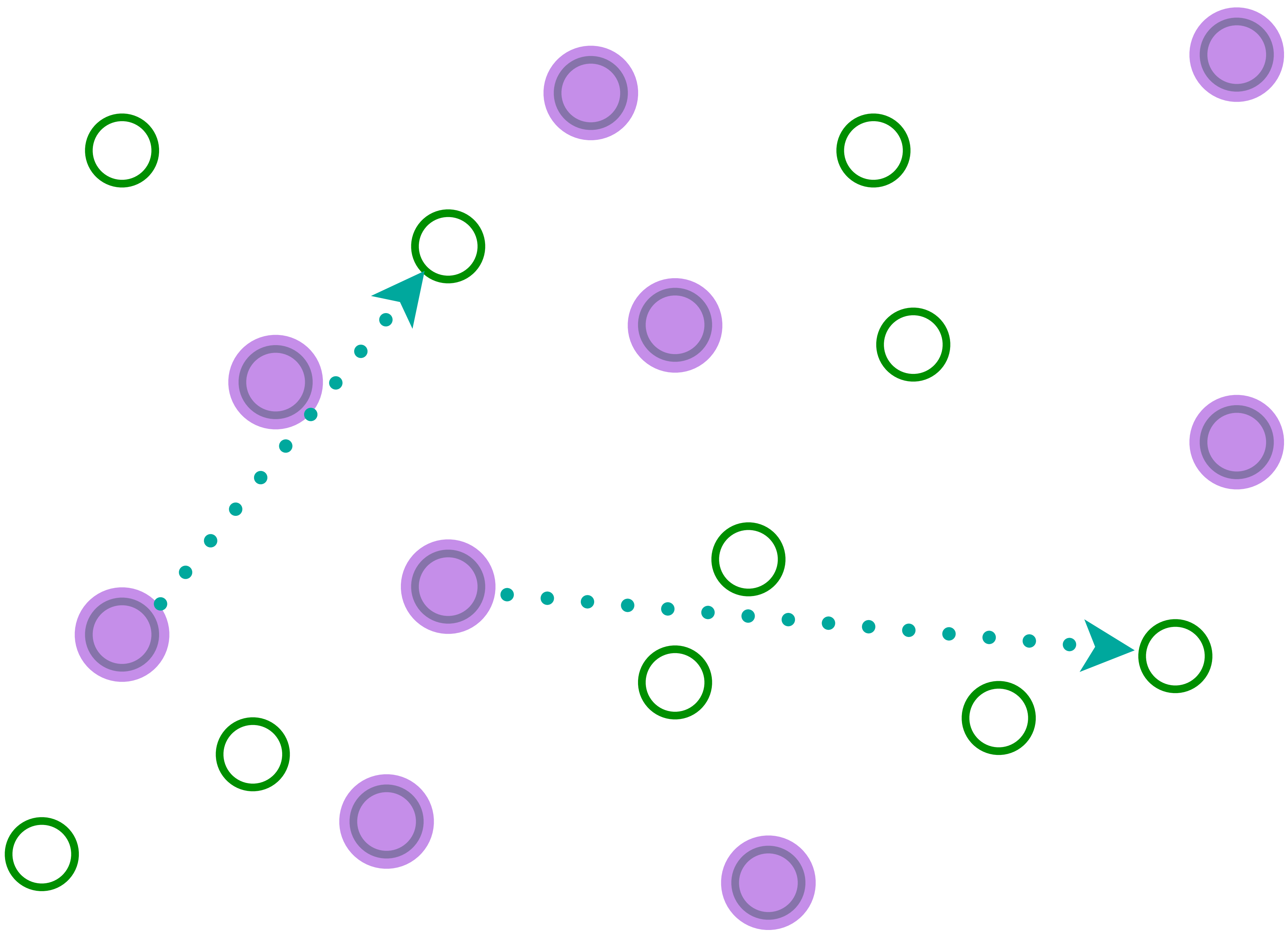
The SYK model



Place electrons randomly on some sites

The SYK model



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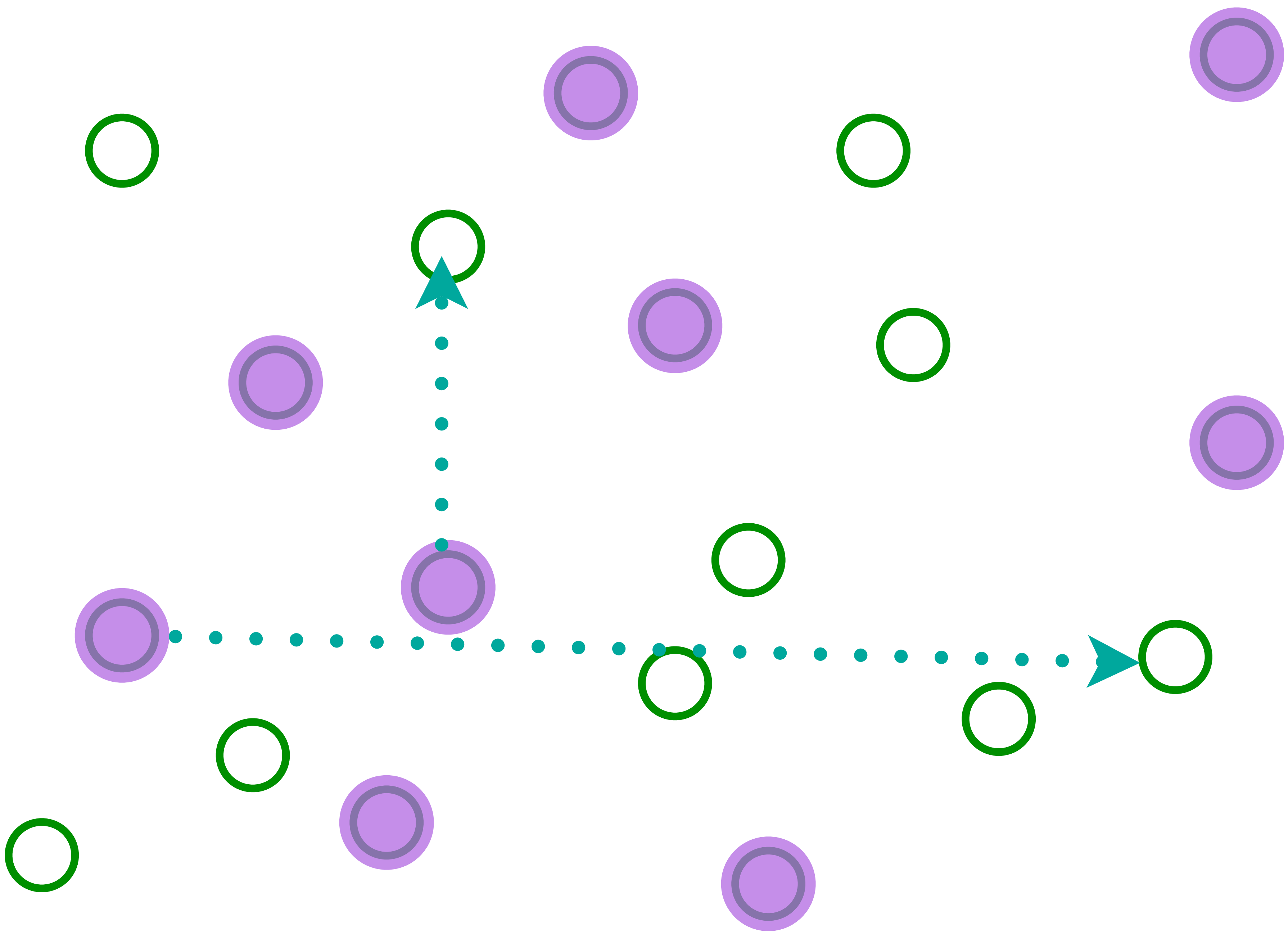

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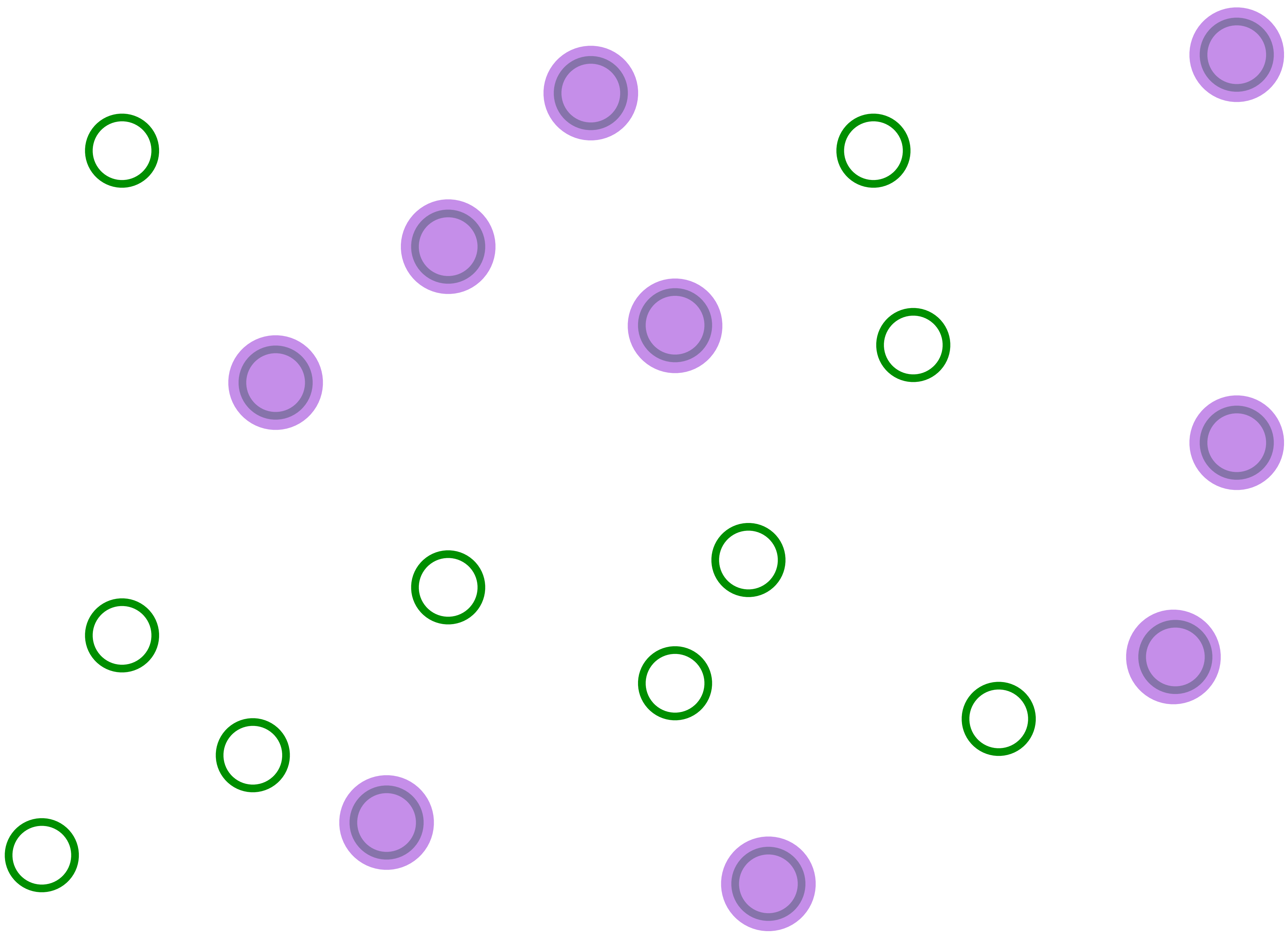

Entangle electrons pairwise randomly

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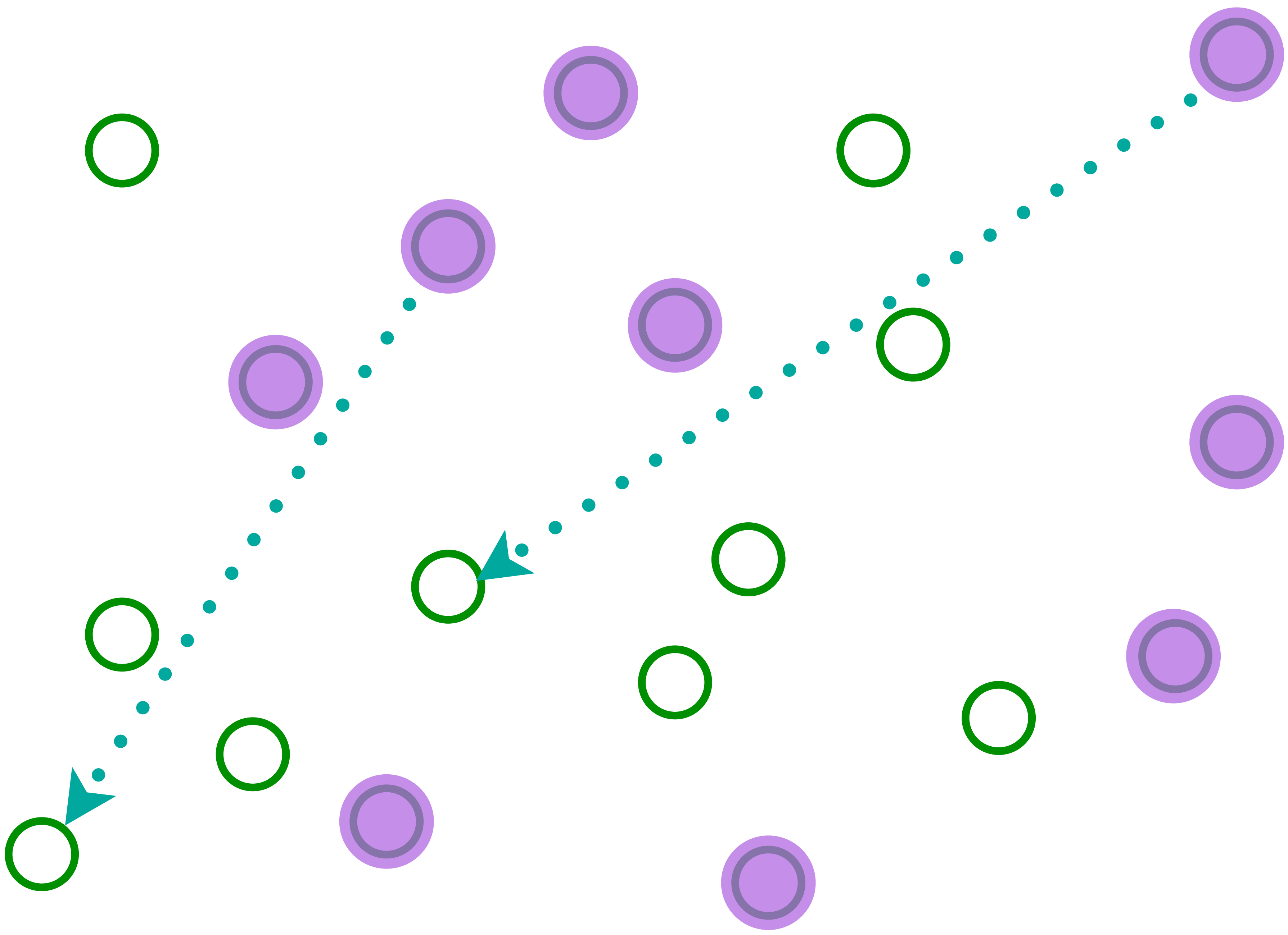

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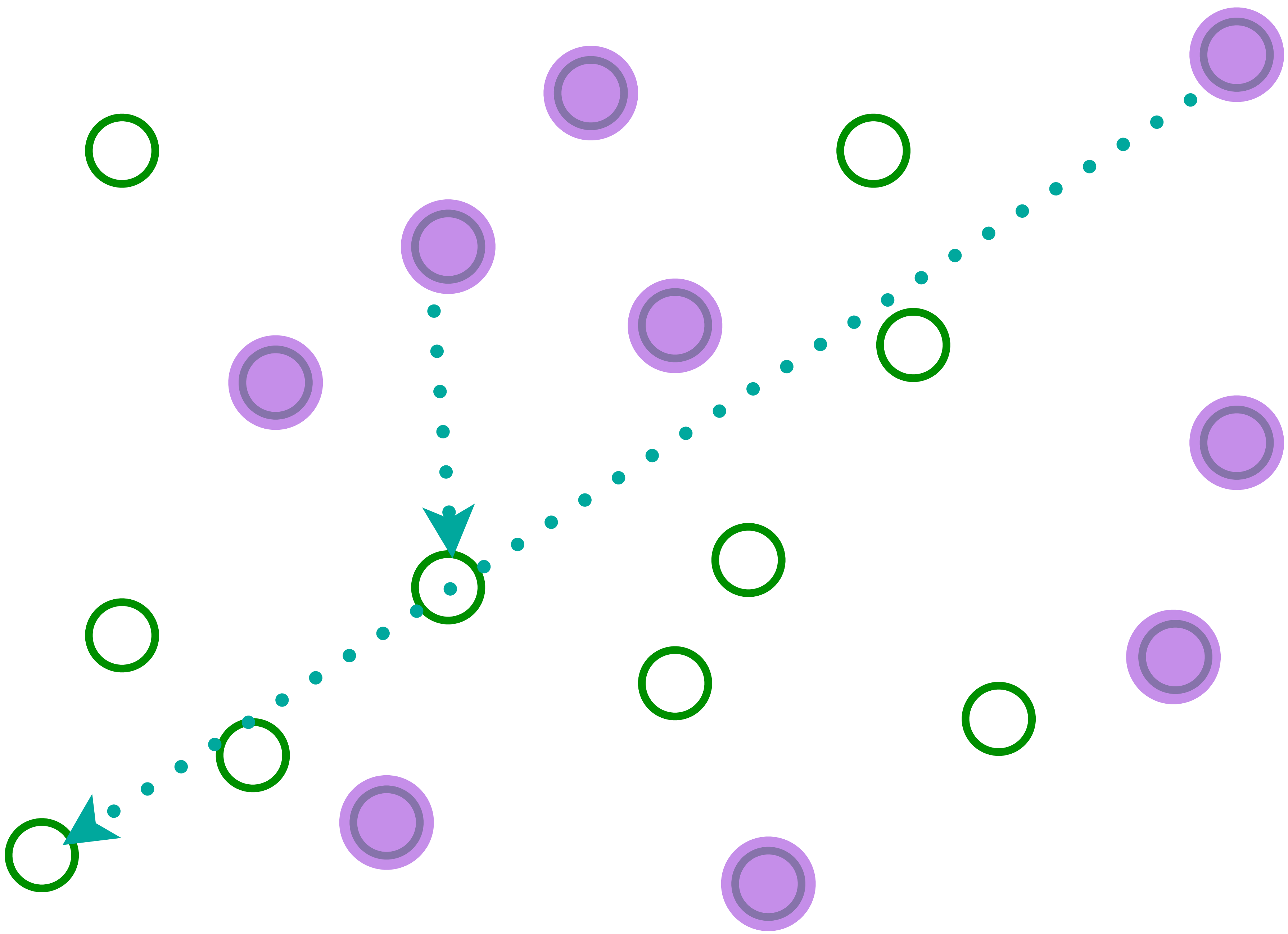

