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Action of hedgehog-instantons in the disordered phase of the 2+1 dimensional CP^{N-1} model

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ABSTRACT

The large- N limit of the 2+1 dimensional CP^{N-1} model exhibits hedgehog-like instanton saddle points in its disordered phase. We determine the structure of these saddle points and evaluate the action, S_{core} of a charge- q instanton. We find that $\lim_{\xi/a \rightarrow \infty} \lim_{N \rightarrow \infty} S_{core}/N = 2\rho_q \ln(\xi/a)$ where the order of limits is significant. Here a is the lattice spacing, ξ is the spin correlations length, and ρ_q are a set of universal constants: $\rho_1 = 0.062296\dots$, $\rho_2 = 0.155548\dots$. Free charge- q instantons therefore occur with a density $\sim a^{-3}(\xi/a)^{-2N\rho_q}$ in the disordered phase. Moreover the length scale, ξ_C , with which correlations of $U(1)$ gauge field, A_μ , decay exponentially, is $\xi_C \sim a(\xi/a)^{N\rho_1}$. The length ξ_C is also the scale at which the matter fields, z^α , experience a confining linear potential. Consequences for spin-Peierls ordering in two-dimensional quantum $SU(N)$ antiferromagnets will be discussed elsewhere, and are briefly noted for completeness.

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I. Introduction

Zero temperature properties of two dimensional quantum antiferromagnets have recently been the focus of intense theoretical and experimental interest [1, 2]. This is due in part to the discovery of high temperature superconductivity in doped antiferromagnets like $La_{2-x}Sr_xCuO_4$ and $YBa_2Cu_3O_{6+x}$ [3]. It is by now well known that the semiclassical limit of a two-dimensional $SU(2)$ antiferromagnet is described at long wavelengths by the relativistic, three-dimensional $O(3)$ non-linear sigma model [4, 5], with the spin-wave velocity playing the role of the velocity of light c . However, this mapping ignores Berry phase terms which accompany topologically non-trivial spin configurations [4, 6]. For the square lattice, the Berry phases are zero when the spin, S , is an even integer; the results of this paper can be directly applied to the antiferromagnets only for these special spin values. Upon considering $SU(N)$ antiferromagnets for arbitrary N , it is found that their semiclassical limit is described at long wavelengths by the CP^{N-1} model [7, 8, 9, 10]. (For $N = 2$, the CP^1 model is equivalent to the $O(3)$ non-linear sigma model [11, 12].) There is again an important restriction on the representation of $SU(N)$ which causes the Berry phase terms to vanish [7]. This paper shall examine properties of instantons in the disordered phase of this three dimensional CP^{N-1} model in the large N limit. We will also discuss applications of the results to the disordered phase of the $SU(N)$ antiferromagnet for the special representations with vanishing Berry phases. The application of the results of this paper to other representations is discussed in detail elsewhere [9]. We note that instanton effects in fermionic large- N theories of $SU(N)$ antiferromagnets have also been considered recently [13, 14]. Interest by one of us (S.S.) on the issues addressed in this paper is a direct consequence of questions that arose in a collaboration with N. Read [7, 8, 9].

The d -dimensional CP^{N-1} model is defined by the action

$$S_z = \frac{N}{2g} \int d^d x \left[|\partial_\mu z^\alpha|^2 - |z_\alpha^* \partial_\mu z^\alpha|^2 \right] \quad (1.1)$$

where the z^α are N -component complex fields which satisfy the fixed length constraint $\sum_{\alpha=1}^N |z^\alpha|^2 = 1$, x is the spacetime co-ordinate (time is measured in units of c), a is the lattice spacing of the underlying antiferromagnet, and g is a coupling constant which is inversely proportional to the spin-wave stiffness. A familiar renormalization group analysis in $d = 2 + \epsilon$ dimensions [16, 7] yields the following scaling equation for $\tilde{g} = g\Lambda^{d-2}$ when the ultra-violet cutoff Λ is changed to Λe^{-l}

$$\frac{d\tilde{g}}{dl} = -\epsilon\tilde{g} + K\tilde{g}^2 + \dots \quad (1.2)$$

where K is a phase space integral. Thus for small, positive ϵ and $\tilde{g} < g_c = \epsilon/K$, \tilde{g} flows to zero at long wavelengths indicating that the system is in the magnetically ordered phase. At $\tilde{g} = g_c$, there is a second-order phase transition leading to the destruction of magnetic order; for $\tilde{g} > g_c$, there is a runaway flow to strong coupling

The ordered phase of the $d = 3$ CP^{N-1} model displays topologically non-trivial ‘hedgehog’ instantons [6, 7]. These are tunnelling events which change the ‘skyrmion-number’ of an equal-time z^α configuration. A charge $q = 1$ instanton, centered at $x = 0$, has the following field configuration at distances much larger than the lattice spacing:

$$z^\alpha = U^\alpha \cos \frac{\theta}{2} e^{i\phi} + V^\alpha \sin \frac{\theta}{2}. \quad (1.3)$$

Here U, V are a pair of arbitrary N -component orthonormal vectors and the space-time co-ordinate x_μ has been written in terms of spherical co-ordinates (r, θ, ϕ) . For the case $N = 2$, we define the $O(3)$ field $n_\mu = z^* \sigma_\mu z$ where σ_μ are the three Pauli matrices and obtain the field configuration $n_\mu = x_\mu/x$ which points radially outward from the origin. Because of the

presence of a non-zero spin-stiffness, the hedgehogs are tightly-bound in pairs [6] and do not affect significantly the low-energy properties of the magnetically ordered phase.

In the disordered phase, the vanishing of the spin-stiffness liberates the instantons and they are expected to have many important consequences for the properties of the antiferromagnet [6, 7, 8, 9]. We postpone further discussions of these consequences until Section I.A. We shall first describe the new results obtained in this paper on the structure of the instantons in the large N limit. This may be obtained by expressing the partition function in the following form [11, 12]:

$$Z = \int \mathcal{D}\tilde{z}^\alpha \mathcal{D}A_\mu \mathcal{D}\lambda \exp(-S)$$

$$S = \frac{1}{g} \int d^3x \left[|(\partial_\mu - iA_\mu)\tilde{z}^\alpha|^2 + i\lambda(|\tilde{z}^\alpha|^2 - N) \right] \quad (1.4)$$

We have introduced the rescaled field $\tilde{z} = \sqrt{N}z$, the gauge-field A_μ arises from a Hubbard-Stratanovich decoupling of the quartic term, and the field λ which enforces the fixed length constraint. The action is quadratic in the z quanta, which can therefore be integrated out, yielding

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\lambda \exp(-NS_{eff})$$

$$S_{eff} = \text{Tr} \ln \left[-(\partial_\mu - iA_\mu)^2 + i\lambda \right] - i\frac{1}{g} \int d^3x \lambda \quad (1.5)$$

In this, and all subsequent functional expressions, we will assume that a Pauli-Villars regulator with mass Λ has been used to control the ultraviolet divergences; a lattice regularization, with lattice spacing $a \sim 1/\Lambda$ will be used in the numerical calculation. In the large- N limit, the functional integral is dominated by saddle points of S_{eff} . The saddle point with $A_\mu = 0$ has been examined in Refs. [11, 12]: the field $i\lambda$ acquires spacetime independent saddle point value $i\lambda = \Delta^2$ given by

$$\Delta = \Lambda - \frac{4\pi}{g}. \quad (1.6)$$

As noted above, Λ is the mass of the Pauli-Villars regulator. Two-point correlations of the z^α field decay exponentially with a correlation length ξ

$$\xi = \frac{1}{\Delta} \quad (1.7)$$

We see from Eqn (1.6) that for $g < g_c = 4\pi/\Lambda$, the saddle point with exponentially decaying correlations does not exist and the system undergoes a transition to the ordered phase. To understand the structure of the fluctuations in the disordered-phase we expand S_{eff} about the saddle point. At long distances ($\gg 1/\Delta$) this yields [11, 12]

$$S_{eff} = \frac{1}{4e^2} \int d^3r F_{\mu\nu}^2 \quad (1.8)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $e^2 = 48\pi\Delta$: this is simply the action of a massless $U(1)$ gauge field. The fluctuations of the λ field are massive and can be neglected.