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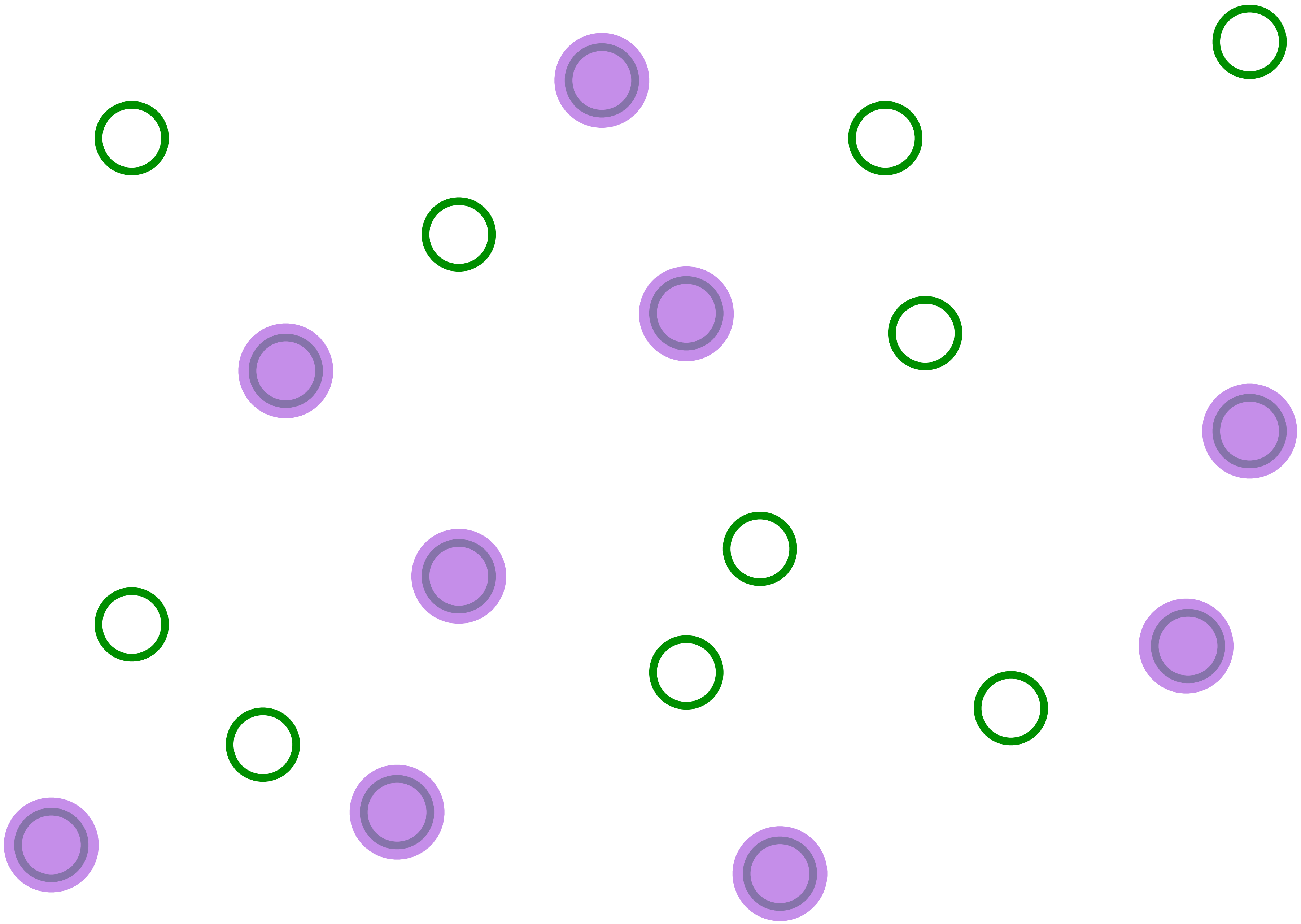

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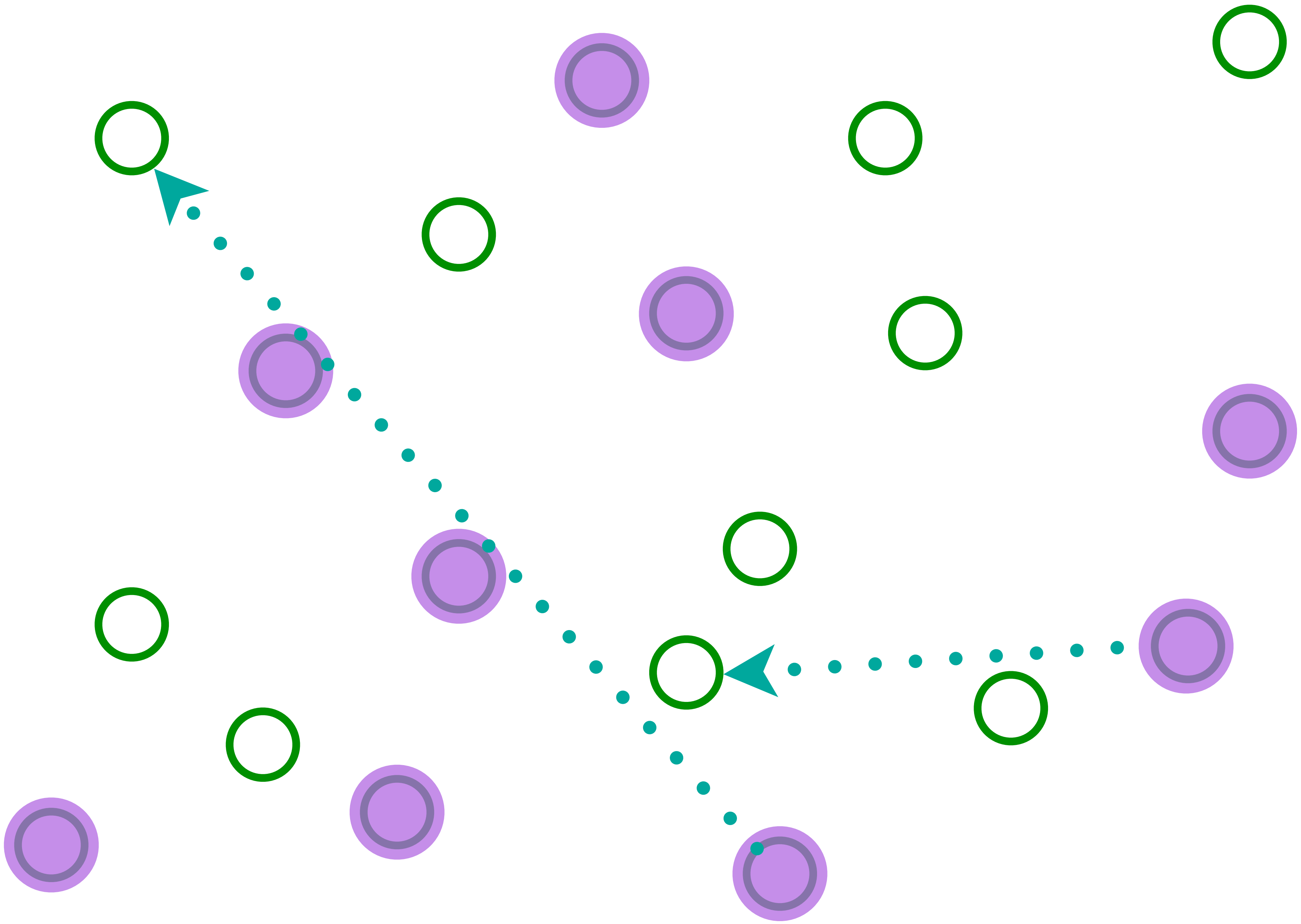

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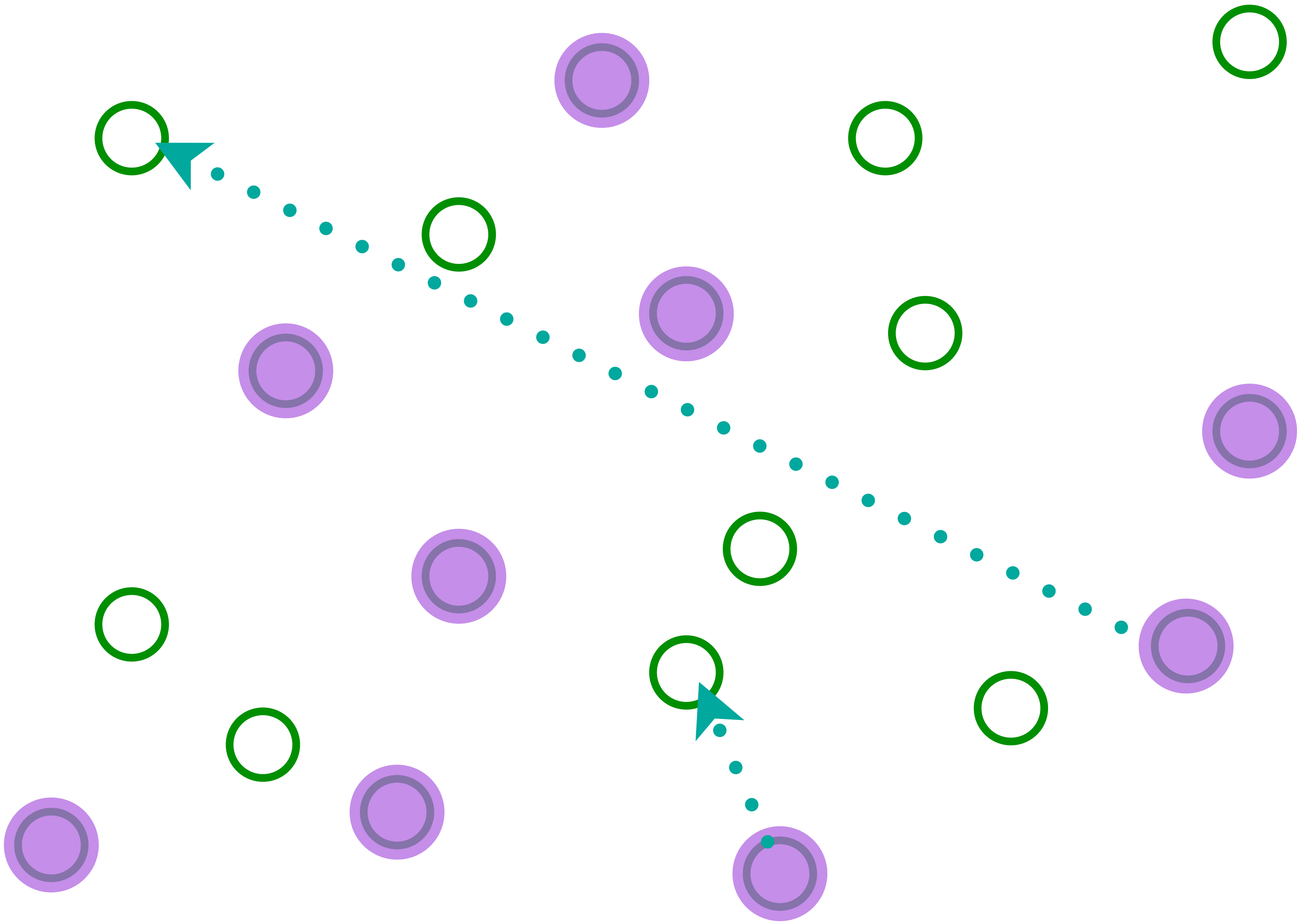

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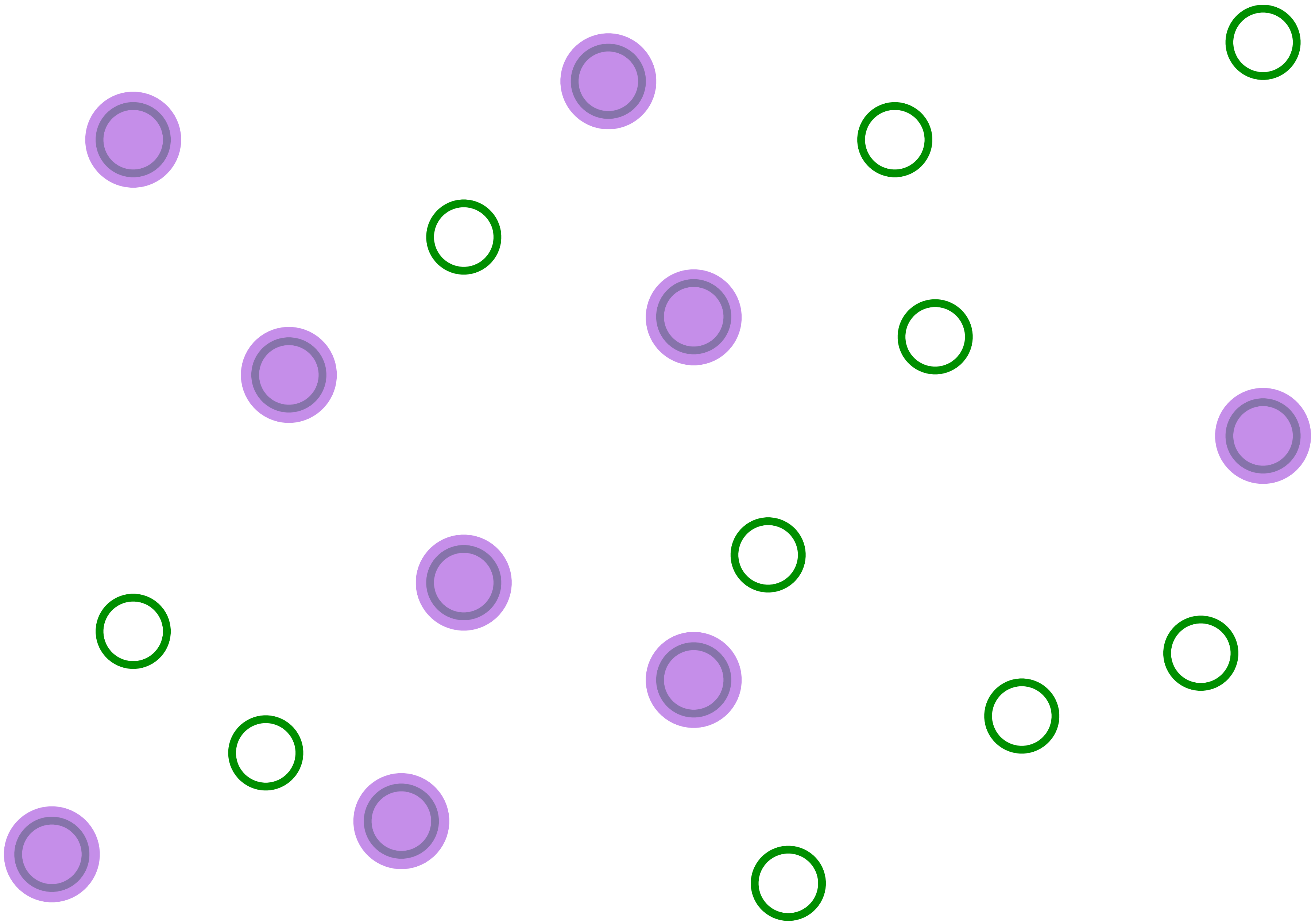

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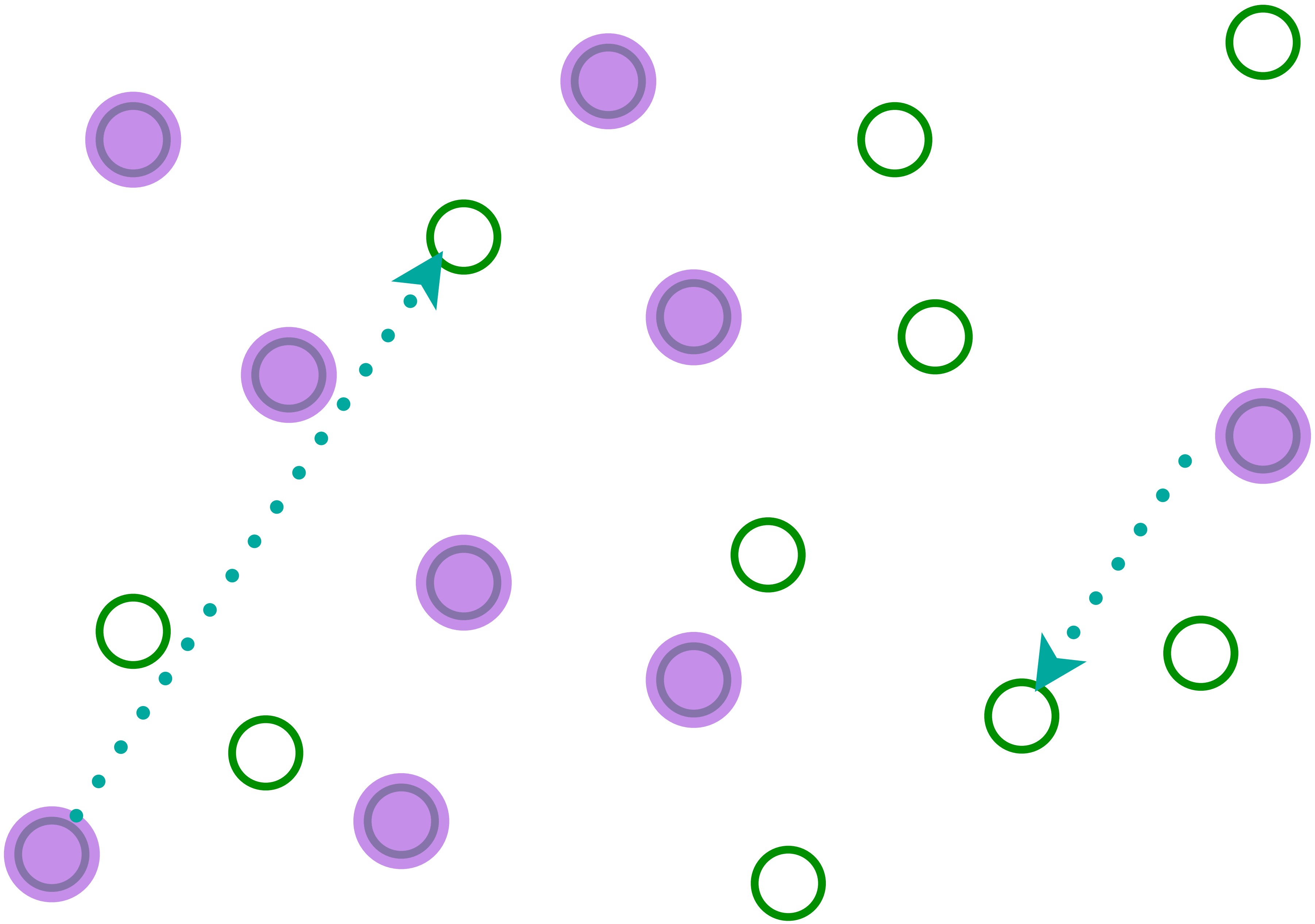

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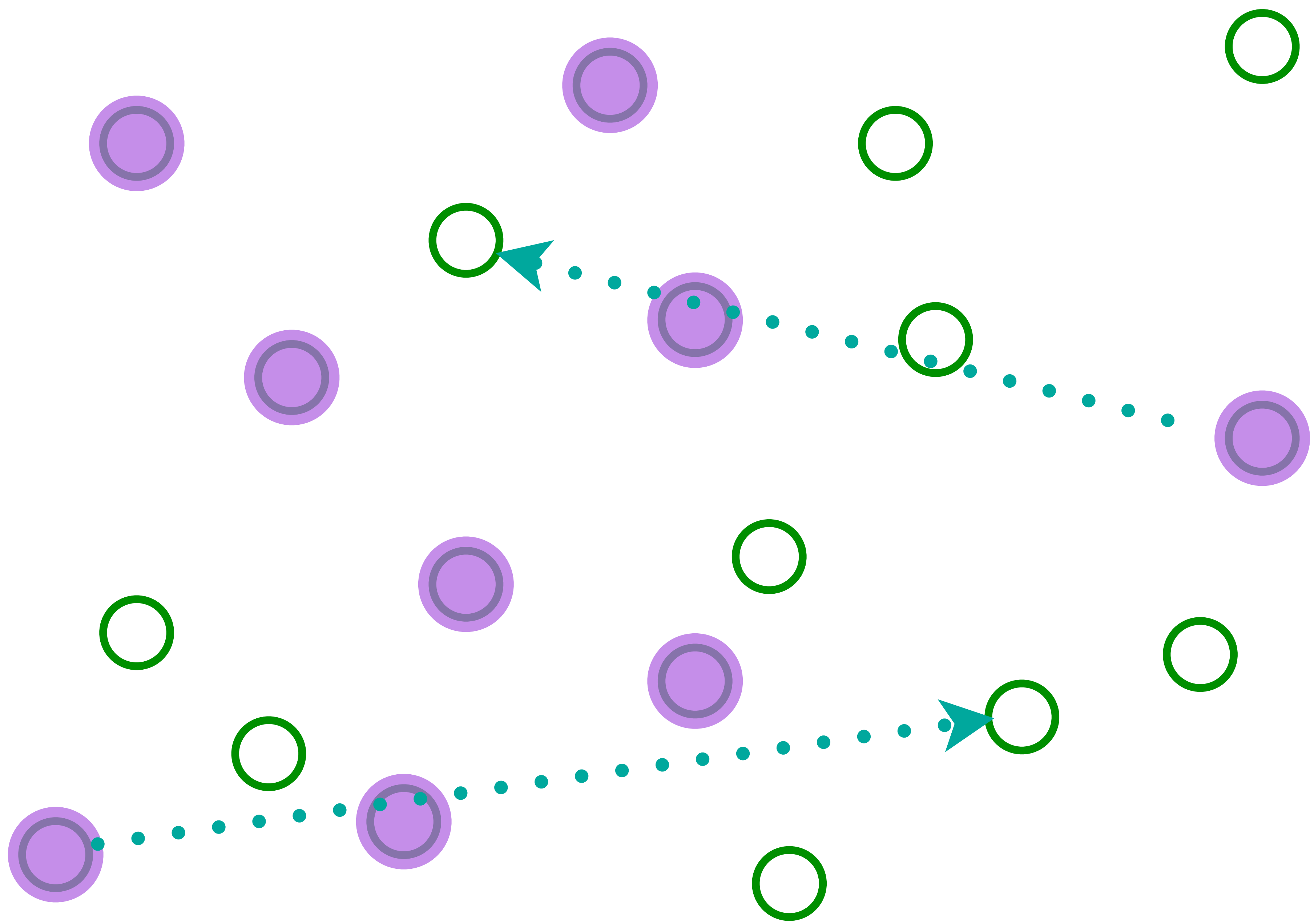

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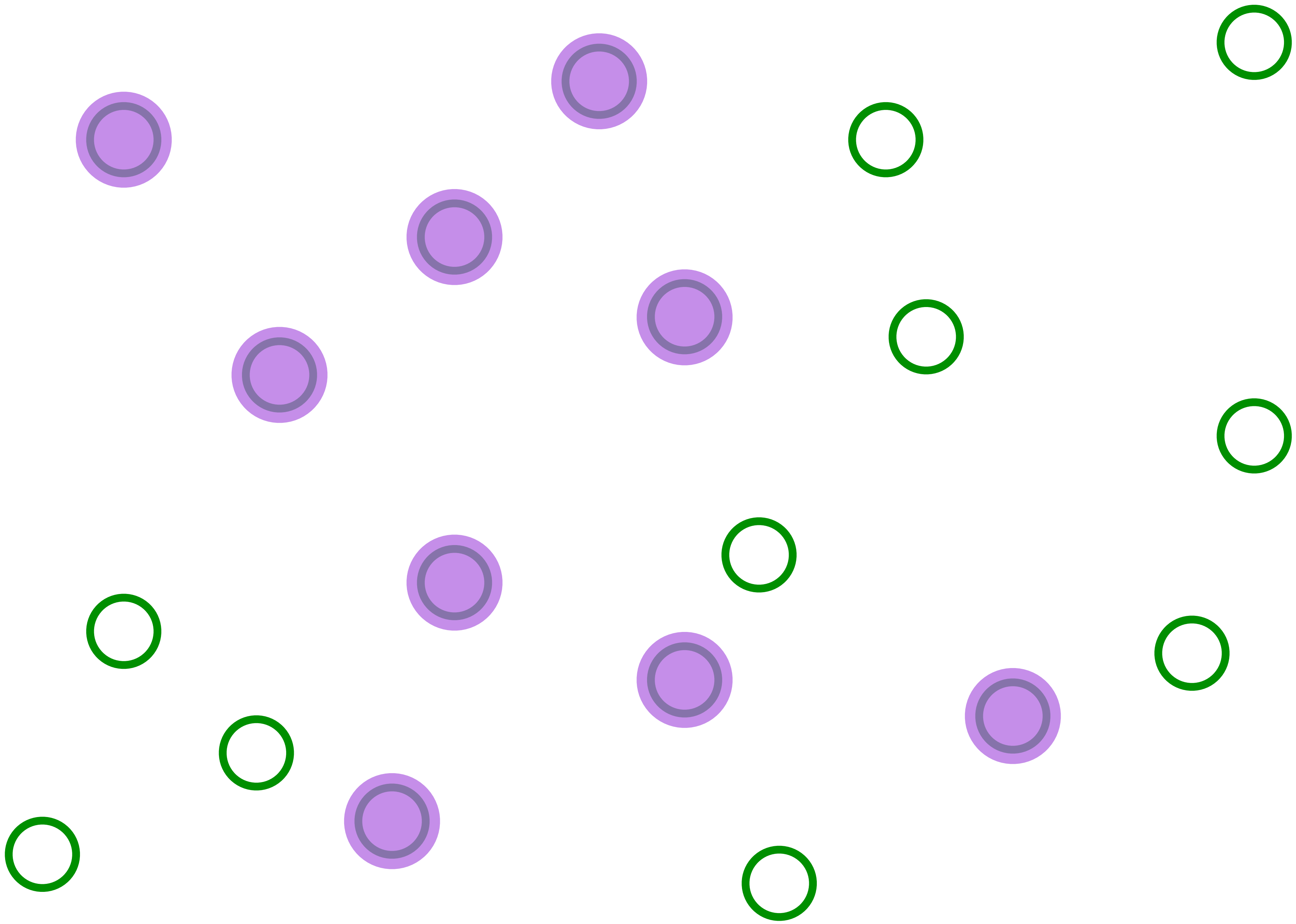

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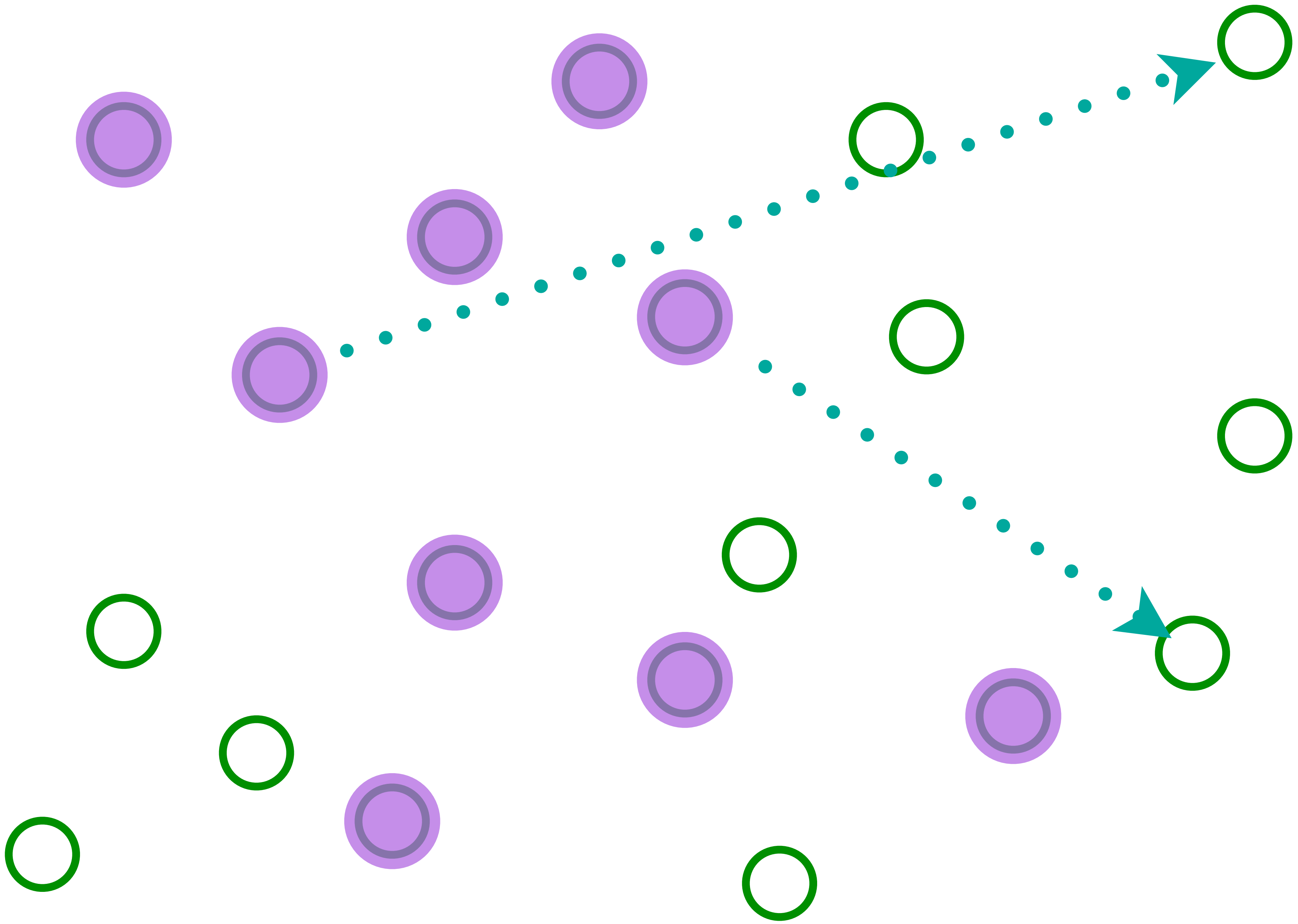

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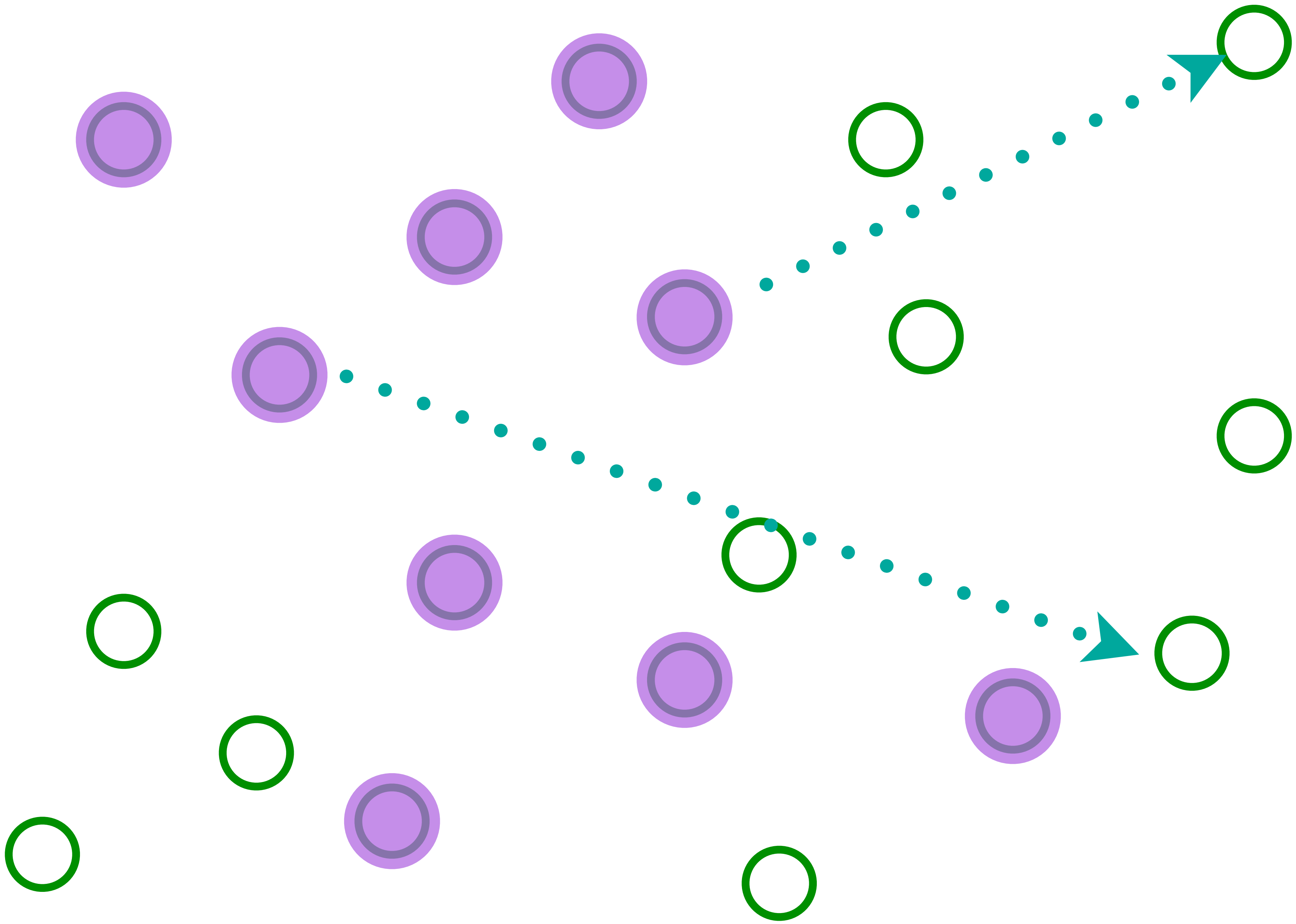

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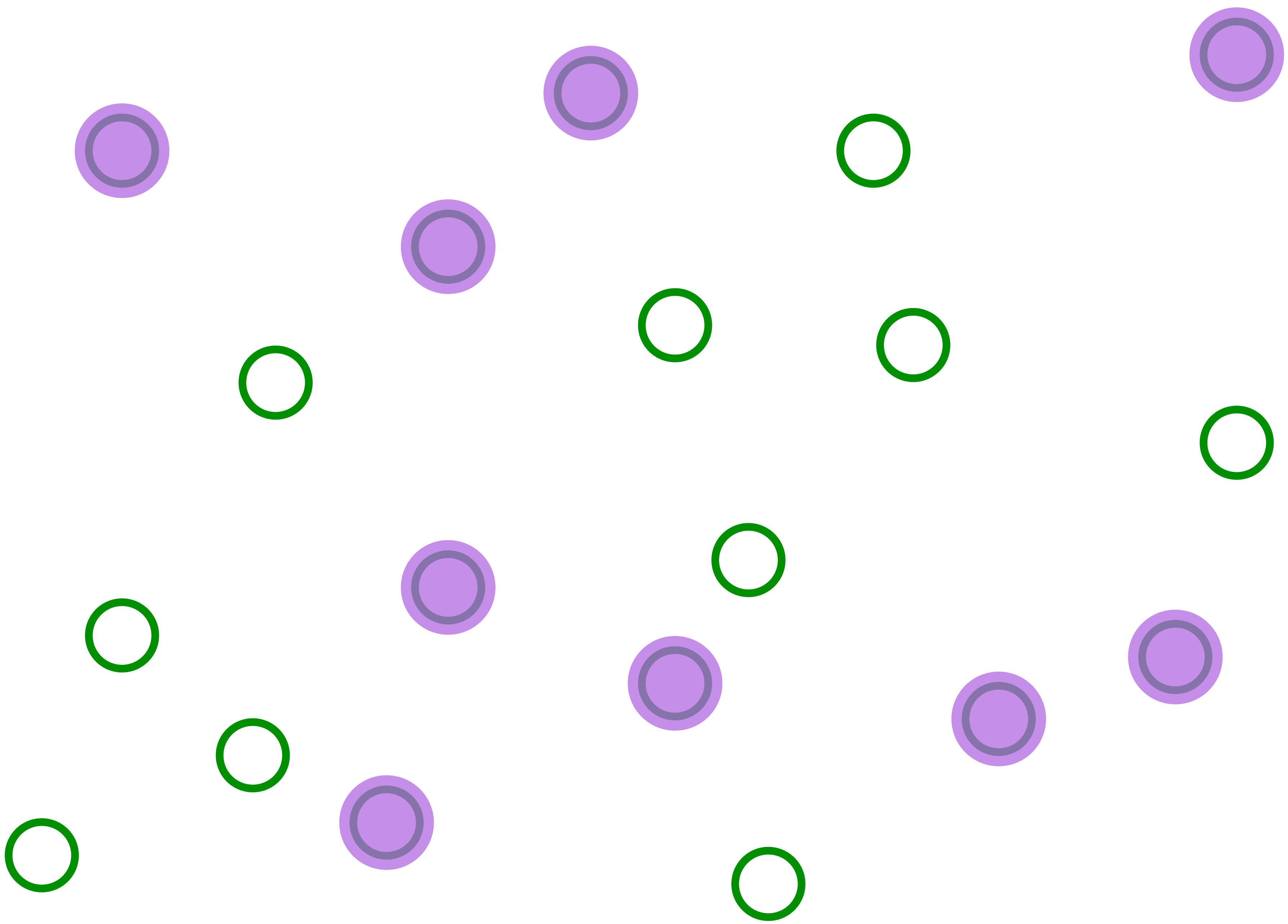