It was pointed out to us by N. Read that the action of the instantons cannot be reliably estimated by the effective action in Eqn (1.8), and it is necessary to return to the full expression for S_{eff} in Eqn (1.5) and search for saddle points with a non-zero expectation value of $F_{\mu\nu}$. The field configuration with a charge- q instanton centered at $x = 0$ has $A_\mu = A_\mu^i$ with [17, 18, 8, 9]:

$$\epsilon_{\mu\nu\lambda} \frac{\partial A_\nu^i}{\partial x_\lambda} = \frac{q x_\mu}{2 x^3} \quad (1.9)$$

This is the vector potential of a Dirac monopole at $x = 0$ and is completely determined (upto gauge transformations) by the requirements of spherical symmetry and flux conservation. The monopole has total flux $2\pi q$ emerging from it. An important property of the saddle point is that field $i\lambda$ now becomes position dependent: we write $i\lambda = \Delta^2 + V(r)$ where spherical symmetry implies that V depends only upon the radial co-ordinate r . The function $V(r)$ is determined by the saddle-point equation

$$\frac{1}{i} \frac{\delta S_{eff}}{\delta \lambda} = \langle x | \frac{1}{-(\partial_\mu - iA_\mu^i)^2 + \Delta^2 + V(r)} | x \rangle - \frac{1}{g} = 0 \quad (1.10)$$

which imposes the constraint $\langle |z^\alpha|^2 \rangle = 1$ at every point in the presence of the monopole. The complete determination of $V(r)$ is an intractable problem; we will show in this paper that it is sufficient to determine the asymptotic values of $V(r)$ for large and small r . We find

$$V(r) \sim \begin{cases} -\frac{q^2}{48\Delta^2 r^4} & r \gg \frac{1}{\Delta} \\ -\frac{\alpha_q}{r^2} & \frac{1}{\Lambda} \ll r \ll \frac{1}{\Delta} \end{cases} \quad (1.11)$$

The universal set of constants, α_q , are determined implicitly by the following equation

$$\sum_{\ell=q/2}^{\infty} \left(\frac{2\ell+1}{\sqrt{(2\ell+1)^2 - q^2 - 4\alpha_q}} - 1 \right) = \frac{q}{2} \quad (1.12)$$

where $q > 0$ and the sum over ℓ extends over the values $q/2, q/2 + 1, q/2 + 2, \dots$. We determine numerically $\alpha_1 = 0.1998063111524528\dots$ and $\alpha_2 = 0.3978297544915455\dots$. The relationship $\alpha_q \approx q/5$ is found to hold to an accuracy of better than 1.4% for all $q < 100$.

We note that despite the attractive form of $V(r)$, the extra centrifugal repulsion provided by $A_\mu^i(r)$ prevents the existence of any negative energy or bound states. We will show that these asymptotic values of $V(r)$ are sufficient to determine the dominant part of the instanton action in the limit of large Λ/Δ . We identify the difference between the saddle-point actions with and without the instanton as the core-action S_{core} of the instanton. We have therefore from Eqn (1.4)

$$S_{core} = N \text{Tr} \ln \left[\frac{-(\partial_\mu - iA_\mu^i)^2 + \Delta^2 + V(r)}{-\partial_\mu^2 + \Delta^2} \right] - \frac{N}{g} \int d^3x V(r) \quad (1.13)$$

We separate the dual difficulties in Eqn (1.13) due to the presence of the non-zero vector potential and the space-dependent mass by performing the following decomposition

$$S_{core} = N(S_1 + S_2) \quad (1.14)$$

where

$$S_1 = \text{Tr} \ln \left[\frac{-(\partial_\mu - iA_\mu^i)^2 + \Delta^2}{-\partial_\mu^2 + \Delta^2} \right] \quad (1.15)$$

and

$$S_2 = \text{Tr} \ln \left[\frac{-(\partial_\mu - iA_\mu^i)^2 + \Delta^2 + V(r)}{-(\partial_\mu - iA_\mu^i)^2 + \Delta^2} \right] - \frac{1}{g} \int d^3x V(r) \quad (1.16)$$

We will begin in Section II with the evaluation of S_1 , the functional determinant of due to the vector potential A_μ^i in the presence of a space-independent mass Δ . The determinant will be evaluated by determining exactly all the eigenvalues of the requisite operators. The system-size dependent contributions will be regulated by compactifying spacetime onto the surface of a four-dimensional sphere: S^3 (Section II.A). The eigenvalues will be used to calculate S_1 for $q = 1$ in Section II.B and for general q in Section II.C. We find in the limit $\Delta \ll \Lambda$

$$S_1 = \Upsilon_q \ln \left(\frac{\Lambda}{\Delta} \right) + C + \mathcal{O} \left(\frac{\Delta}{\Lambda} \right) \quad (1.17)$$

where Υ_q are a set of *universal* numbers and C is a constant which depends upon the details of ultraviolet regularization procedure (for Pauli-Villars regularization $C = 0$). The numbers Υ_q are given by

$$\Upsilon_q = \frac{q^3}{24} + \frac{q}{12} - \Omega_q \quad (1.18)$$

with

$$\Omega_q = \frac{q^4}{4} \sum_{\ell=q/2}^{\infty} \frac{1}{\left(\sqrt{(2\ell+1)^2 - q^2} + 2\ell + 1 \right)^2} \quad (1.19)$$

We find $\Upsilon_1 = 0.09680740430261567\dots$, and $\Upsilon_2 = 0.2264273679853038\dots$. Finally in Section II.D we perform a numerical evaluation of S_1 by calculating the determinant on a lattice of size $30 \times 30 \times 60$. The result of the lattice regularization is in close agreement with Eqn (1.17) thus offering independent evidence of the universality of S_1 .

Next Section III will address the evaluation of S_2 , the change in S_{core} due to the presence of the space-dependent potential $V(r)$. We will begin in Section III.A by a determination of the asymptotic forms of $V(r)$ quoted above in Eqn (1.11). Because $V(r)$ falls off as $1/r^4$

at large r , the asymptotic form of the eigenfunctions of the two operators in S_2 differ only by the presence of a phase-shift $\eta_\ell(k)$ in the radial wave-function; here $\ell = q/2, q/2 + 1, q/2 + 2, \dots$ is a quantum number characterizing the angular dependence of the wavefunction and k is a radial ‘momentum’. We show in Section III.B that for an infinite system S_2 can be expressed solely in terms of the phase-shifts [15]:

$$S_2 = -\frac{2}{\pi} \sum_{\ell=q/2}^{\infty} (2\ell + 1) \int_0^{\infty} dk \eta_\ell(k) \left[\frac{k}{k^2 + \Delta^2} - \frac{k}{k^2 + \Lambda^2} \right] - \frac{1}{g} \int d^3x V(r) \quad (1.20)$$

where we have used a Pauli-Villars regulator to evaluate the integral over k . Note that the second term in Eqn (1.20) is clearly of order Λ/Δ ; we show in Section III.B that the first term in Eqn (1.20) contains a contribution which exactly cancels the leading Λ/Δ term in the spatial integral of $V(r)$. The phase-shifts are evaluated in Section III.C. We also show in Section III.C that *only the leading term* $V(r) \sim -\alpha_q/r^2$ in the asymptotic form of $V(r)$ at small r contributes a $\ln(\Lambda/\Delta)$ term to S_2 . This permits an *exact* evaluation of the coefficient. The final result for S_2 has the form

$$\lim_{\Lambda/\Delta \rightarrow \infty} S_2 = \Xi_q \ln \left(\frac{\Lambda}{\Delta} \right) \quad (1.21)$$

where the universal set of numbers Ξ_q are given by

$$\Xi_q = - \sum_{\ell=q/2}^{\infty} (2\ell + 1) \left(\nu_\ell - \nu'_\ell - \frac{\alpha_q}{2\nu'_\ell} \right) \quad (1.22)$$

with

$$\nu_\ell = \sqrt{\left(\ell + \frac{1}{2}\right)^2 - \frac{q^2}{4}} \quad ; \quad \nu'_\ell = \sqrt{\left(\ell + \frac{1}{2}\right)^2 - \frac{q^2}{4} - \alpha_q} \quad (1.23)$$

We find $\Xi_1 = 0.02778477609820585\dots$ and $\Xi_2 = 0.08466787712667854\dots$

Recent studies of instantons in fermionic large- N theories of $SU(N)$ antiferromagnets also noted the presence of logarithms [13, 14]. As these theories have massless fermionic excitations, the instanton action grew logarithmically with the size of the system. However

these calculations were low-order perturbations in the field strength, and it remains to be seen if the logarithm survives a full functional calculation of the type carried out in this paper.

I.A Physical Consequences

We now briefly explore the consequences of our results on S_{core} on the properties of the CP^{N-1} model and quantum antiferromagnets. We emphasize that all of the results were obtained in the limit $N \rightarrow \infty$, while the ratio of the spin-correlation length to the lattice spacing $\xi/a \sim \Lambda/\Delta$ was fixed at a finite, but large value. It is not clear to what extent the results continue to be valid in the opposite limit of fixed N but $\xi \rightarrow \infty$.

An immediate consequence of our results is the following estimate of the density n_q of free charge- q instantons in the disordered phase

$$\begin{aligned} n_q \sim \frac{1}{a^3} e^{-S_{core}} &= \frac{1}{a^3} e^{-N(S_1+S_2)} \\ &= \frac{1}{a^3} \left(\frac{\xi}{a} \right)^{2N\varrho_q} \end{aligned} \tag{1.24}$$

where we have introduced the exponents ϱ_q which are given in the limit of large N by

$$2\varrho_q = \Upsilon_q + \Xi_q \tag{1.25}$$

We have $\varrho_1 = 0.06229609020041076\dots$ and $\varrho_2 = 0.15554762255599117\dots$. Tightly bound instanton-anti-instanton pairs occur in both the magnetically ordered and disordered phases, and are not included in the above estimate. In obtaining this result we have glossed over potential complications in changing the measure of integration from $\mathcal{D}A_\mu$ to the co-ordinates of the instantons [17]: we have simply assumed that in a theory regularized with a lattice, distinct instantons must be at least a distance a apart. In any event, such effects will not change the value of the exponents to leading order in large N .

We now consider the consequences of this density of instantons on the CP^{N-1} model. Instantons are not expected to significantly alter correlation functions of the z^α quanta: these correlations decay exponentially with the length ξ . They however have significant consequences for the correlations of the ‘electromagnetic’ field $F_{\mu\nu}$. These are described at long distances by the action (1.8). We find it convenient to introduce the field H_μ

$$H_\mu = \frac{1}{2}\epsilon_{\mu\nu\sigma}F_{\nu\sigma} \quad (1.26)$$

Equation (1.8) now implies

$$\langle H_\mu(k)H_\nu(k) \rangle^{(0)} = e^2 \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad (1.27)$$

where the (0) indicates that instanton effects have not yet been included. The singularity at $k = 0$ leads to power-law decay of correlations in space-time. Instanton effects in 2+1 dimensional electrodynamics have been considered by Polyakov [17]. He found that they transformed the field correlations to

$$\langle H_\mu(k)H_\nu(k) \rangle = e^2 \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 + \xi_C^{-2}} \right) \quad (1.28)$$

Field correlations now decay exponentially with the length ξ_C . The central consequence of the new results in this paper is that in the limit $N \rightarrow \infty$

$$\frac{\xi_C}{a} \sim \left(\frac{\xi}{a} \right)^{N_{\varrho_1}} \quad (1.29)$$

The length ξ_C is also the scale at which oppositely charged z -quanta will experience an attractive confining linear potential [17].

A crucial ingredient in Polyakov’s analysis [17] was the use of a $1/r$ interaction between instantons. In the $CP^{(N-1)}$ model the interaction has this form only at distances larger than ξ . The mean spacing between the instantons is of order $n_1^{-1/3}$, which for sufficiently large

N , is much larger than ξ . This justifies the use of Polyakov's analysis in the context of the $CP^{(N-1)}$ model.

Finally we briefly address the application of the results to two-dimensional quantum $SU(N)$ antiferromagnets. This issue is addressed in detail in Refs [8, 9], and for completeness we restate the results here. It has been argued in these papers that the Berry phases which accompany the instantons lead to the appearance of spin-Peierls order for certain representations of $SU(N)$. For the case of $SU(2)$ antiferromagnets on a square lattice, these representations correspond to all spins S which satisfy $2S \pmod{4} \neq 0$. The spin-Peierls order parameter Ψ is found to have the following magnitude

$$\langle \Psi \rangle \sim \begin{cases} \xi_C^{-2} & \text{for } S = 1, 3, 5, 7, 9 \dots \\ \xi_C^{-4} & \text{for } S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \dots \end{cases} \quad (1.30)$$

Use of Eqn (1.29) implies that $\langle \Psi \rangle$ is suppressed by a power of the spin-correlation length.

II. Calculation of S_1

This section will present the details of the analytical calculation of S_1 , the functional determinant defined in Eqn (1.15) with the space-independent mass Δ^2 . An important issue that arises at the outset is the question of regularization of infrared and ultraviolet divergences. Physically the ultraviolet fluctuations are controlled by the presence of an underlying lattice. While this is the procedure used in the numerical evaluation of Section II.D, we use here the gauge-invariant Pauli-Villars regularization. The Pauli-Villars mass Λ will then be of order the inverse lattice spacing $1/a$. The infrared divergences may be controlled by evaluating the functional determinants in a finite system, taking their ratio and then letting the system size go to infinity. A possibility that immediately suggests itself is to use a spherical cavity about the instanton-center of radius R and demand that the eigenfunctions vanish on the surface of the sphere. However this procedure leads to eigenvalues depending upon the zeros of Bessel

functions of irrational order and we have been unable to evaluate the subsequent sum over the eigenvalues. The method we shall use is to place an instanton anti-instanton pair on opposite poles of S^3 , the surface of a four-dimensional sphere of radius R . The eigenvalue equation

$$-\left(\vec{\nabla} - i\vec{A}^i\right)^2 \Psi_\sigma(u) = \epsilon_\sigma^q(R)\Psi_\sigma(u) \quad (2.1)$$

will be solved exactly. Here $\vec{\nabla}$ is the gradient operator on S^3 , \vec{A}^i is the vector-potential of an instanton anti-instanton pair of charges $\pm q$ located on the north and south poles, u is a co-ordinate on S^3 and $\epsilon_\sigma^q(R)$ is the eigenvalue on a sphere of radius R . Thus the eigenvalues in the absence of instantons are represented by ϵ_σ^0 . In both cases the index σ is used to order the eigenvalues in ascending order. We introduce the quantity

$$D_m(N_{max}, R) = \sum_{\sigma=1}^{N_{max}} \ln \left[\frac{\epsilon_\sigma^q(R) + m^2}{\epsilon_\sigma^0(R) + m^2} \right] \quad (2.2)$$

which calculates the action over the N_{max} lowest eigenvalues from both sets. The action S_1 is then clearly given by

$$S_1 = \frac{1}{2} \lim_{R \rightarrow \infty} \lim_{N_{max} \rightarrow \infty} \left[D_\Delta(N_{max}, R) - D_\Lambda(N_{max}, R) \right] \quad (2.3)$$

where, as noted before, the Pauli-Villars mass Λ is of order $1/a$. The order of the limits two limits above cannot be interchanged.