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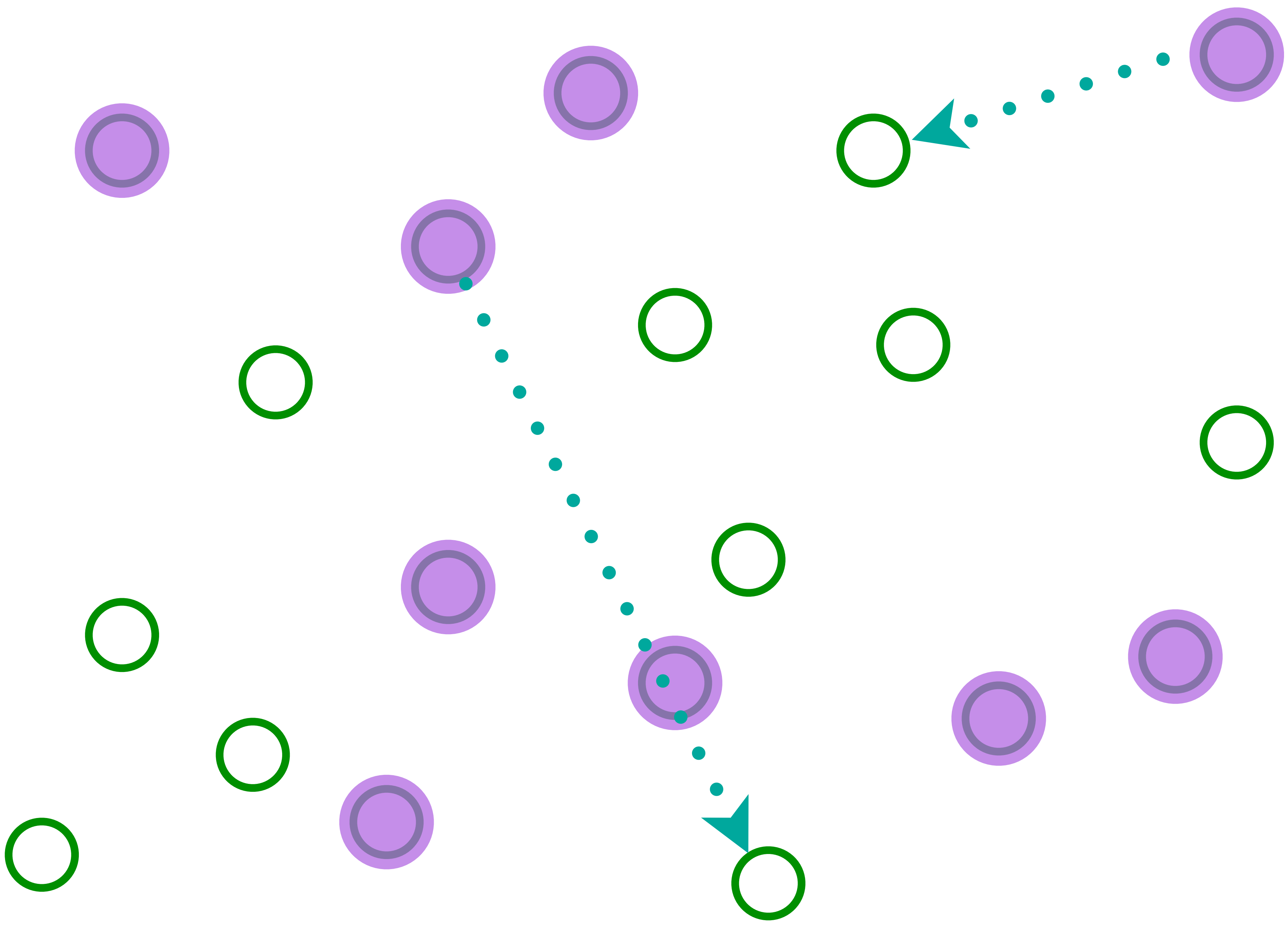

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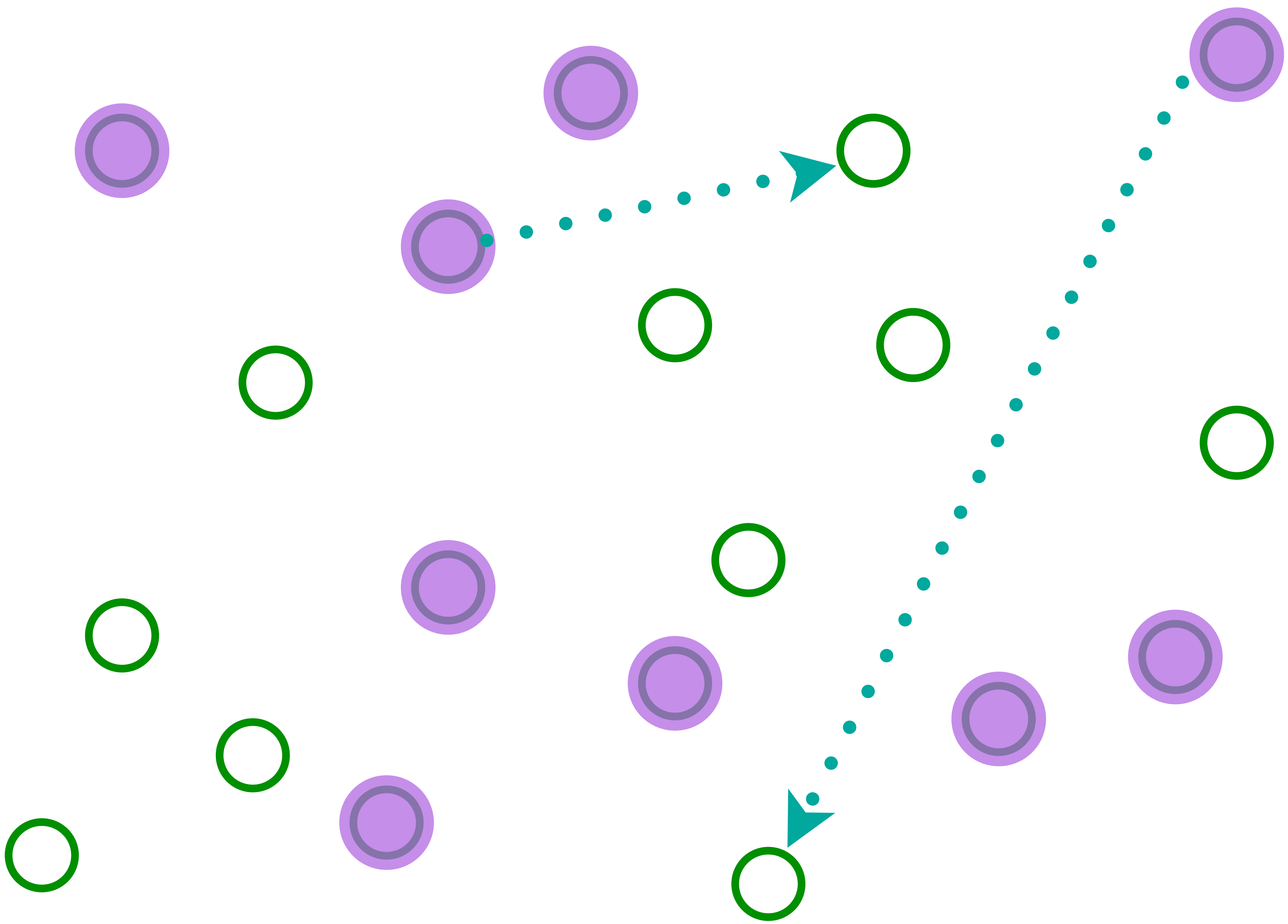

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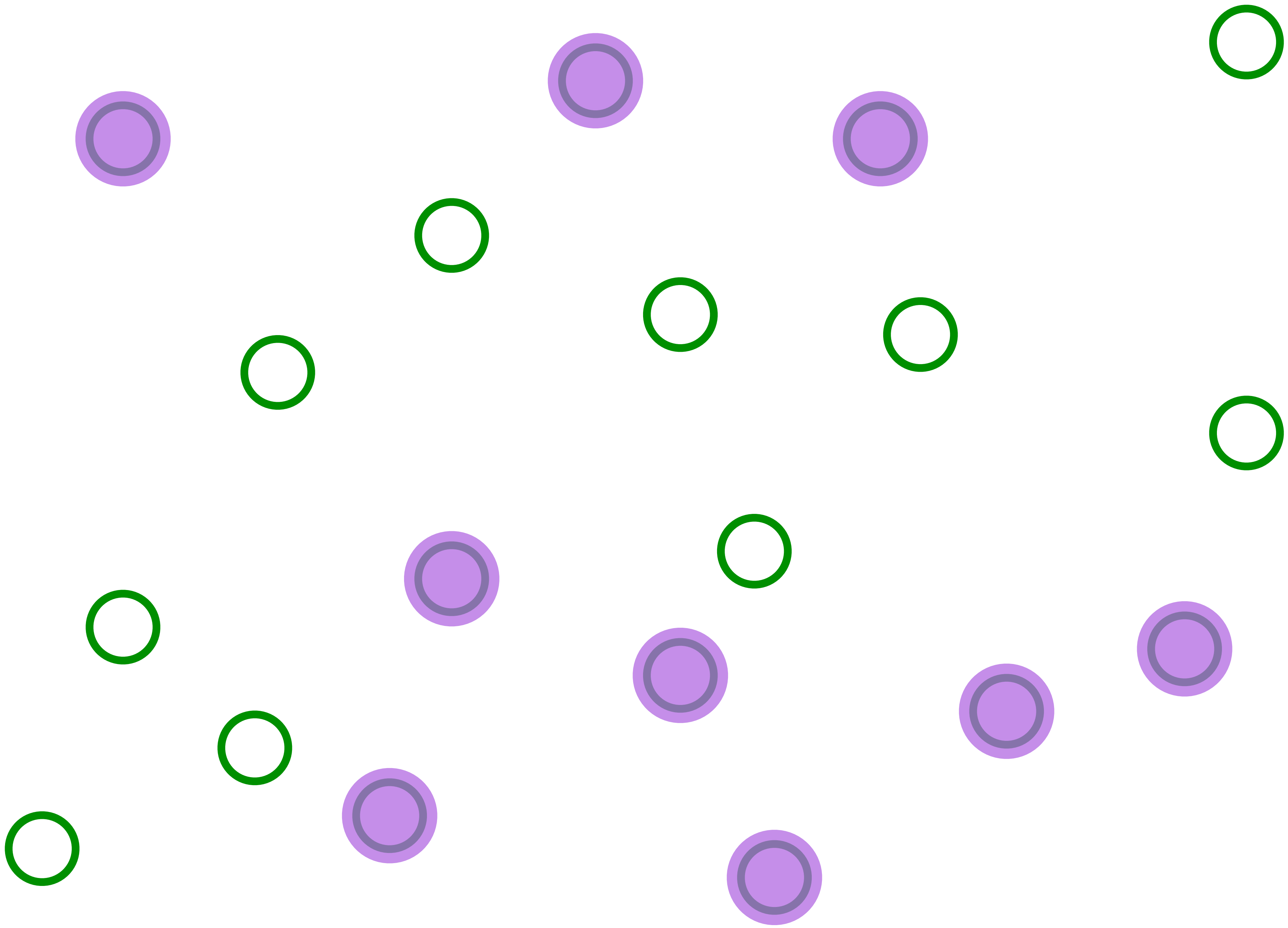

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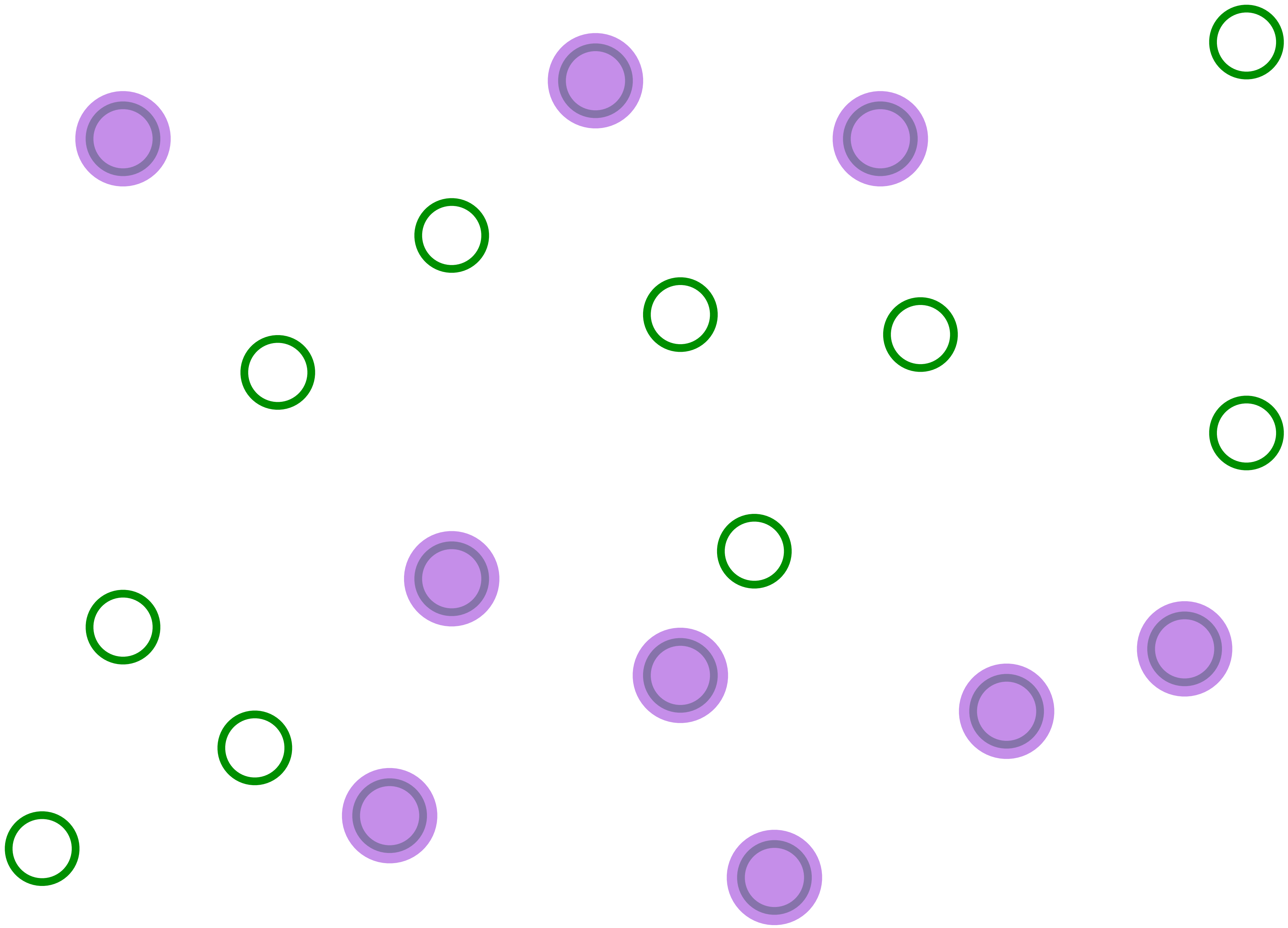

This describes both a strange metal and a black hole!

The SYK model

(See also: the "2-Body Random Ensemble" in nuclear physics; did not obtain the large N limit;
T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981))

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} U_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell - \mu \sum_i c_i^\dagger c_i$$

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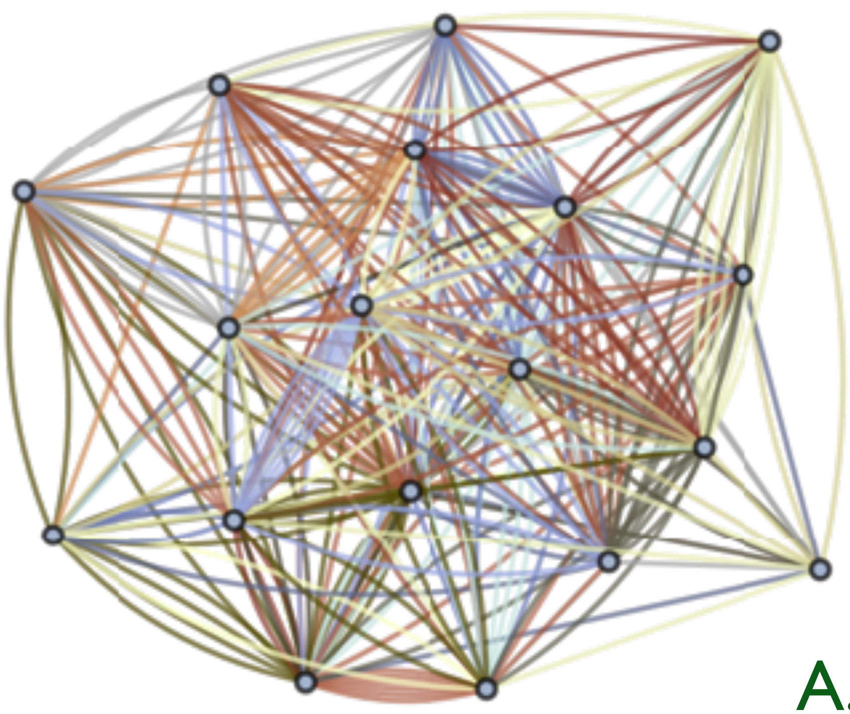
$$\mathcal{Q} = \frac{1}{N} \sum_i c_i^\dagger c_i$$

$U_{ij;k\ell}$ are independent random variables with $\overline{U_{ij;k\ell}} = 0$ and $\overline{|U_{ij;k\ell}|^2} = U^2$
$N \to \infty$ yields critical strange metal.