II.A Eigenvalues on S^3

We will use angular co-ordinates (ψ, θ, ϕ) with $0 \leq \psi, \theta \leq \pi$ and $0 \leq \phi < 2\pi$ to represent points on the surface of a sphere. The angles θ, ϕ are analogous to those used in spherical co-ordinates on R^3 , while ψ measures geodesic distance from the south pole. If we embed S^3 in R^4 , the four Cartesian co-ordinates will be

$$R(\cos \psi, \sin \psi \cos \theta, \sin \psi \sin \theta \cos \phi, \sin \psi \sin \theta \sin \phi) \quad (2.4)$$

and the metric on S^3 is given by

$$ds^2 = R^2 d\psi^2 + R^2 \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2) \quad (2.5)$$

We place an instanton of charge q ($q > 0$) at $\psi = 0$ (south pole) and of charge $-q$ at $\psi = \pi$ (north pole). A convenient choice for the vector potential associated with this configuration is

$$A_\phi^i = \frac{q}{2R \sin \psi} \frac{1 - \cos \theta}{\sin \theta} \quad (2.6)$$

with $A_\theta^i = A_\psi^i = 0$. We have then

$$\begin{aligned} & (\vec{\nabla} - i\vec{A}^i)^2 = \\ & \frac{1}{R^2 \sin^2 \psi} \frac{\partial}{\partial \psi} \left(\sin^2 \psi \frac{\partial}{\partial \psi} \right) + \frac{1}{R^2 \sin^2 \psi} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left(\frac{\partial}{\partial \phi} - i \frac{q}{2} (1 - \cos \theta) \right)^2 \right\} \end{aligned} \quad (2.7)$$

The operator in the curly brackets is precisely the operator that was diagonalized by Wu and Yang [19] in their paper on the wavefunctions of an electron in the presence of a Dirac monopole. Its eigenfunctions are the ‘monopole harmonics’ $Y_{q/2, \ell, m}$ and the eigenvalues are $\ell(\ell + 1) - (q/2)^2$ where $\ell = q/2, q/2 + 1, q/2 + 2, \dots$ and the degeneracy of each eigenvalue is $2\ell + 1$. These monopole harmonics also appeared in Marston’s [14] analysis of instanton effects in fermionic large- N theories of $SU(N)$ antiferromagnets. The problem of diagonalizing the operator in Eqn 2.7 is therefore reduced to that of diagonalizing

$$\frac{1}{R^2 \sin^2 \psi} \frac{\partial}{\partial \psi} \left(\sin^2 \psi \frac{\partial}{\partial \psi} \right) + \frac{1}{R^2 \sin^2 \psi} \left\{ \ell(\ell + 1) - \frac{q^2}{4} \right\} \quad (2.8)$$

If we change variables to $\cos \psi = y$, the equation for the eigenvalue ϵ becomes

$$\left[\frac{d^2}{dy^2} + \frac{3y}{y^2 - 1} \frac{d}{dy} - \frac{\ell(\ell + 1) - q^2/4}{(y^2 - 1)^2} - \frac{R^2 \epsilon}{y^2 - 1} \right] w(y) = 0 \quad (2.9)$$

where w is any function which is integrable at the poles ($y = \pm 1$). This differential equation is of the hypergeometric type and can be readily solved using standard techniques [20]. The

condition that w be integrable simultaneously at $y = \pm 1$ quantizes the allowed values of ϵ

$$\epsilon_{m,\ell} = \frac{1}{R^2} \left(m + \frac{\sqrt{(2\ell+1)^2 - q^2} - 1}{2} \right) \left(m + \frac{\sqrt{(2\ell+1)^2 - q^2} + 3}{2} \right) \quad (2.10)$$

where $m = 0, 1, 2, 3, \dots$ is a non-negative integer. We introduce the integer $n = \ell + m$, and obtain the result for the eigenvalues $\epsilon_{n,\ell}$ in its final form

$$\epsilon_{n,\ell}(R) = \frac{1}{R^2} (n - \gamma_q(\ell)) (n - \gamma_q(\ell) + 2) \quad (2.11)$$

where

$$\gamma_q(\ell) = \frac{q^2/2}{\sqrt{(2\ell+1)^2 - q^2} + 2\ell + 1}, \quad (2.12)$$

$n = q/2, q/2 + 1, q/2 + 2, \dots, \infty$, ℓ is restricted to the values $\ell = q/2, q/2 + 1, \dots, n$ and the degeneracy of each state is $2\ell + 1$.

We may obtain the eigenvalues in the absence of the instantons by putting $q = 0$ in the expressions above. The eigenvalues are *independent* of ℓ and take the values

$$\epsilon_n^0 = \frac{1}{R^2} n(n+2) \quad (2.13)$$

and the degeneracy of the n 'th state is $\sum_{\ell=0}^n (2\ell + 1) = (n+1)^2$.

II.B S_1 for $q = 1$

In this section we will use the eigenvalues computed in Section II.A to explicitly compute the core-action S_1 for the special case $q = 1$. The generalization to arbitrary values of q will be presented in Section II.C.

We show in Fig. 1, a schematic of the eigenvalues both with and without the instantons. In the presence of the instantons, eigenvalues with the same value of n but differing values of ℓ cluster together into separate groups. Each cluster is shown as a shaded block, along with the total number of states in the cluster. In the absence of the instanton, the eigenvalues

are independent of ℓ and are therefore shown as a single line. In the evaluation of D_m , the eigenvalues have to be placed in a one-to-one correspondence in the argument of the logarithm: we use the pairing shown in Fig. 1 which splits the states in cluster with quantum number n equally between the states with $n - 1/2$ and $n + 1/2$. Other pairings can also be used and the final answer is independent of the specific choice.

An important issue which arises now is the placement of the ultraviolet cutoff N_{max} . The value of $D_m(N_{max}, R)$ will clearly be dependent on this choice, but the arbitrariness should disappear after the Pauli-Villars subtraction. We will choose N_{max} such that the all states in the presence of the instanton in clusters with quantum number n ranging from $1/2$ to $N_{max} - 1/2$ are included. (There is a slight abuse of notation here: we are now using N_{max} to denote the total number of clusters and not the total number of states.) We see from Fig. 1 that a little less than *half* of the degenerate states with $n = N_{max}$ without the instanton will be included. Later in this section we will state the result of a calculation which has the upper cutoff chosen to include *all* degenerate states in with every value of $n \leq N_{max}$ without the instantons: in this case only *half* of a cluster of states with the instanton with $n = N_{max} + 1/2$ will be included. The change in the cutoff will change the value of D_m but will leave S_1 unchanged.

In the flat space limit ($R \rightarrow \infty$), it can be shown [21] that the cutoff procedures discussed above are equivalent to choosing a hard-cutoff in momentum space with wavevector $\Lambda = N_{max}/R$

Using the first of the cutoff procedures outlined above we obtain

$$D_m(N_{max}, R) = \sum_{n=1/2}^{N_{max}-1/2} \sum_{\ell=1/2}^n \left(\ell + \frac{1}{2} \right) \left[\ln \left(\frac{\epsilon_{n,\ell}^q(R) + m^2}{\epsilon_{n-1/2}^0(R) + m^2} \right) + \ln \left(\frac{\epsilon_{n,\ell}^q(R) + m^2}{\epsilon_{n+1/2}^0(R) + m^2} \right) \right] \quad (2.14)$$

We now insert the expressions for $\epsilon_{n,\ell}$ and ϵ_n^0 obtained in Section II.A and perform an

expansion in inverse powers of R . This is equivalent to expanding D_m in inverse powers of $m^2 + (n+1)^2/R^2$. Performing this on the first term of D_m we obtain

$$\begin{aligned} & \sum_{n=1/2}^{N_{max}-1/2} \sum_{\ell=1/2}^n \left(\ell + \frac{1}{2}\right) \ln \left(\frac{\epsilon_{n,\ell}^q(R) + m^2}{\epsilon_{n-1/2}^0(R) + m^2} \right) = \\ & \sum_{n=1/2}^{N_{max}-1/2} \sum_{\ell=1/2}^n \left(\ell + \frac{1}{2}\right) \left[-\frac{1}{R^2} \frac{2(n+1)\gamma_1(\ell) - n - 3/4}{(n+1)^2/R^2 + m^2} + \frac{1}{2R^4} \frac{(n+7/4)^2}{((n+1)^2/R^2 + m^2)^2} \right] + \mathcal{O}\left(\frac{1}{R}\right) \end{aligned} \quad (2.15)$$

All omitted terms in the expression can be shown to be of order $1/R$ or smaller.

We examine first the sum over ℓ . Expressions of the form $\sum_{\ell=q/2}^n (2\ell+1)\gamma_q(\ell)$ are in Eqn (2.15) (with $q=1$) and in many of the subsequent analyses. From the definition of $\gamma_q(\ell)$ in Eqn (2.12) we obtain

$$\begin{aligned} \sum_{\ell=q/2}^n (2\ell+1)\gamma_q(\ell) &= \frac{q^2}{4} \sum_{\ell=q/2}^n \frac{4\ell+2}{\sqrt{(2\ell+1)^2 - q^2} + 2\ell+1} \\ &= \frac{q^2}{4} \sum_{\ell=q/2}^n \left(1 + \frac{q^2}{\left(\sqrt{(2\ell+1)^2 - q^2} + 2\ell+1\right)^2} \right) \\ &= \frac{q^2}{4} \left(n - \frac{q}{2} + 1 \right) + \Omega_q + \mathcal{O}\left(\frac{1}{n}\right) \end{aligned} \quad (2.16)$$

where we have introduced the set of irrational numbers Ω_q defined by

$$\Omega_q \equiv \sum_{\ell=q/2}^{\infty} \frac{q^4}{4} \frac{1}{\left(\sqrt{(2\ell+1)^2 - q^2} + 2\ell+1\right)^2} \quad (2.17)$$

We have $\Omega_1 = 0.02819259569738433\dots$ and $\Omega_2 = 0.2735726320146962\dots$. The $\mathcal{O}(1/n)$ in Eqn (2.16) can be shown to contribute to D_m only at order $1/R$.

We now perform the remaining sums over ℓ in Eqn (2.15) and the second term in Eqn (2.14), collect terms, and obtain the following expression for D_m

$$D_m(N_{max}, R) = \sum_{n=1/2}^{N_{max}-1/2} \left(-\frac{1}{4R^2} \frac{2(n+1)(n+4\Omega_1+1/2) + ((n+1)^2 - 1/4)}{(n+1)^2/R^2 + m^2} \right)$$

$$+ \frac{1}{4R^4} \frac{(n + 7/4)^2 + (n + 1/4)^2}{((n + 1)^2/R^2 + m^2)^2} \left[(n + 1)^2 - 1/4 \right] + \mathcal{O}\left(\frac{1}{R}\right) \quad (2.18)$$

Finally, we use the Euler-Maclaurian formula to evaluate the summation in the limit of large N_{max} .

$$\begin{aligned} D_m(N_{max}, R) = & -\frac{3R}{4} \int_0^{N_{max}/R} dx \frac{x^2}{x^2 + m^2} + \left(\frac{1}{4} - 2\Omega_1\right) \int_0^{N_{max}/R} dx \frac{x}{x^2 + m^2} \\ & + \frac{R}{2} \int_0^{N_{max}/R} dx \frac{x^4}{(x^2 + m^2)^2} + C + \mathcal{O}\left(\frac{1}{R}\right) + \mathcal{O}\left(\frac{mR}{N_{max}}\right) \end{aligned} \quad (2.19)$$

Here C is an uninteresting constant independent of m , R , and N_{max} which will not appear in the final expression for S_1 . Performing the integrals we obtain

$$D_m(N_{max}, R) = -\frac{N_{max}}{4} + \left(\frac{1}{4} - 2\Omega_1\right) \ln\left(\frac{N_{max}}{Rm}\right) + C + \mathcal{O}\left(\frac{1}{R}\right) + \mathcal{O}\left(\frac{mR}{N_{max}}\right) \quad (2.20)$$

We note that terms of the form mR appear at intermediate stages of the calculation. However they cancel among themselves in the final result. A term of this form would have led to a R dependence in the final answer for S_1 .

We use our result for D_m in the expression (2.3) for the core action to obtain

$$S_1 = \left(\frac{1}{8} - \Omega_1\right) \ln\left(\frac{\Lambda}{\Delta}\right) \quad (2.21)$$

This is the central result of this section. Notice that the constant C has disappeared. There are also no correction of order Δ/M , but this is clearly an artifact of the Pauli-Villars regularization. If we had used the second procedure for the placement of the ultraviolet cutoff N_{max} discussed earlier in this section we would have obtained

$$D_m(N_{max}, R) = +\frac{N_{max}}{4} + \left(\frac{1}{4} - 2\Omega_1\right) \ln\left(\frac{N_{max}}{Rm}\right) + C' + \mathcal{O}\left(\frac{1}{R}\right) + \mathcal{O}\left(\frac{mR}{N_{max}}\right) \quad (2.22)$$

From Eqn (2.3) it is clear that the expression for S_1 would have been unchanged.

We can also conclude that the co-efficient of the $\ln(\Lambda/\Delta)$ term in *universal* and independent of the Pauli-Villars regularization. The logarithm appeared from long-wavelength

log-divergence of the integral over the normal modes n . A different regularization procedure would then renormalize the value of Λ but will leave the co-efficient unchanged.

II.C S_1 for Arbitrary q

The method of section II.B can be extended to obtain S_1 for arbitrary q . However, as was apparent from that section, careful account had to be taken of the ultraviolet divergences at intermediate stages of the calculation. Having shown that a well-defined answer emerges at the end of calculation for $q = 1$, we choose now to work with expressions that are finite in the ultraviolet by taking derivatives with respect to m^2 . Consider:

$$\begin{aligned} -\frac{\partial^2}{\partial(m^2)^2} D_m(N_{\max}, R) &= \sum_{\sigma=1}^{N_{\max}} \left\{ \frac{1}{(\epsilon_{\sigma}^q(R) + m^2)^2} - \frac{1}{(\epsilon_{\sigma}^0(R) + m^2)^2} \right\} \\ &\equiv A(q, R) - A(0, R) \end{aligned} \quad (2.23)$$

where the second equation defines the quantity $A(q, R)$. The bulk of this section will concentrate upon an evaluation of $A(q, R)$ in inverse powers of R , upto order $1/R$. The two m^2 derivatives ensure that the sum converges in the ultraviolet. Thus, we can let N_{\max} go to ∞ in the expression for $A(q, R)$. Thus

$$\begin{aligned} A(q, R) &= \sum_{\sigma=1}^{\infty} \frac{1}{(\epsilon_{\sigma}^q(R) + m^2)^2} \\ &= \sum_{n=\frac{q}{2}}^{\infty} \sum_{l=\frac{q}{2}}^n (2l+1) \frac{1}{\left[m^2 + (n - \gamma_q(l))(n - \gamma_q(l) + 2)/R^2 \right]^2} \end{aligned} \quad (2.24)$$

We decompose the denominator as:

$$\begin{aligned} m^2 + \frac{(n - \gamma_q(l))(n - \gamma_q(l) + 2)}{R^2} &= \mu + \frac{1}{R^2} [-1 - 2(n+1)\gamma_q(l) + \gamma_q(l)^2] \\ \mu &\equiv m^2 + \frac{(n+1)^2}{R^2} \end{aligned} \quad (2.25)$$

where we have introduced the label μ for brevity of notation. It is clear that for all values of n , and R large, μ will be much larger than the remaining terms in Eqn (2.24). We therefore

expand in inverse powers of μ and keep all terms which will be shown later to contribute to $A(q, R)$ upto order $1/R$. This yields

$$\begin{aligned}
A(q, R) &= \sum_{n=q/2}^{\infty} \sum_{l=q/2}^n \frac{(2l+1)}{\mu^2} \left\{ 1 + \frac{2}{R^2} \frac{(1+2(n+1)\gamma_q(l) - \gamma_q(l)^2)}{\mu} \right. \\
&\quad \left. + \frac{3}{R^4} \frac{(1+4(n+1)\gamma_q + 4(n+1)^2\gamma_q^2(l))}{\mu^2} + \dots \right\} \quad (2.26)
\end{aligned}$$

We first do the sum over l , using the following identities:

$$\begin{aligned}
\sum_{l=q/2}^n (2l+1) &= (n+1)^2 - (q/2)^2 \\
\sum_{l=q/2}^n (2l+1)\gamma_q(l) &= \left(\frac{q}{2}\right)^2 \left(n+1 - \frac{q}{2}\right) + \Omega_q - \frac{(q/2)^4}{2(2n+3)} - \dots \\
\sum_{l=q/2}^n (2l+1)\gamma_q(l)^2 &= \frac{(q/2)^4}{2} \ln(n+1) + \Omega'_q + \dots \quad (2.27)
\end{aligned}$$

where the ellipses indicate terms higher order in $1/n$ which will modify $A(q, R)$ at order $1/R^2$. The set of q -dependent numbers Ω_q was introduced earlier in Eqn (2.17) and the new set Ω'_q are defined by

$$\Omega'_q = \frac{q^4}{4} \sum_{l=q/2}^{\infty} (2l+1) \left(\frac{1}{(\sqrt{(2l+1)^2 - q^2} + 2l+1)^2} - \frac{1}{4(2l+1)^2} \right) \quad (2.28)$$