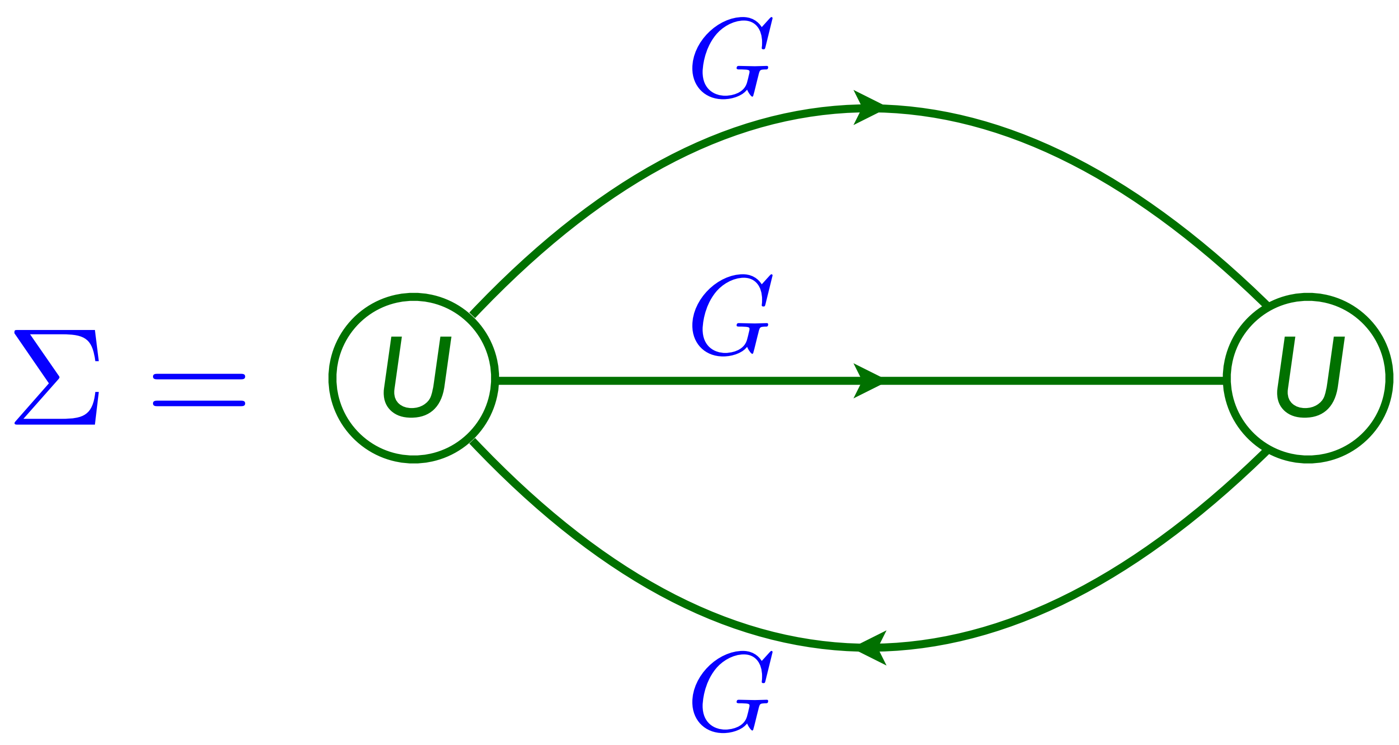
S. Sachdev and J. Ye, PRL **70**, 3339 (1993)

A. Kitaev, unpublished; S. Sachdev, PRX **5**, 041025 (2015)

The SYK model

Feynman graph expansion in $U_{ijk\ell}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$



S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

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$$G(\tau = 0^-) = \mathcal{Q}.$$

Low frequency analysis shows that the solutions must be gapless and obey

$$\Sigma(z) = \mu - \frac{e^{i(\pi/4+\theta)}}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A e^{-i(\pi/4+\theta)}}{\sqrt{z}}$$

where $A = (\pi/U^2 \cos(2\theta))^{1/4}$. The value of θ is universally related to $\mathcal{Q}$ by a Luttinger-Ward functional analysis similar to that used to establish the Luttinger theorem of Fermi liquid theory:

$$\boxed{\mathcal{Q} = \frac{1}{2} - \frac{\theta}{\pi} - \frac{\sin(2\theta)}{4}}$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

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At $T > 0$, we obtain a solution with a conformal structure

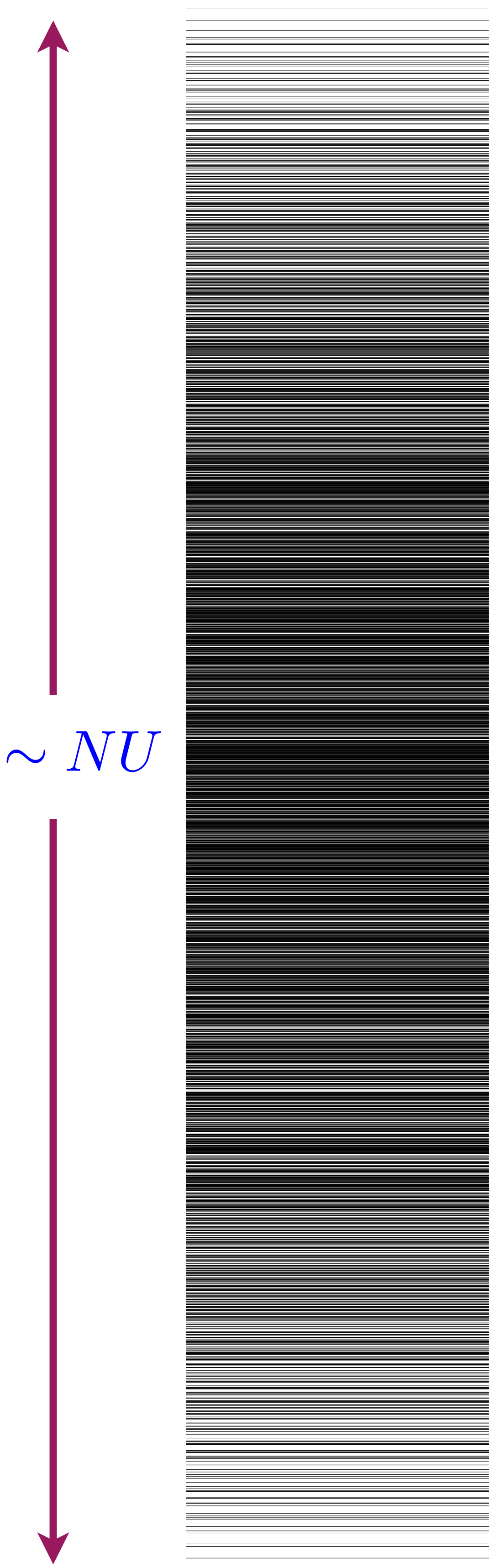
$$G(\tau) = -A \frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1 + e^{-4\pi\mathcal{E}}}} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2} \quad , \quad 0 < \tau < 1/T \, ,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$

$$\boxed{e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}}\,.$$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
S. Sachdev, PRX **5**, 041025 (2015)

The SYK model



Many-body level spacing $\sim$ $2^{-N} = e^{-N\ln 2}$

Non-quasiparticle excitations with spacing $\sim e^{-N s_0}$

There are 2^N many body levels with energy E. Shown are all values of E for a single cluster of size $N = 12$. The $T \to 0$ state has an entropy $S_{GPS} = N s_0$, where $s_0 < \ln 2$ is determined by integrating

$$\boxed{\frac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}\,.}$$

At $\mathcal{Q} = 1/2$,

$$s_0 \quad = \quad \frac{G}{\pi} + \frac{\ln(2)}{4} = 0.464848\ldots$$

where G is Catalan's constant.

GPS: A. Georges, O. Parcollet, and S. Sachdev, PRB **63**, 134406 (2001)

W. Fu and S. Sachdev, PRB **94**, 035135 (2016)

The SYK model

$$\Omega(T) - E_0 = N\left[-s_0 T - \frac{1}{2}(\gamma + 4\pi^2\mathcal{E}^2 K)T^2 + \mathcal{O}(T^3)\right] + 2T\ln\left(\frac{U}{T}\right)\ldots$$

is the grand potential, where $K = d\mathcal{Q}/d\mu \sim 1/U$ is the compressibility$/N$, $\gamma \sim 1/U$ will appear later in the co-efficient of the Schwarzian, and the N^0 term arises from fluctuations about the large N theory described by the Schwarzian.

The inversion from $\Omega(T)$ to the *many*-body density of states, $D(E)$,

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

requires terms in $\Omega(T)$ which are exponentially small in N (not shown above) from the Schwarzian action, yielding terms which are not small in $D(E)$. We obtain

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, arXiv:1611.04650;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849 ;
A.M. Garcia-Garcia and J.J.M. Verbaarschot, arXiv:1701.06593; D. Bagrets, A. Altland, and A. Kamenev, arXiv:1702.08902;
D. Stanford and E. Witten, arXiv:1703.04612; A. Kitaev and S.J. Suh, arXiv:1711.08467; Yingfei Gu and S. Sachdev, unpublished.

The SYK model

$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}}\, d\left(E - \frac{p^2}{2NK}\right)$$

where $N\mathcal{Q} + p$ is the integer fermion number, $d(E) = 0$ for $E < E_0$, and

$$\boxed{d(E) \sim \exp\left(N s_0\right) \sinh\left(\sqrt{2N\gamma(E - E_0)}\right) \quad , \quad E > E_0\ ,\ e^{-cN} \ll \gamma(E - E_0) \ll N}$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N.

We can understand the dependence on the integer charge p by the relationship $ds_0/d\mathcal{Q} = 2\pi\mathcal{E}$, and hence $N s_0(\mathcal{Q} + p/N) \approx N s_0 + 2\pi p \mathcal{E}$.

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, arXiv:1611.04650;
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A simple model of a metal with quasiparticles

The grand potential $\Omega(T)$ at low T is (from the Sommerfeld expansion)

$$\Omega(T) - E_0 = N\left(-\frac{\pi^2}{6}\rho_0 T^2 + \mathcal{O}(T^4)\right) + \ldots$$

where $\rho_0 \equiv \rho(0)$ is the *single* particle density of states at the Fermi level.

We can also define the *many* body density of states, $D(E)$, via

$$Z = e^{-\Omega(T)/T} = \int_{-\infty}^{\infty} dE D(E) e^{-E/T}$$

The inversion from $\Omega(T)$ to $D(E)$ has to performed with care (it need not commute with the $1/N$ expansion), and we obtain

$$\boxed{D(E) \sim \exp\left(\pi\sqrt{\frac{2N\rho_0(E-E_0)}{3}}\right) \quad , \quad E > E_0 \; , \; \frac{1}{N} \ll \rho_0(E-E_0) \ll N}$$

and $D(E) = 0$ for $E < E_0$. This is related to the asymptotic growth of the partitions of an integer, $p(n) \sim \exp(\pi\sqrt{2n/3})$. Near the lower bound, there are large sample-to-sample fluctuations due to variations in the lowest quasiparticle energies.

The SYK model

$$D(E) = \sum_{p=-\infty}^{\infty} e^{2\pi p \mathcal{E}} \, d\left(E - \frac{p^2}{2NK}\right)$$

where $N\mathcal{Q} + p$ is the integer fermion number, $d(E) = 0$ for $E < E_0$, and

$$d(E) \sim \boxed{\exp\left(N s_0\right)} \sinh\left(\sqrt{2N\gamma(E - E_0)}\right) \quad , \quad E > E_0 \ , \ e^{-cN} \ll \gamma(E - E_0) \ll N$$

There are exponentially more low energy states than for the quasiparticle case, and $D(E)$ self-averages down to energies exponentially small in N.

We can understand the dependence on the integer charge p by the relationship $ds_0/d\mathcal{Q} = 2\pi\mathcal{E}$, and hence $N s_0(\mathcal{Q} + p/N) \approx N s_0 + 2\pi p \mathcal{E}$.

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The SYK model

No quasiparticles

- Rapid local thermal equilibration (of fermion correlators) in a 'Planckian' time

$$\tau_{\rm eq} \sim \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Eberlein, V. Kasper, S. Sachdev, and
J. Steinberg, PRB **96**, 205123 (2017)

Established by solution of Schwinger-Keldysh equations for a quench.

- Presence of quasiparticles should slow down thermalization, so *all* quantum systems obey

S. Sachdev, *Quantum Phase Transitions*,
Cambridge (1999)

$$\tau_{\rm eq} > C \frac{\hbar}{k_B T} \quad , \quad \text{as } T \to 0.$$

Absence of quasiparticles $\Leftrightarrow$ Fastest possible thermalization

1. Random matrix quasiparticle model

q=2, complex SYK

2. Matter without quasiparticles

q=4, complex SYK

3. The Schwarzian theory

4. Connections to black holes with AdS_2 horizons

The SYK model

Feynman graph expansion in $J_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

$$\boxed{\begin{gathered} G(i\omega) = \frac{1}{\cancel{i\omega} + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau) \\ G(\tau = 0^-) = \mathcal{Q}. \end{gathered}}$$

Low frequency analysis shows that the solutions must be gapless and obey

$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

where $A = e^{-i\pi/4}(\pi/U^2)^{1/4}$ at half-filling. The ground state is a non-Fermi liquid, with a continuously variable density $\mathcal{Q}$.

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

The SYK model

The equations for the Green's function can also be solved at non-zero T. At half-filling, $\mathcal{Q} = 1/2$, we "guess" the particle-hole symmetric solution

$$G(\tau) = B\,\mathrm{sgn}(\tau)\left|\frac{\pi T}{\sin(\pi T \tau)}\right|^{\rho}$$

Then the self-energy is

$$\Sigma(\tau) = U^2 B^3 \mathrm{sgn}(\tau)\left|\frac{\pi T}{\sin(\pi T \tau)}\right|^{3\rho}$$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

The SYK model

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Taking Fourier transforms, we have as a function of the Matsubara frequency ω_n

$$\begin{aligned} G(i\omega_n) &= [iB\Pi(\rho)]\,\frac{T^{\rho-1}\,\Gamma\left(\dfrac{\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}{\Gamma\left(1-\dfrac{\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)} \\ \Sigma_{\mathrm{sing}}(i\omega_n) &= \left[iU^2B^3\Pi(3\rho)\right]\,\frac{T^{3\rho-1}\,\Gamma\left(\dfrac{3\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}{\Gamma\left(1-\dfrac{3\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}\,, \end{aligned}$$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

The SYK model

$$G(i\omega_n) \;=\; [iB\Pi(\rho)]\,\frac{T^{\rho-1}\,\Gamma\left(\dfrac{\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}{\Gamma\left(1-\dfrac{\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}$$

$$\Sigma_{\rm sing}(i\omega_n) \;=\; \left[iU^2B^3\Pi(3\rho)\right]\frac{T^{3\rho-1}\,\Gamma\left(\dfrac{3\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)}{\Gamma\left(1-\dfrac{3\rho}{2}+\dfrac{\omega_n}{2\pi T}\right)},$$

where we have dropped a less-singular term in Σ, and

$$\Pi(s) \equiv \pi^{s-1}2^s\cos\left(\frac{\pi s}{2}\right)\Gamma(1-s).$$

Now the singular part of Dyson's equation is

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)

$$G(i\omega_n)\Sigma_{\rm sing}(i\omega_n) = -1$$

Remarkably, the Γ functions appear with just the right arguments, so that there is a solution of the Dyson equation at $\rho = 1/2$!

So the Green's functions display thermal 'damping' at a scale set by T alone, which is independent of U.