Inserting these expressions into Eqn (2.26) we get

$$\begin{aligned}
A(q, R) &= \sum_{n=q/2}^{\infty} \frac{1}{\mu^2} \left\{ (n+1)^2 - \frac{q^2}{4} \right\} \\
&\quad + \sum_{n=q/2}^{\infty} \frac{2}{R^2 \mu^3} \left\{ (n+1)^2 - \frac{q^2}{4} + 2(n+1) \left[\frac{q^2}{4} \left(n+1 - \frac{q}{2} \right) + \Omega_q - \frac{q^4}{32(2n+3)} \right] \right. \\
&\quad \quad \quad \left. - \frac{q^4}{32} \ln(n+1) - \Omega'_q \right\} \\
&\quad + \sum_{n=q/2}^{\infty} \frac{3}{R^4 \mu^4} \left\{ (n+1)^2 - \frac{q^2}{4} + q^2(n+1) \left(n+1 - \frac{q}{2} \right) \right. \\
&\quad \quad \quad \left. + 4(n+1)^2 \left(\frac{q^4}{32} \ln(n+1) + \Omega'_q \right) \right\} \quad (2.29)
\end{aligned}$$

Once again we use the Euler-Maclaurin formula to convert the sums into integrals. As the manipulations are tedious and not very illuminating, we merely present the final result.

$$A(q, R) = \frac{R^3\pi}{4m} + \frac{R\pi}{8m^3} + \frac{1}{m^4} \left\{ -\frac{q^3}{24} - \frac{q}{12} + \Omega_q \right\} + \frac{3\pi}{32Rm^5} + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (2.30)$$

Several cancellations occur at order $1/R$: in particular all terms dependent upon Ω'_q have dropped out. Notice also that the first two terms, which blow up as $R \rightarrow \infty$, are q -independent. This will cause them to drop out of the final expression for S_1 . The q -independence of the $1/R$ term is more unusual: we will comment on this later in this section. Inserting our result for $A(q, R)$ into Eqn (2.23) we obtain therefore

$$-\frac{\partial^2}{\partial m^2} D_m(R) = \frac{1}{m^4} \left\{ -\frac{q^3}{24} - \frac{q}{12} + \Omega_q \right\} + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (2.31)$$

Integrating twice with respect to m^2 , we get

$$D_m(R) = 2 \left\{ \frac{q^3}{24} + \frac{q}{12} - \Omega_q \right\} \ln \frac{C_1}{mR} + C_2 m^2 R^2 + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (2.32)$$

where C_1 and C_2 are dimensionless constants of integration, and the additional factors of R are determined from purely dimensional considerations. Comparing with the results of Section II.B for $q = 1$ (and equivalent calculations for arbitrary q) we may conclude that $C_2 = 0$ in the limit $N_{max} \rightarrow \infty$. Finally, we insert this result into the expression (2.3) for S_1 and obtain

$$S_1 = \left\{ \frac{q^3}{24} + \frac{q}{12} - \Omega_q \right\} \ln \frac{\Lambda}{\Delta} + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (2.33)$$

which is precisely the result quoted in Eqn (1.17) of the introduction.

A property of this expression for S_1 which appears unusual at first sight is the absence of a $1/R$ term. In flat-space the long-distance action in Eqn (1.8) leads to a $1/R$ interaction between instanton charges. On S^3 , the long-distance action (1.8) would be generalized to

$$S_{eff} = \frac{R^3}{4e^2} \int \sin^2 \psi \sin \theta d\psi d\theta d\phi F_{\mu\nu}^2 \quad (2.34)$$

We now insert (2.6), the expression for the vector-potential for an instanton-anti-instanton pair on the north and south poles, into the expression above. The spatial integration is cut-off at a geodesic distance $1/\Lambda'$ from the poles. The long-distance estimate for the action is then

$$S_{eff} = 4\pi q^2 \frac{\Lambda'}{m} - \frac{4\pi q^3}{3\Lambda' m} \frac{1}{R^2} + \dots \quad (2.35)$$

Notice the absence of a $1/R$ term. This result is therefore consistent with the full calculation which yielded Eqn (2.33). Of course the long-distance action cannot accurately reproduce the R -independent term arising from physics at distances shorter than $1/\Delta$.

II.D Numerical Results

This section will present independent numerical verification of the results obtained in Section II. The calculations will be performed in flat-space using a lattice regularization and thus provide a strong verification of the form of S_1 in Eqn. (1.17) and the universality of Υ_1 .

We use a cubic lattice of $2L \times 2L \times 4L$ points with lattice-spacing a and free boundary conditions. The continuum operator $-(\partial_\mu - iA_\mu^i)^2 + \Delta^2$ is now regularized to the matrix M_{ij} of dimension $16L^3$ with site labels i, j . The matrix M_{ij} is defined by

$$\sum_{i,j} z_i^* M_{ij} z_j \equiv \sum_{\langle kl \rangle} \left| z_k - \exp \left(i \int_{\mathbf{r}_k}^{\mathbf{r}_l} \mathbf{A}^i(\mathbf{r}) \cdot d\mathbf{r} \right) z_l \right|^2 + \Delta^2 a^2 \sum_k |z_k|^2 \quad (2.36)$$

where z_i is an arbitrary complex number associated with the site i , the sum over i, j on the left-hand-side is a free sum over all sites on the lattice, and the sum over k, l on the right-hand-side is over nearest-neighbor pairs of sites. The vector-potential \mathbf{A}^i is determined in the continuum and it is important that the line-integral be evaluated exactly; approximation of the line-integral will spoil the invisibility of the Dirac string. If we define the matrix M^0 as above but with $\mathbf{A}^i = 0$ then clearly

$$S_1 = \frac{1}{2} \lim_{L \rightarrow \infty} \ln \frac{\det M}{\det M^0} \quad (2.37)$$

Before describing the evaluation of determinants, we will specify the continuum gauge potential $\mathbf{A}^i(\mathbf{r})$. We place the lattice in box, B , defined by the planes $x = La$, $x = -La$, $y = La$, $y = -La$, $z = -2La$ and $z = 2La$. A instanton of charge $q = 1$ was placed at $(x = 0, y = 0, z = -La)$ and of charge $q = -1$ at $(0, 0, La)$. An infinite set of ‘image’ charges were placed outside the box, B , so that the ‘magnetic field’ $\mathbf{B} = \nabla \times \mathbf{A}^i$ has a vanishing *normal* component on the surface of B . As a result all the field lines emerging from the positive charge at $(0, 0, -La)$ end at the negative charge at $(0, 0, La)$ without leaving the box B - see Fig. 2. A two-dimensional section of the image charges and the box B is shown in Fig. 2. The field \mathbf{B} was determined by an Ewald sum [22] over the charges

$$\mathbf{B}(\mathbf{r}) = \sum_i' q_i \frac{\mathbf{r} - \mathbf{R}_i}{|\mathbf{r} - \mathbf{R}_i|^3} \quad (2.38)$$

where the sum extends over charges q_i at the points \mathbf{R}_i and the prime indicates that Ewald’s method was used to obtain a convergent result. Finally, the vector potential \mathbf{A}^i is determined in the gauge $A_z^i = 0$ by the integrals

$$A_x^i(x, y, z) = \int_{-L}^z B_y(x, y, t) dt \quad ; \quad A_y^i(x, y, z) = - \int_{-L}^z B_x(x, y, t) dt \quad (2.39)$$

The determinants of M and M^0 were evaluated by a judicious modification of standard methods [23]. We performed a Cholesky decomposition of the matrices $M = R^\dagger R$ where all the non-zero matrix elements of R are in upper- right triangle. The determinant of M is then the square of the product of the diagonal matrix elements of R . The calculations were performed on sizes as large as $L = 15$. In this case M is matrix of order 54000 and it is clearly impossible to store all the $(54000)^2$ matrix elements of M . Crucial use of was made of the fact that all the non- zero matrix elements of M are along 7 diagonal rows close to the central diagonal of the matrix. (The use of open boundary conditions was crucial for this; periodic boundary conditions would have produced non-zero entries along the edges of the

matrix.) The routine for Cholesky decomposition made explicit use of the particular sparse form of M and were optimized to minimize the total storage requirements. All of the above operations were repeated for M^0 and S_1 was then evaluated using Eqn (2.37).