The SYK model

Away from half-filling, the $T = 0$ solution has the low frequency form

$$\Sigma(z) = \mu - \frac{e^{i(\pi/4+\theta)}}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{Ae^{-i(\pi/4+\theta)}}{\sqrt{z}}$$

where $A = (\pi/U^2\cos(2\theta))^{1/4}$. The value of θ is universally related to $\mathcal{Q}$ by a Luttinger-Ward functional analysis similar to that used to establish the Luttinger theorem of Fermi liquid theory:

$$\boxed{\mathcal{Q} = \frac{1}{2} - \frac{\theta}{\pi} - \frac{\sin(2\theta)}{4}}$$

At $T > 0$, we obtain a solution with a conformal structure

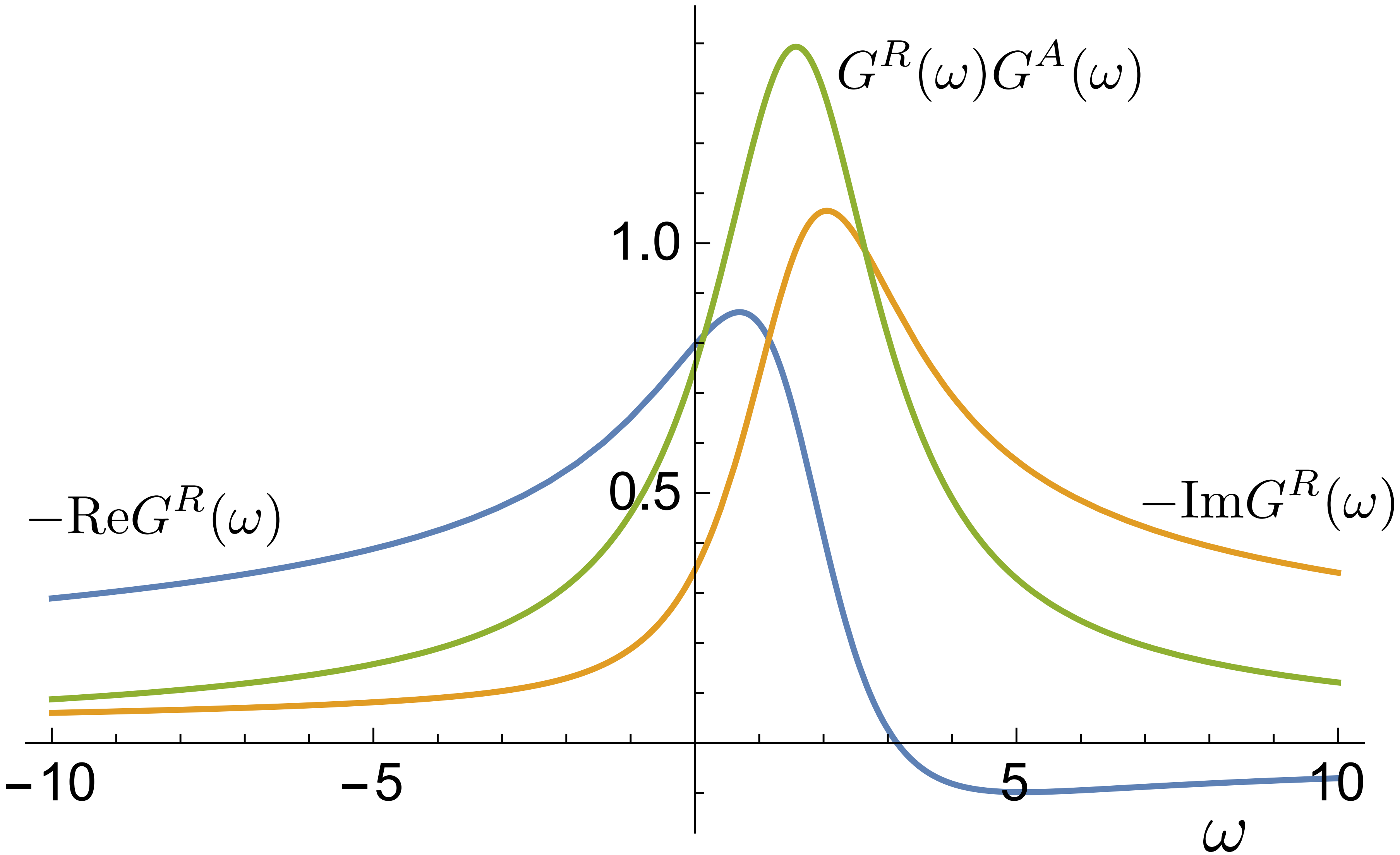
$$G(\tau) = -A\frac{e^{-2\pi\mathcal{E}T\tau}}{\sqrt{1+e^{-4\pi\mathcal{E}}}}\left(\frac{T}{\sin(\pi T\tau)}\right)^{1/2} \quad , \quad 0 < \tau < 1/T \,,$$

where the 'particle-hole asymmetry' is determined by $\mathcal{E}$

$$\boxed{e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4+\theta)}{\sin(\pi/4-\theta)}\,.}$$

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Georges, O. Parcollet, and S. Sachdev,
PRB **63**, 134406 (2001)
S. Sachdev, PRX **5**, 041025 (2015)

The SYK model



Green's functions away from half-filling

So the Green's functions display thermal 'damping' at a scale set by T alone, which is independent of U.

The SYK model

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \ldots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

The SYK model

$$G(i\omega) = \frac{1}{\cancel{i\omega} + \cancel{\mu} - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$$

$$\Sigma(z) = \cancel{\mu} - \frac{1}{A}\sqrt{z} + \dots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

At frequencies $\ll U$, the $i\omega + \mu$ can be dropped, and without it equations are invariant under the reparametrization and gauge transformations. The singular part of the self-energy and the Green's function obey

$$\int_0^\beta d\tau_2 \, \Sigma_{\text{sing}}(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma_{\text{sing}}(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

The SYK model

$$\int_0^\beta d\tau_2 \, \Sigma_(\tau_1, \tau_2) G(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -U^2 G^2(\tau_1, \tau_2) G(\tau_2, \tau_1)$$

These equations are invariant under

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \widetilde{\Sigma}(\sigma_1, \sigma_2)$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.
By using $f(\sigma) = \tan(\pi T \sigma)/(\pi T)$ we can
now obtain the $T > 0$ solution from the $T = 0$ solution.

A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

The SYK model

Let us write the large N saddle point solutions of S as

$$\begin{aligned} G_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-1/2} \\ \Sigma_s(\tau_1 - \tau_2) &\sim (\tau_1 - \tau_2)^{-3/2}. \end{aligned}$$

The saddle point will be invariant under a reperamaterization $f(\tau)$ when choosing $G(\tau_1, \tau_2) = G_s(\tau_1 - \tau_2)$ leads to a transformed $\widetilde{G}(\sigma_1, \sigma_2) = G_s(\sigma_1 - \sigma_2)$ (and similarly for Σ). It turns out this is true only for the SL(2, R) transformations under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken down to SL(2, R) by the saddle point.

A. Kitaev

Basics of conformal field theory

In a space with metric tensor $g_{\mu\nu}$ and proper distance

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu$$

after the co-ordinate transformation $x_\mu \to x'_\mu$, the new metric tensor is

$$g'_{\mu\nu} = g_{\rho\lambda} \frac{\partial x_\rho}{\partial x'_\mu} \frac{\partial x_\lambda}{\partial x'_\nu} \,.$$

A conformal transformation is one which preserves all angles and so

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \,.$$

In a conformal field theory, two-point correlators of scalar fields transform as

$$\langle \phi(x_1) \phi(x_2) \rangle = \left| \det \left[\frac{\partial x'_1}{\partial x_1} \right] \right|^{\Delta/d} \left| \det \left[\frac{\partial x'_2}{\partial x_2} \right] \right|^{\Delta/d} \langle \phi(x'_1) \phi(x'_2) \rangle$$

Infinite-range (SYK) model without quasiparticles

After introducing replicas $a = 1 \ldots n$, and integrating out the disorder, the partition function can be written as

$$Z = \int \mathcal{D}c_{ia}(\tau) \exp \left[-\sum_{ia} \int_0^\beta d\tau \, c_{ia}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) c_{ia} \right.$$
$$\left. - \frac{U^2}{4N^3} \sum_{ab} \int_0^\beta d\tau d\tau' \left| \sum_i c_{ia}^\dagger(\tau) c_{ib}(\tau') \right|^4 \right] .$$

For simplicity, we neglect the replica indices, and introduce the identity

$$1 = \int \mathcal{D}\Sigma(\tau_1, \tau_2) \exp \left[-N \int_0^\beta d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2) \left(G(\tau_2, \tau_1) \right. \right.$$
$$\left. \left. + \frac{1}{N} \sum_i c_i(\tau_2) c_i^\dagger(\tau_1) \right) \right] .$$

Infinite-range (SYK) model without quasiparticles

Then the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$
\begin{aligned}
Z &= \int \mathcal{D}G(\tau_1,\tau_2)\mathcal{D}\Sigma(\tau_1,\tau_2)\exp(-NS) \\
S &= \ln\det\left[\delta(\tau_1-\tau_2)(\partial_{\tau_1}+\mu)-\Sigma(\tau_1,\tau_2)\right] \\
&\quad + \int d\tau_1 d\tau_2 \Sigma(\tau_1,\tau_2)\left[G(\tau_2,\tau_1)+(U^2/2)G^2(\tau_2,\tau_1)G^2(\tau_1,\tau_2)\right]
\end{aligned}
$$

At frequencies $\ll U$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

A. Georges and O. Parcollet
PRB **59**, 5341 (1999)
A. Kitaev, 2015
S. Sachdev, PRX **5,** 041025 (2015)

$$
\begin{aligned}
\tau &= f(\sigma) \\
G(\tau_1,\tau_2) &= \left[f'(\sigma_1)f'(\sigma_2)\right]^{-1/4}\frac{g(\sigma_1)}{g(\sigma_2)}\,G(\sigma_1,\sigma_2) \\
\Sigma(\tau_1,\tau_2) &= \left[f'(\sigma_1)f'(\sigma_2)\right]^{-3/4}\frac{g(\sigma_1)}{g(\sigma_2)}\,\Sigma(\sigma_1,\sigma_2)
\end{aligned}
$$

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

The SYK model

Reparametrization and phase zero modes

We can write the path integral for the SYK model as

$$\mathcal{Z} = \int \mathcal{D}G(\tau_1, \tau_1)\mathcal{D}\Sigma(\tau_1, \tau_2) e^{-NS[G,\Sigma]}$$

for a known action $S[G, \Sigma]$. We find the saddle point, G_s, Σ_s, and only focus on the "Nambu-Goldstone" modes associated with breaking reparameterization and U(1) gauge symmetries by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1)f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)}$$

(and similarly for Σ). Then the path integral is approximated by

$$\mathcal{Z} = \int \mathcal{D}f(\tau)\mathcal{D}\phi(\tau) e^{-NS_{\text{eff}}[f,\phi]}.$$

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
S. Sachdev, PRX **5**, 041025 (2015); J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098; J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

The Schwarzian theory of the SYK model

Symmetry arguments, and explicit computations, show that the effective action is

$$\boxed{S_{\rm eff}[f,\phi] = \frac{K}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi \mathcal{E} T)\partial_\tau f)^2 - \frac{\gamma}{4\pi^2}\int_0^{1/T} d\tau \left\{\tan(\pi T f(\tau)), \tau\right\},}$$

where $f(\tau)$ is a monotonic map from $[0, 1/T]$ to $[0, 1/T]$, the couplings K, γ, and $\mathcal{E}$ can be related to thermodynamic derivatives and we have used the Schwarzian:

$$\{g, \tau\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .$$

Specifically, an argument constraining the effective at $T = 0$ is

$$S_{\rm eff}\left[f(\tau) = \frac{a\tau + b}{c\tau + d}, \phi(\tau) = 0\right] = 0,$$

and this is origin of the Schwarzian.

J. Maldacena and D. Stanford, arXiv:1604.07818;
R. Davison, Wenbo Fu, A. Georges, Yingfei Gu, K. Jensen, S. Sachdev, arXiv.1612.00849;
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746;
Yingfei Gu and S. Sachdev, unpublished

1. Random matrix quasiparticle model

q=2, complex SYK

2. Matter without quasiparticles

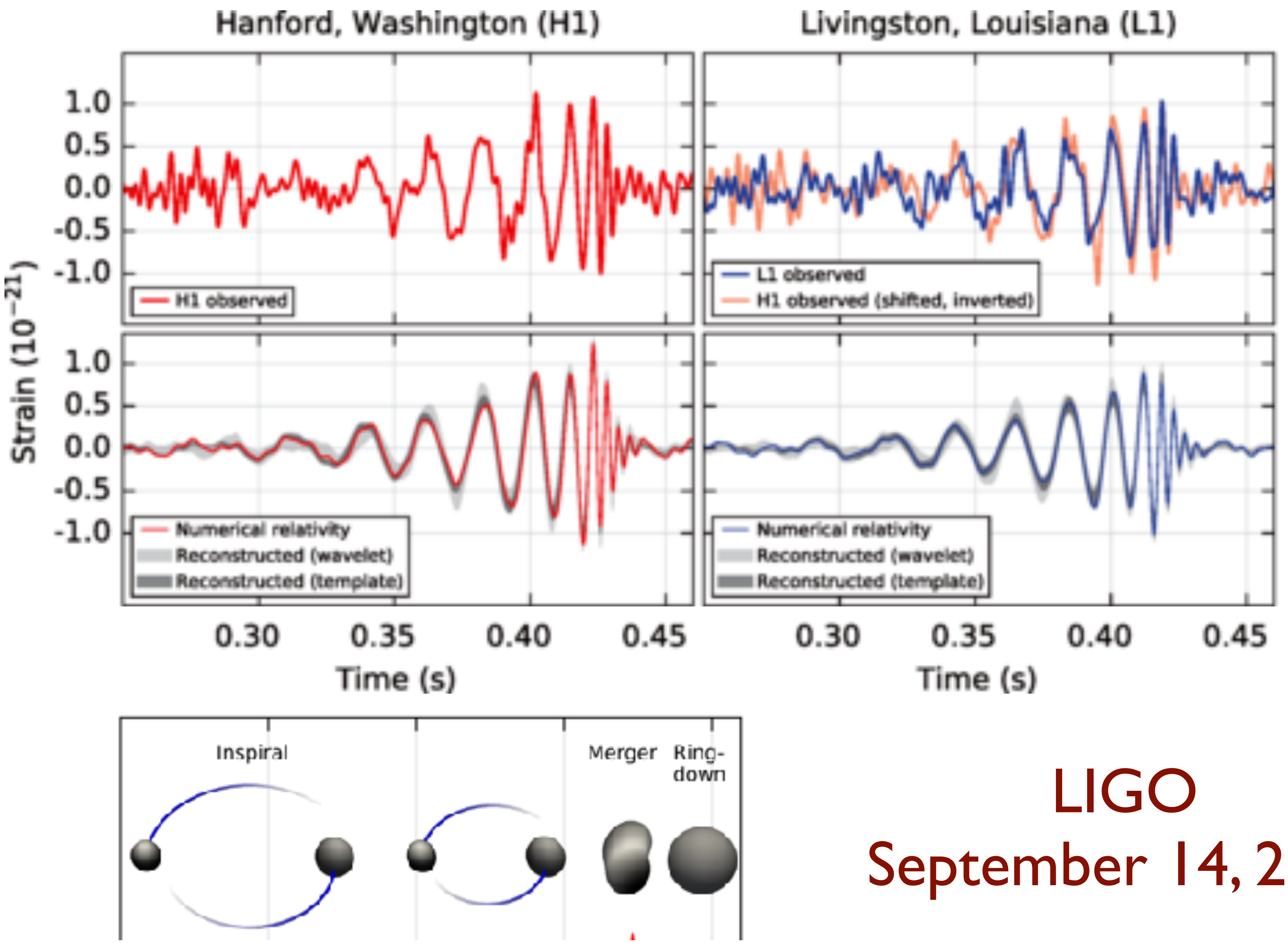
q=4, complex SYK

3. The Schwarzian theory

4. Connections to black holes with AdS_2 horizons

Black holes

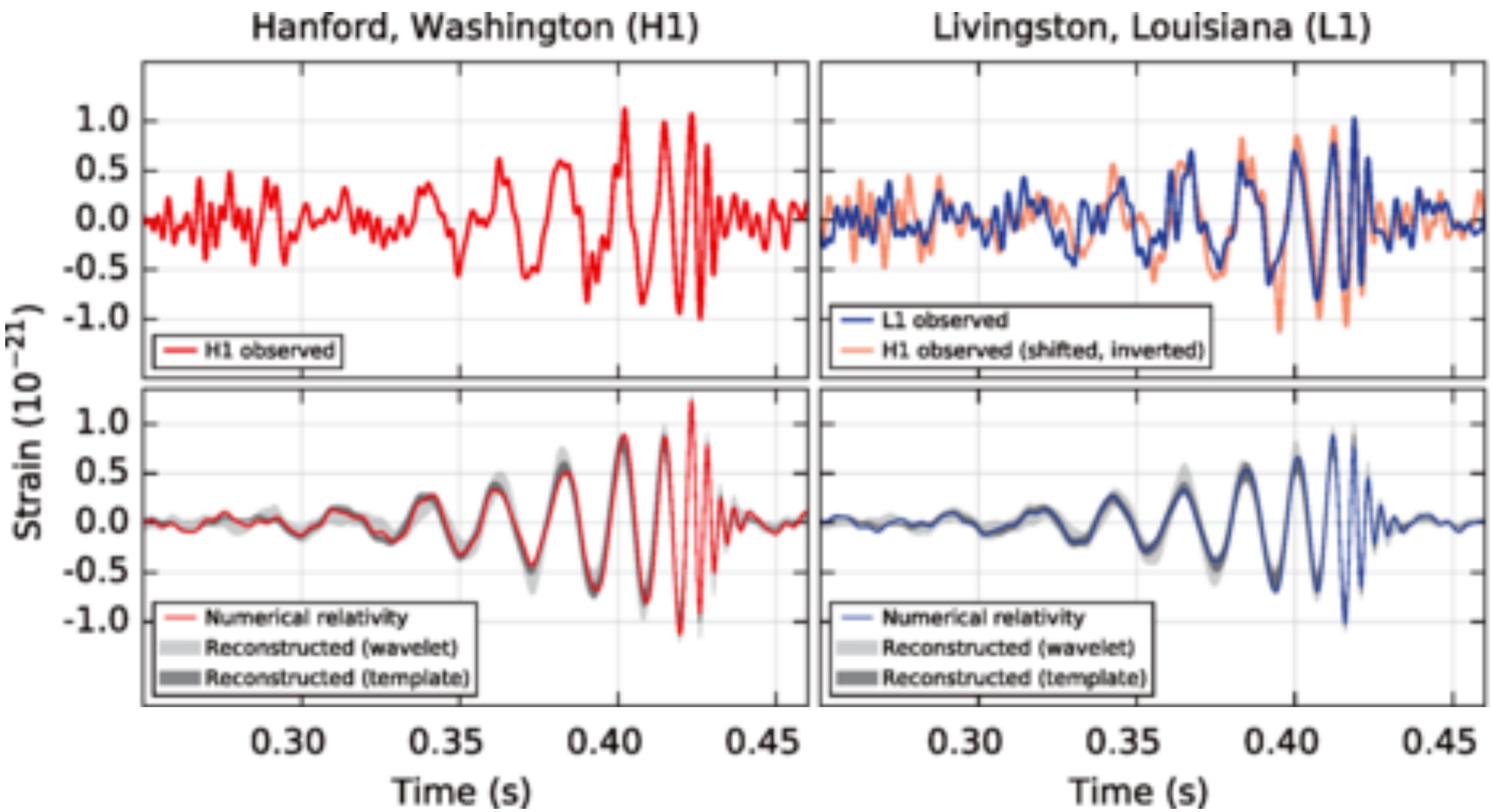
- Black holes have an entropy and a temperature, $T_H = \hbar c^3/(8\pi G M k_B)$.
- The entropy is proportional to their surface area.

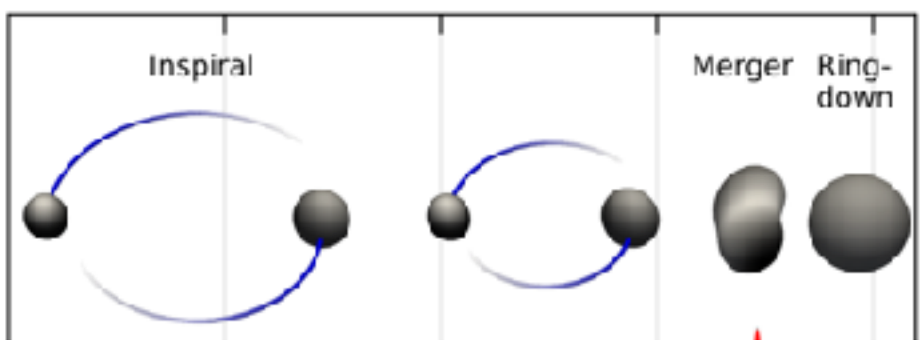


LIGO
September 14, 2015

- The ring-down is predicted by General Relativity to happen in a time $\dfrac{8\pi GM}{c^3} \sim 8$ milliseconds.

C. V. Vishveshwara, Nature **227**, 936 (1970)





LIGO
September 14, 2015

- The ring-down is predicted by General Relativity to happen in a time $\dfrac{8\pi GM}{c^3} \sim 8$ milliseconds. Curiously this happens to equal $\boxed{\dfrac{\hbar}{k_B T_H}}$: so the ring down can also be viewed as the approach of a quantum system to thermal equilibrium at the fastest possible rate!

Black holes

- Black holes have an entropy and a temperature, $T_H = \hbar c^3/(8\pi G M k_B)$.
- The entropy is proportional to their surface area.
- They relax to thermal equilibrium in a time $\sim \hbar/(k_B T_H)$.

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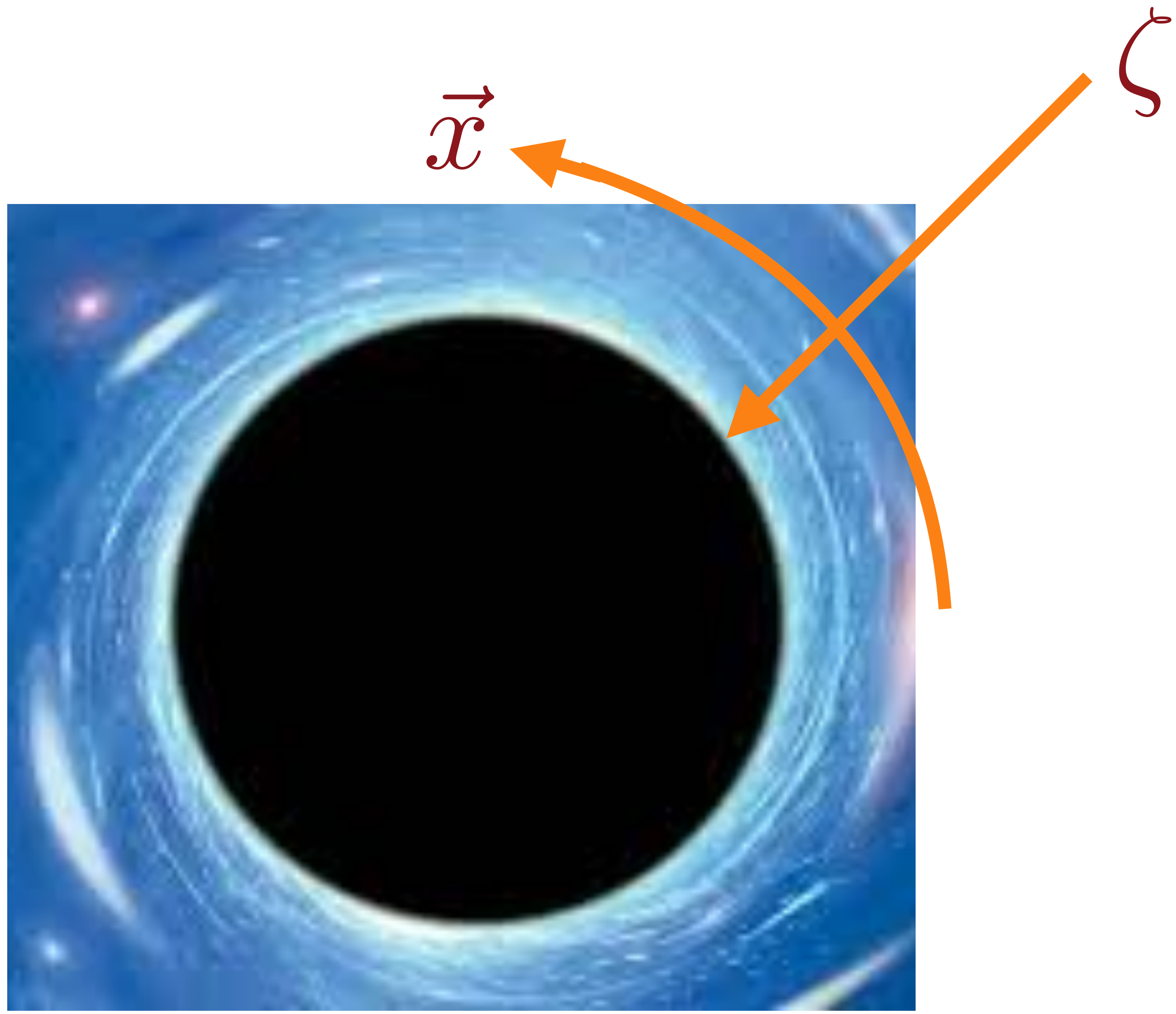
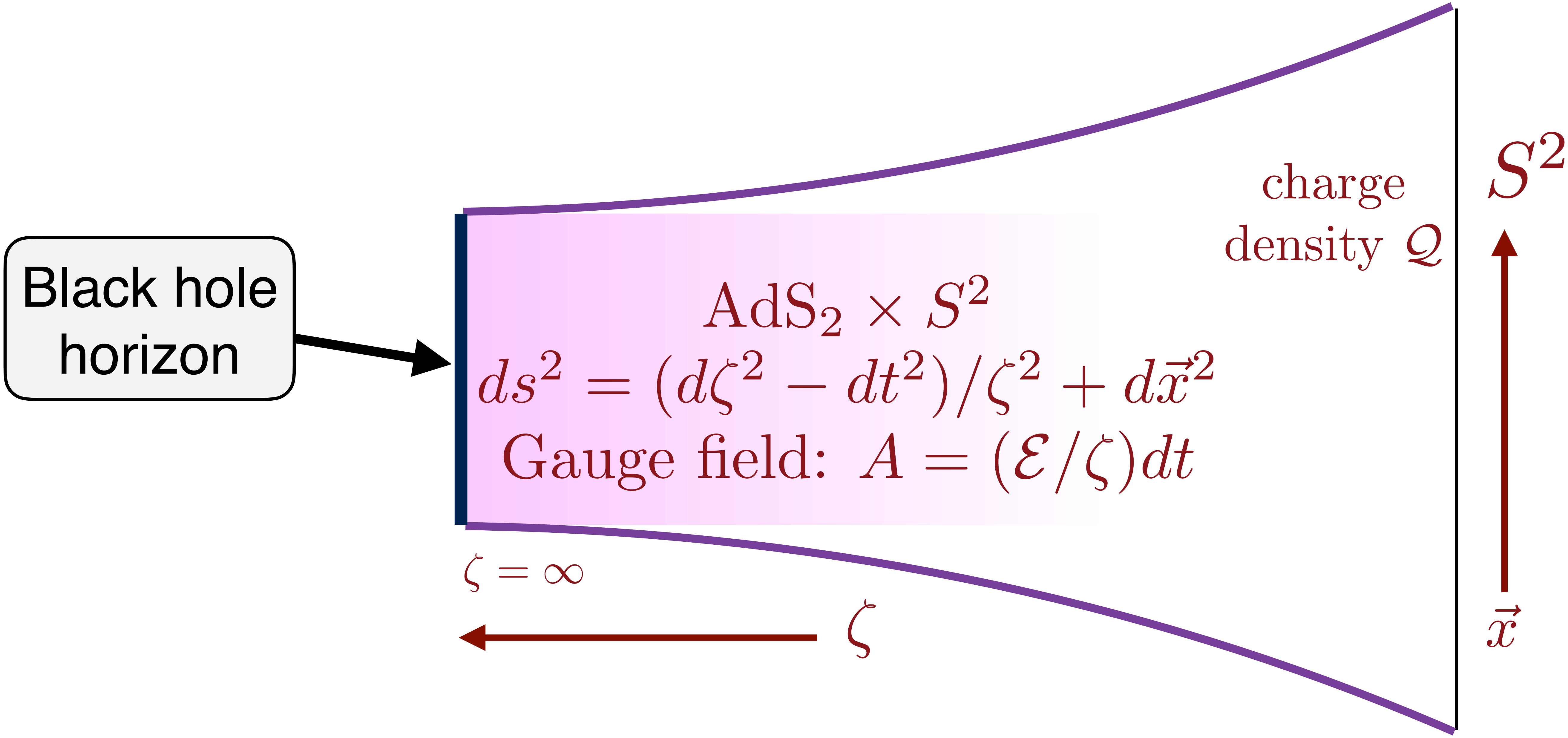
<u>Holography</u>:
Quantum black holes "look like" quantum many-particle systems without quasiparticle excitations, residing "on" the surface of the black hole



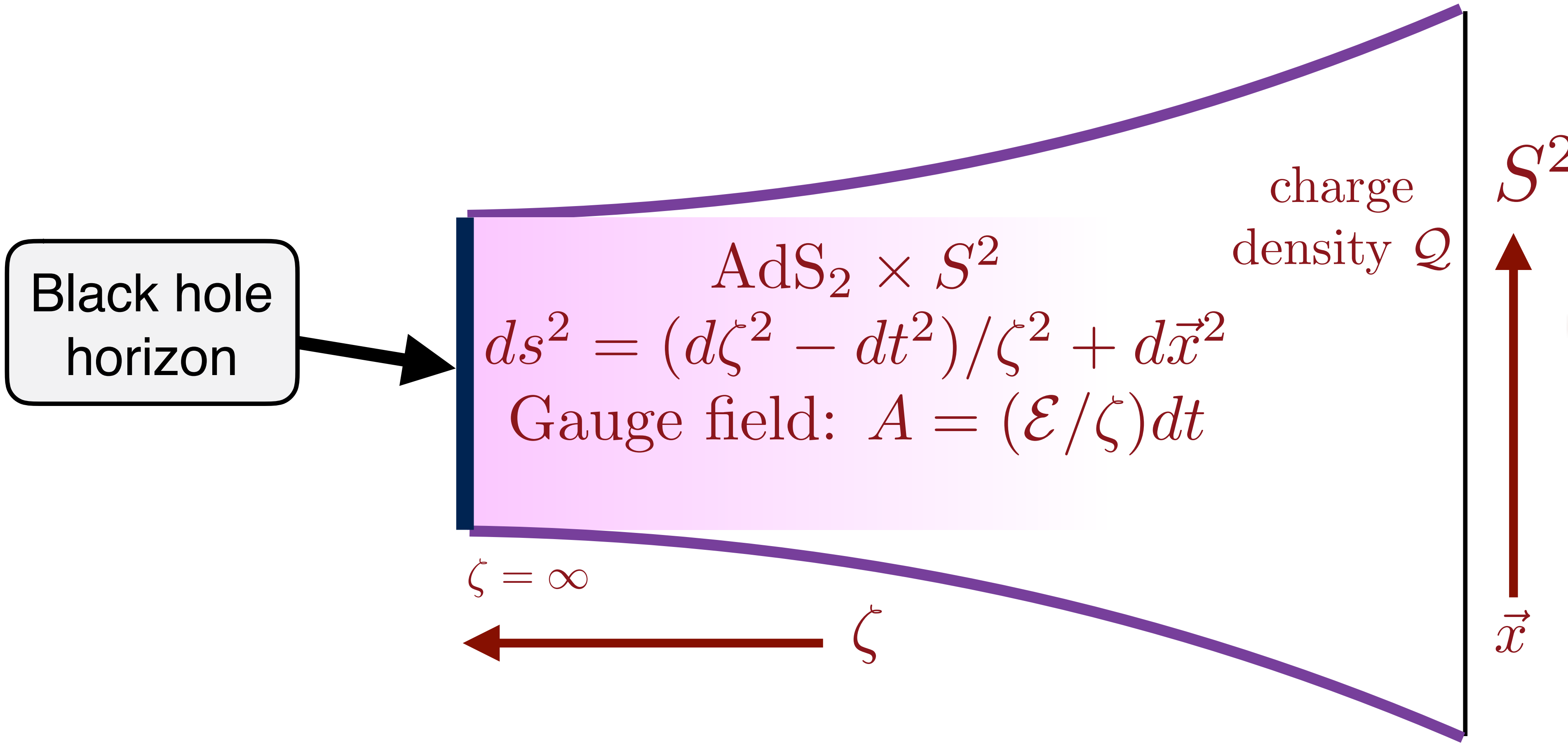
Consider a charged black hole with the smallest possible mass: the extremal limit. Zoom in to the near-horizon region at low energies. In this limit, the quantum theory lives in one space (ζ) and one time dimension

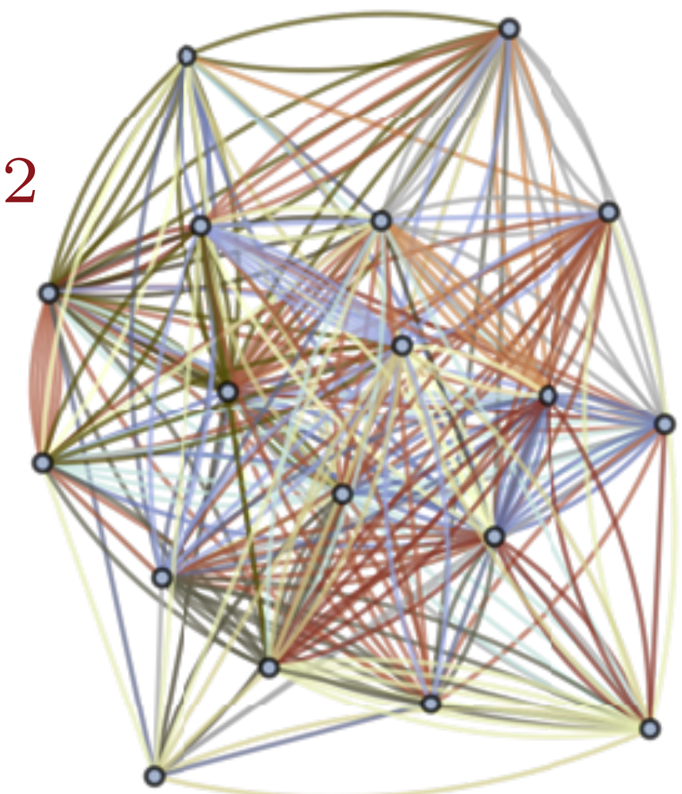
SYK models and black holes



The near-horizon region of an extremal charged black hole has the geometry of (1+1)-dimensional anti-de Sitter spacetime. By holography, this should map to a zero-dimensional quantum system: this turns out to be the SYK model

SYK models and black holes



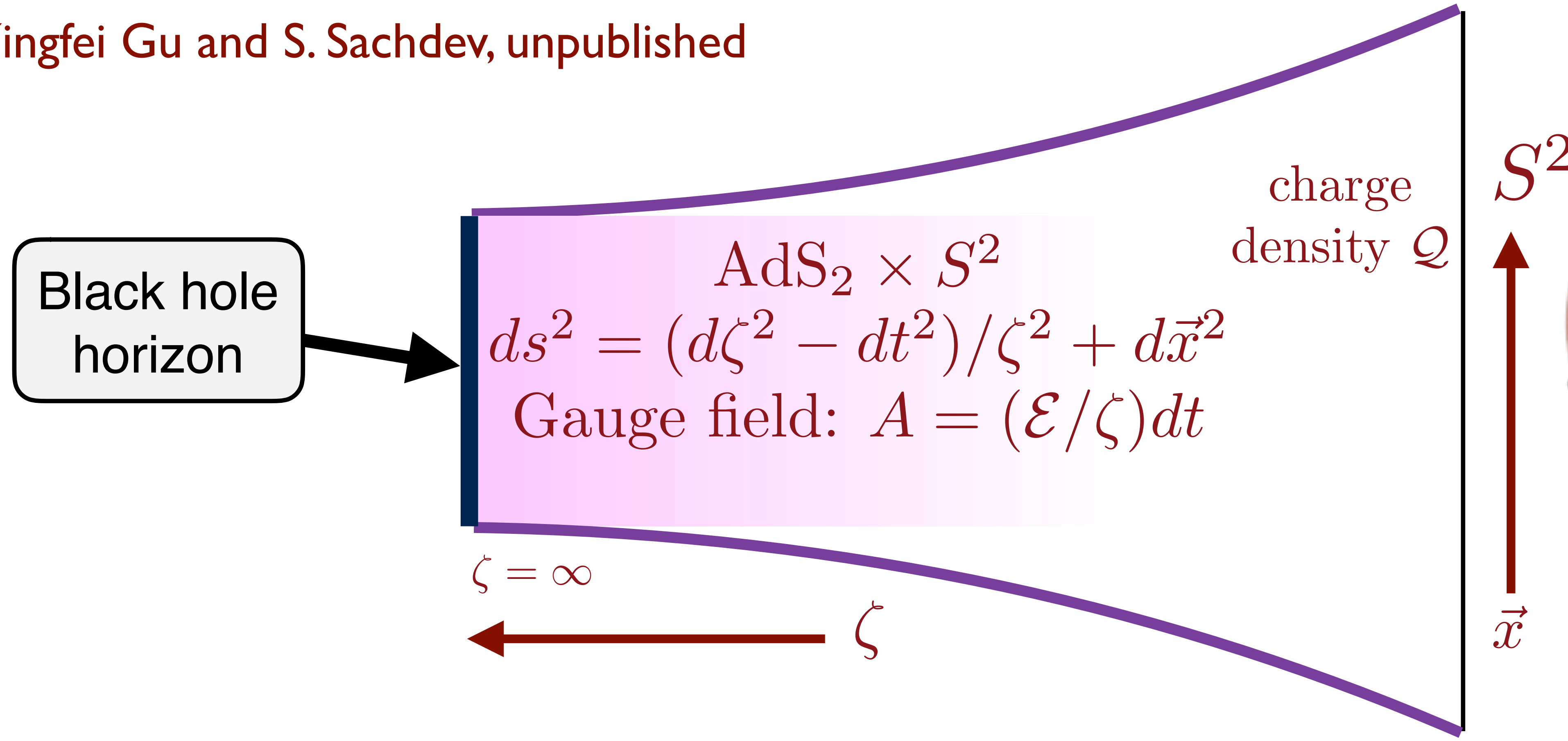


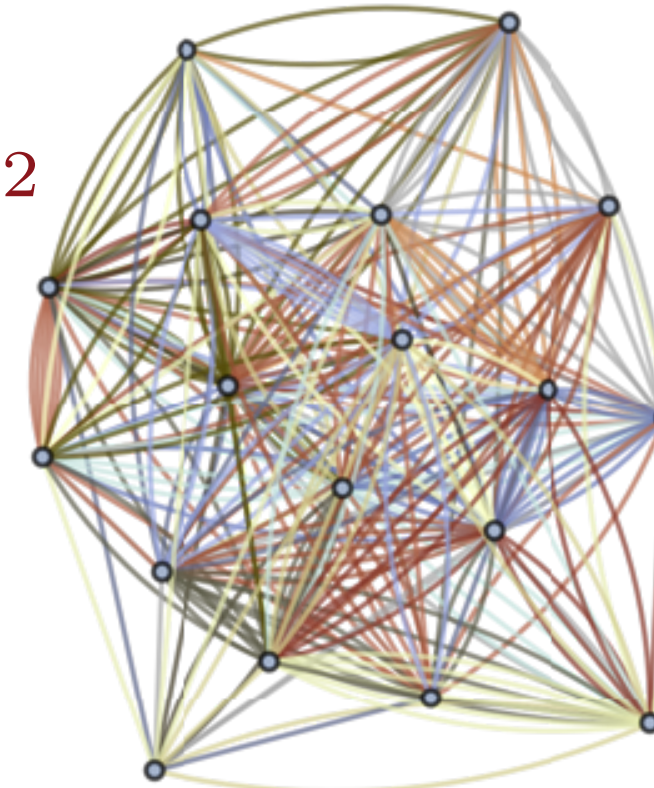
Bekenstein-Hawking entropy of AdS_2 horizon at $T = 0 \Leftrightarrow Ns_0$ entropy of SYK model