To obtain the infinite size limit of the determinants, two requirements have to be enforced: (i) to minimize the $\mathcal{O}(a\Delta)$ corrections in the expression for S_1 we require $\Delta a \ll 1$; (ii) to minimize interactions between instantons and finite-size corrections we require $\Delta a \gg 1/L$. These two requirements limit considerably the range of values of Δa that can be examined numerically. Numerical results for

$$S(L) = \ln \det M - \ln \det M^0 \quad (2.40)$$

as a function of L for a range of values of Δ are shown in Table 1. For each value of Δ , we fit $S(L)$ to the functional form $2S_1 + c_1/L + c_2/L^2$ where S_1 , c_1 and c_2 were arbitrary fitting parameters. The probable error in the value of S_1 was determined by two methods: (i) an independent fit of $S(L)$ to $2S'_1 + c'_1/L + c'_2/L^2 + c'_3/L^3$ was performed and the error was set to $S_1 - S'_1$; (ii) for the larger values of Δa , the point $S(L = 15)$ was omitted in the fit and the change in the value of $S(L)$ was noted.

The final results for S_1 are shown in Table 2. We test the theoretical predictions by fitting S_1 to the functional form $\alpha \ln(\beta/\Delta a)$ at the four largest values of Δa . We found the best fit

$$S_1 \approx 0.094 \ln \frac{0.81}{\Delta a}. \quad (2.41)$$

The accuracy of the fit is shown in Fig 3. The predictions of Eqn (2.41) at the two smallest values of Δa (which were *not* used in determining the fitting parameters) are consistent with the numerically determined points. The coefficient of the logarithm is remarkably close to the value of 0.0968 determined in Section II and is thus strong evidence for its universality.

Finally, the data for S_1 at the four largest values of Δ were also fit to the functional form $S_1 = \alpha' + \beta'/\Delta$. The results are denoted by the dashed line in Fig 3. Note that the predictions at the two smallest values of Δa are now considerably different from the theoretical predictions.

III. Calculation of S_2

This section will calculate the modification of the core- energy, S_2 , due to the presence of a space-time dependent potential $V(r)$ (Eqn (1.16)). We will begin in Section III.A by determining the potential $V(r)$ necessary to maintaining a space-independent value of $\langle |z^\alpha|^2 \rangle$ in the region $r\Lambda \gg 1$. At distances $r \sim 1/\Lambda$, the singularity in the field-strength $F_{\mu\nu}$ at the core of the instanton drives the density of the z^α quanta to zero. A lattice regularization clearly removes the singularity in the field. An alternative is embed the $U(1)$ gauge group of the $CP^{(N-1)}$ model into a compact Lie group which is spontaneously broken [17, 24]. The action S_2 is not affected by the physics at these length scales, so we will concentrate henceforth on length scales $\gg 1/\Lambda$.

In Section III.B we will, following 't Hooft [15], derive an expression for S_2 in terms of the scattering phase-shifts of the potential $V(r)$. Finally Section III.C will compute the scattering phase shifts and obtain our final expression for S_2 . We note that, unlike Section II, all calculations in this section will be carried out in flat three-dimensional space.

III.A Calculation of $V(r)$

We begin by introducing the operator \mathcal{M}

$$\mathcal{M} = -(\partial_\mu - iA_\mu^i)^2 + \Delta^2 + V(r) \tag{3.1}$$

and its Green's function

$$\mathcal{M}G(x, x') = \delta^3(x_\mu - x'_\mu) \quad (3.2)$$

The potential $V(r)$ must be determined to constrain $\langle |z^\alpha|^2 \rangle$

$$\langle |z^\alpha|^2 \rangle = gG(x, x) = 1 \quad (3.3)$$

where $G(x, x)$ is regulated by an appropriate Pauli- Villars subtraction. We also introduce the operator \mathcal{M}_0 as above but with $V = 0$, and the corresponding Green's function G_0 . We make the following expansions for G and G_0 in spherical co- ordinates (r, θ, ϕ)

$$G(x, x') = \sum_{\ell, m} g_\ell(r, r') Y_{q, \ell, m}(\theta, \phi) Y_{q, \ell, m}(\theta', \phi') \quad (3.4)$$

and similarly for G_0 but with the radial functions $g_{0\ell}$. Then the radial functions satisfy

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\kappa_\ell^2}{r^2} - \Delta^2 - V(r) \right) g_\ell(r, r') = -\frac{1}{r^2} \delta(r - r') \quad (3.5)$$

with

$$\kappa_\ell^2 = \ell(\ell + 1) - \frac{q^2}{4} \quad (3.6)$$

The function $g_{0\ell}$ satisfies an identical equation, but with $V = 0$.

We now turn to the determination of G_0 . Standard methods can be used to solve the differential equation for $g_{0\ell}$:

$$g_{0\ell}(r, r') = \frac{1}{\sqrt{rr'}} I_{\nu_\ell}(\Delta r_{<}) K_{\nu_\ell}(\Delta r_{>}) \quad (3.7)$$

where we recall the constant ν_ℓ introduced in Eqn (1.23)

$$\nu_\ell = \sqrt{\left(\ell + \frac{1}{2}\right)^2 - \frac{q^2}{4}}, \quad (3.8)$$

$r_{<}$ ($r_{>}$) is the smaller (larger) of r, r' , and $I_{\nu_\ell}, K_{\nu_\ell}$ are modified Bessel functions [25]. The boson density in the absence of $V(r)$ is thus proportional to

$$G_0(x, x) = \sum_{\ell=q/2}^{\infty} \frac{2\ell + 1}{4\pi r} (I_{\nu_\ell}(\Delta r) K_{\nu_\ell}(\Delta r) - I_{\nu_\ell}(\Lambda r) K_{\nu_\ell}(\Lambda r)) \quad (3.9)$$

where we have explicitly displayed the Pauli-Villars regularization.

We will not be able to determine an exact solution for $V(r)$, but as will become clear in this section, it will suffice to determine the leading asymptotic term for $V(r)$ at small and large r . We will consider these two regions in turn in the following subsections:

III.A.1 Large r : $r \gg 1/\Delta$

We perform a direct evaluation of G_0 by performing the summation over ℓ in Eqn (3.9) in the limit of large r . The details of this calculation are shown in Appendix A. We obtain

$$G_0(x, x) = \frac{\Lambda - \Delta}{4\pi} - \frac{q^2}{384\pi\Delta^3 r^4} + \mathcal{O}\left(\frac{1}{\Delta^5 r^6}\right) + \mathcal{O}\left(\frac{1}{\Lambda^3 r^4}\right) \quad (3.10)$$

It is however also possible to obtain this result by a simpler, more physical argument which also has the virtue of allowing us to determine G and $V(r)$. The crucial point is that in the region $r \gg 1/\Delta$ we may consider the system as being placed in a *uniform magnetic field* of strength B , given by

$$B = \frac{q}{2r^2} \quad (3.11)$$

All effects arising from the gradients of B are expected to be suppressed by powers of $1/(\Delta r)$. In a uniform field, the eigenvalues of \mathcal{M}_0 split into Landau levels [26], and are characterized by a momentum p_z and a Landau level index n :

$$p_z^2 + (2n + 1)B + \Delta^2 \quad (3.12)$$

The Green's function in a uniform field is easily determined and we find

$$G_0(x, x) = \frac{B}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \sum_{n=0}^{\infty} \left(\frac{1}{p_z^2 + (2n + 1)B + \Delta^2} - \frac{1}{p_z^2 + (2n + 1)B + \Lambda^2} \right) \quad (3.13)$$

Evaluating the sum over n by the Euler Maclaurian expansion we find

$$\begin{aligned} G_0(x, x) &= \frac{\Lambda - \Delta}{4\pi} - \frac{B^2}{96\pi\Delta^3} + \mathcal{O}\left(\frac{B^3}{\Delta^4}\right) \\ &= \frac{\Lambda - \Delta}{4\pi} - \frac{q^2}{384\pi\Delta^3 r^4} + \mathcal{O}\left(\frac{1}{\Delta^4 r^5}\right) \end{aligned} \quad (3.14)$$

which agrees completely with the directly evaluated expression (3.10) upto order $1/r^4$.