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010)

SYK models and black holes

Yingfei Gu and S. Sachdev, unpublished





The correspondence between the complex SYK model and extremal black holes holds also for the low T thermodynamics and low energy density of states. Both obey

$$\Omega(T) - E_0 = N\left[-s_0 T - \frac{1}{2}(\gamma + 4\pi^2\mathcal{E}^2 K)T^2 + \mathcal{O}(T^3)\right] + 2T\ln\left(\frac{U}{T}\right)\ldots$$

for the grand potential, and for the density of states at a fixed charge $\mathcal{Q}$

$$d(E) \sim \exp\left(N s_0\right)\sinh\left(\sqrt{2N\gamma(E-E_0)}\right) \quad , \quad E > E_0 \ , \ e^{-cN} \ll \gamma(E-E_0) \ll N$$

with the relation $\dfrac{ds_0}{d\mathcal{Q}} = 2\pi\mathcal{E}$ also obtained from Einstein's equations

A Sen, JHEP **0509**, 038 (2005)

SYK models and black holes

- Reparameterization invariance is a defining property of Einstein's theory of gravity

- In imaginary time, AdS_2 is the homogeneous hyperbolic space: two-dimensional surface of constant negative curvature. Its metric is invariant under SL(2,R)

$ds^2 = (d\tau^2 + d\zeta^2)/\zeta^2$ is invariant under

$$\tau' + i\zeta' = \frac{a(\tau + i\zeta) + b}{c(\tau + i\zeta) + d} \text{ with } ad - bc = 1.$$

A. Kitaev, 2015
J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857

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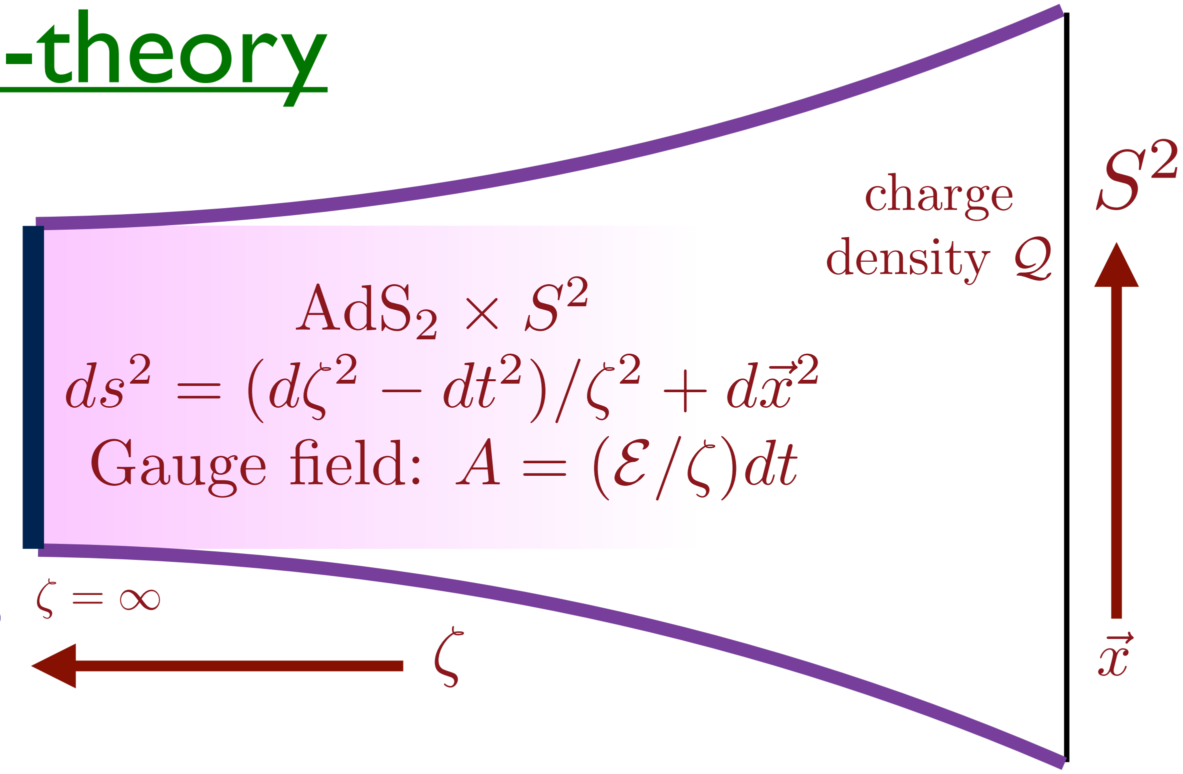
Their identical symmetries lead to the same low energy quantum theory for the SYK model and extremal charged black holes !

A. Kitaev, 2015
J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857

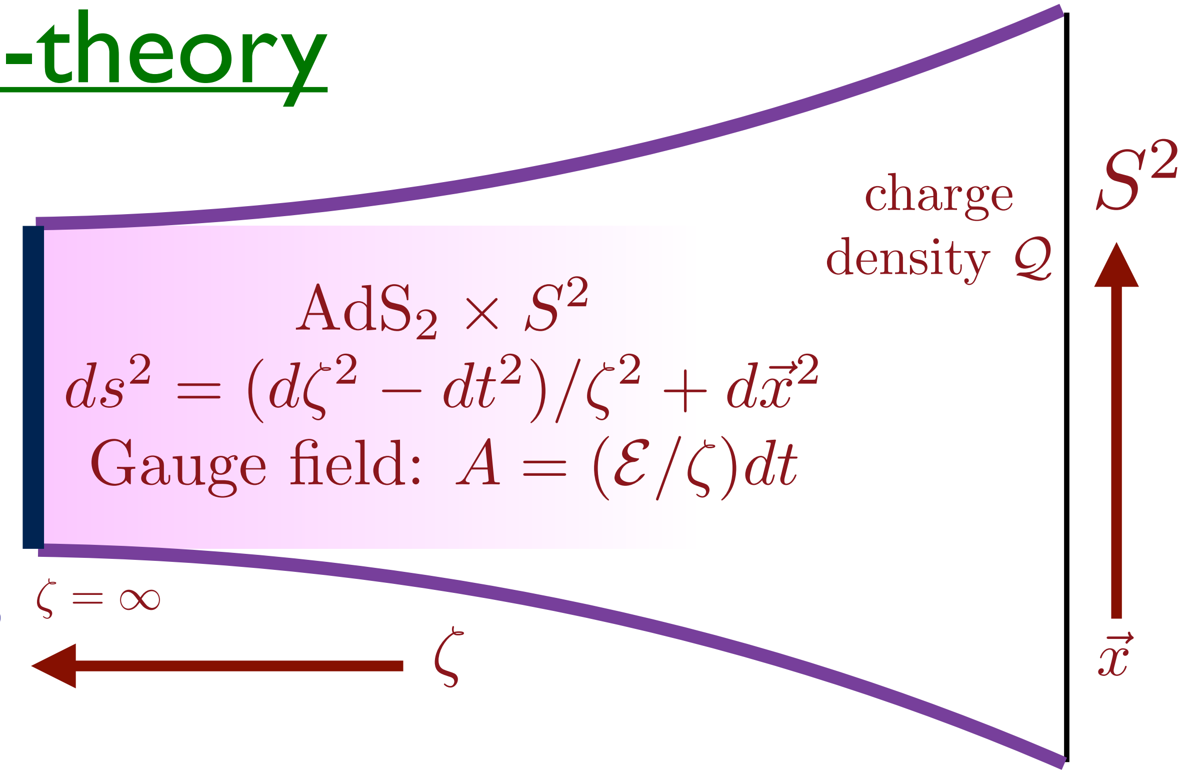
Einstein-Maxwell-theory

$$S_{4D} = \int d^4x \sqrt{-\hat{g}} \left(\hat{\mathcal{R}} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right),$$



- Has Reissner-Nördstrom-AdS charged black hole solution, with charge density $\mathcal{Q}$, a near-horizon $AdS_2 \times S^2$ geometry, and surface electric field $\mathcal{E}$. (This analysis also applies in asymptotically Minkowski spacetime ($L \to \infty$) provided the black hole mass is extremal.)

Einstein-Maxwell-theory



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- From Einstein's equations, the Bekenstein-Hawking black hole entropy S_{4D} is found to obey the same relation as the entropy of the SYK model

$$\boxed{\frac{\partial S_{4D}}{\partial \mathcal{Q}} = 2\pi \mathcal{E}\,,}$$

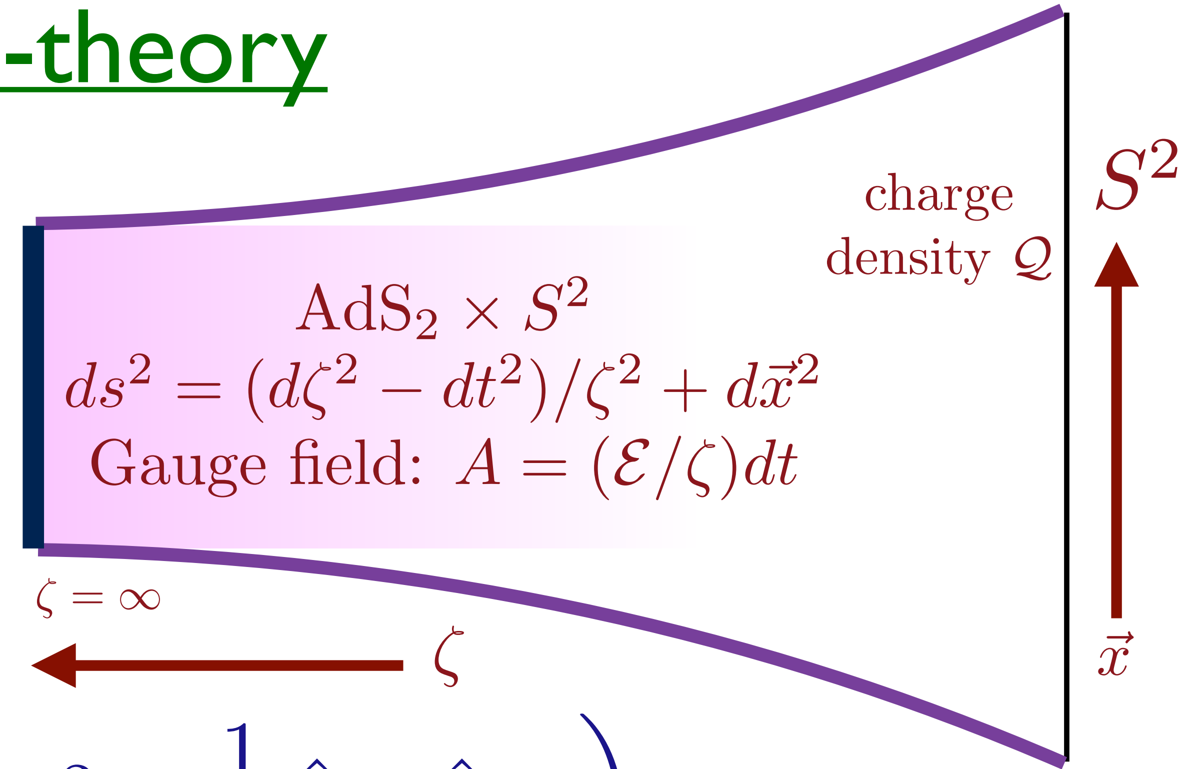
A Sen, JHEP **0509**, 038 (2005)

where $\mathcal{E}$ is identified from the spectral asymmetry of probe particle Green's functions in both cases. This establishes that the SYK entropy $N s_0$ maps onto (Area of horizon)/(4G)

S. Sachdev, PRX **5,** 041025 (2015)

Einstein-Maxwell-theory

P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal, arXiv:1802.09547;
A. Gaikwad, L.K. Joshi, G. Mandal, and S.R. Wadia, arXiv:1802.07746



$$S_{4D} = \int d^4x \sqrt{-\hat{g}} \left(\hat{\mathcal{R}} + 6/L^2 - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right),$$

In the small black hole size limit, $T \ll 1/R$, where R is the radius of the black hole, the theory dimensionally reduces to an Einstein-Maxwell-dilaton theory in two dimensions (the Jackiw-Teitelbaum model), along with Maxwell term

$$S_{2D} = N s_0 + \int d^2x \sqrt{-g} \left(\Phi(\mathcal{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab} F^{ab} \right).$$

The dilaton Φ represents the radial oscillations of the small black hole.

Einstein-Maxwell-theory

P. Nayak, A. Shukla, R.M. Soni, S.P. Trivedi, and V. Vishal, arXiv:1802.09547;
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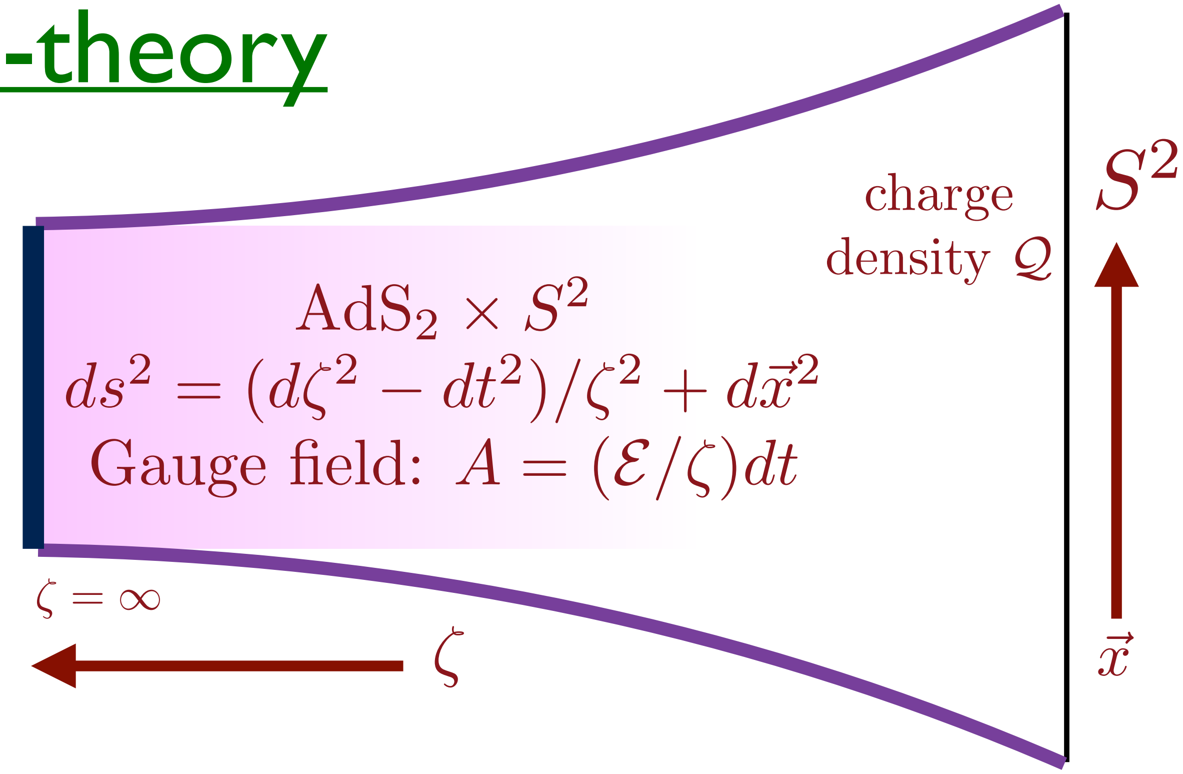

$$S_{2D} = N s_0 + \int d^2x\sqrt{-g}\left(\Phi(\mathcal{R} - \Lambda) - \frac{Z(\Phi)}{4} F_{ab}F^{ab}\right) .$$

There are no bulk quantum fluctuations of the metric in two-dimensional gravity, and there a further dimensional reduction to a $0+1$ dimensional theory representing fluctuations of the AdS_2 boundary: this $0+1$ dimensional turns out to be *precisely the Schwarzian theory obtained for the SYK model.*

$$\boxed{S_{\rm eff}[f,\phi] = \frac{K}{2}\int_0^{1/T} d\tau (\partial_\tau \phi + i(2\pi\mathcal{E}T)\partial_\tau f)^2 - \frac{\gamma}{4\pi^2}\int_0^{1/T} d\tau \left\{\tan(\pi T f(\tau)), \tau\right\},}$$

J. Maldacena, D. Stanford, and Zhenbin Yang, arXiv:1606.01857;
K. Jensen, arXiv:1605.06098;
J. Engelsoy, T.G. Mertens, and H. Verlinde, arXiv:1606.03438

Quantum matter without quasiparticles

- Planckian dynamics is realized in the 'solvable' SYK models
- Black holes thermalize in a time $\sim \hbar/(k_B T_H)$, where T_H is the Hawking temperature.
- A Schwarzian theory of a time reparameterization mode, with SL(2,R) symmetry, describes the quantum dynamics of
  - the SYK models
  - black holes with near-extremal AdS_2 horizons