In the same spirit, we now evaluate G by assuming that the system is in a uniform magnetic field B and a *constant* potential V . We find

$$G(x, x) = \frac{\Lambda - \sqrt{\Delta^2 + V}}{4\pi} - \frac{B^2}{96\pi\Delta^3} + \mathcal{O}\left(\frac{V^2}{\Delta^3}\right) + \mathcal{O}\left(\frac{B^3}{\Delta^4}\right) \quad (3.15)$$

Demanding that G satisfy the constraint $gG(x, x) = 4\pi G(x, x)/(\Lambda - \Delta) = 1$ we find

$$\begin{aligned} V &\sim -\frac{B^2}{12\Delta^2} \\ &\sim -\frac{q^2}{48\Delta^2 r^4} \end{aligned} \quad (3.16)$$

We have thus obtained the leading term in the asymptotic expansion of V at large r ; all subsequent terms will be suppressed by factors of $1/(\Delta r)$.

III.A.2 Small r : $1/\Lambda \ll r \ll 1/\Delta$

The expression (3.9) for G_0 now has to be evaluated using a small argument expansion for the first term and a large argument expansion for the Pauli-Villars term. We use the results in Appendix A for the Pauli-Villars term and the standard power series expansion for the modified Bessel functions [25] for the first term. This procedure yields

$$G_0(x, x) = \lim_{L_{max} \rightarrow \infty} \left\{ \left[\frac{1}{4\pi r} \sum_{\ell=q/2}^{L_{max}} \frac{2\ell+1}{2\nu_\ell} \right] - \left[\frac{L_{max}+1-\Lambda r}{4\pi r} \right] \right\} + \mathcal{O}(r\Delta^2) + \mathcal{O}\left(\frac{1}{\Lambda r^2}\right) \quad (3.17)$$

where the term in the first square bracket arises from the small argument expansion of $I_{\nu_\ell}(\Delta r)K_{\nu_\ell}(\Delta r)$, and the second square bracket from the large argument expansion in Appendix A of the Pauli-Villars term in Eqn (3.9). Rearranging terms we find

$$G_0(x, x) = \frac{\Lambda}{4\pi} - \frac{1}{4\pi r} \left[\sum_{\ell=q/2}^{\infty} \left(1 - \frac{2\ell+1}{2\nu_\ell} \right) + \frac{q}{2} \right] \quad (3.18)$$

The coefficient of $1/r$ can be easily shown to be non-zero and negative. Thus in the absence of $V(r)$ there is a boson-density deficit $\sim -g/r$ at small r . The origin of this deficit can

be understood to be the extra centrifugal repulsion in the radial equation due to the presence of the monopole: the angular quantum number begins at the value $\ell = q/2$, unlike $\ell = 0$ in the absence of the monopole.

We assert that the centrifugal repulsion may be compensated by an attractive potential

$$V(r) \sim -\frac{\alpha_q}{r^2} \quad (3.19)$$

at small r . For this potential G can be determined from the differential equation (3.5) and we find

$$G(x, x) = \sum_{\ell=q/2}^{\infty} \frac{2\ell+1}{4\pi r} \left(I_{\nu'_\ell}(\Delta r) K_{\nu'_\ell}(\Delta r) - I_{\nu'_\ell}(\Lambda r) K_{\nu'_\ell}(\Lambda r) \right) \quad (3.20)$$

for small r where we recall the constant ν'_ℓ introduced in Eqn (1.23)

$$\nu'_\ell = \sqrt{\left(\ell + \frac{1}{2}\right)^2 - \frac{q^2}{4} - \alpha_q} \quad (3.21)$$

Proceeding as above for G_0 , we find that in the region of r under consideration

$$G(x, x) = \frac{\Lambda}{4\pi} - \frac{1}{4\pi r} \left[\sum_{\ell=q/2}^{\infty} \left(1 - \frac{2\ell+1}{2\nu'_\ell} \right) + \frac{q}{2} \right] \quad (3.22)$$

To leading order in Δ/Λ and $r\Delta$, the constraint $gG(x, x) = 1$ can be imposed by requiring that the coefficient of the $1/r$ term in the above equation vanish. This leads to following condition on α_q

$$\sum_{\ell=q/2}^{\infty} \left(\frac{2\ell+1}{\sqrt{(2\ell+1)^2 - q^2 - 4\alpha_q}} - 1 \right) = \frac{q}{2}, \quad (3.23)$$

which was quoted earlier in Section I (Eqn (1.12)). We have thus completely determined the leading term in $V(r)$ at small r .

III.B Phase Shift Representation of S_2

We begin this section by providing the missing steps between the expressions for S_2 in Eqn (1.16) and (1.20). This involves determining the relative phase-shifts of the operators in Eqn

(1.16) and then expressing the determinant in terms of the phase shifts [15]. We consider the eigenfunctions Ψ_k and eigenvalues $k^2 + \Delta^2$ of the operator \mathcal{M}

$$\mathcal{M}\Psi_k \equiv \left[-(\partial_\mu - iA_\mu^i)^2 + \Delta^2 + V(r) \right] \Psi_k = (k^2 + \Delta^2)\Psi_k \quad (3.24)$$

and the operator \mathcal{M}_0 defined as above but with $V = 0$. In spherical co-ordinates we write

$$\Psi_k(x) = \frac{Y_{q,\ell,m}(\theta, \phi) f(r)}{\sqrt{r}} \quad (3.25)$$

where the $Y_{q,\ell,m}$ are the monopole harmonics [19]. We perform the identical decomposition for \mathcal{M}_0 but with a different radial function $f_0(r)$. The radial eigenvalue equation becomes

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left[k^2 - V(r) - \frac{\nu_\ell^2}{r^2} \right] f = 0 \quad (3.26)$$

The function f_0 satisfies an identical equation but with $V = 0$. We temporarily quantize the eigenvalues by demanding that the wavefunctions vanish on the surface of a large sphere of radius R . To do this we need the large r behaviour of f and f_0 . As the potential $V(r)$ falls off as $1/r^4$ at large r , we may conclude from standard scattering theory [26]

$$\lim_{r \rightarrow \infty} f(r) \sim \frac{1}{\sqrt{r}} \sin \left(kr - \frac{\nu_\ell \pi}{2} + \frac{\pi}{4} + \eta_\ell(k) \right) \quad (3.27)$$

and

$$\lim_{r \rightarrow \infty} f_0(r) \sim \frac{1}{\sqrt{r}} \sin \left(kr - \frac{\nu_\ell \pi}{2} + \frac{\pi}{4} \right) \quad (3.28)$$

where $\eta_\ell(k)$ is the phase-shift. The eigenvalue associated with the n 'th zero of the sin will be labeled by $k(n)$. We have then

$$\begin{aligned} k(n)R - \frac{\nu_\ell \pi}{2} + \frac{\pi}{4} + \eta_\ell(k(n)) &= n\pi \\ k_0(n)R - \frac{\nu_\ell \pi}{2} + \frac{\pi}{4} &= n\pi \end{aligned} \quad (3.29)$$

from which we may obtain

$$\frac{k(n)}{k_0(n)} = 1 - \frac{1}{R} \frac{\eta_\ell(k(n))}{k(n)} \quad (3.30)$$

and the eigenvalue spacing $\delta k = k(n+1) - k(n)$

$$\delta k = \frac{\pi}{R} + \mathcal{O}\left(\frac{1}{R^2}\right) \quad (3.31)$$

We now insert this information into the determinants

$$\begin{aligned} \text{Tr} \ln \frac{\mathcal{M}}{\mathcal{M}_0} &= \sum_{\ell=q/2}^{\infty} (2\ell+1) \sum_n \ln \left(\frac{k(n)^2 + \Delta^2}{k_0(n)^2 + \Delta^2} \right) \\ &= -\frac{2}{\pi} \sum_{\ell=q/2}^{\infty} (2\ell+1) \int_0^{\infty} dk \eta_{\ell}(k) \frac{k}{k^2 + \Delta^2} + \mathcal{O}\left(\frac{1}{R}\right) \end{aligned} \quad (3.32)$$

After regulating the integral by the Pauli-Villars method we obtain finally the expression (1.20) for S_2 .

Before closing this section we will show how the term linear in V in the expression for S_2 can be disposed of. We will show in the next section that there is a well defined expansion for the phase-shifts, η_{ℓ} , in successive powers of V . This implies that a similar expansion exists for S_2 . Let us introduce the functional $\mathcal{A}[\Phi]$

$$\mathcal{A}[\Phi] = \text{Tr} \ln \left[-(\partial_{\mu} - iA_{\mu}^i)^2 + \Delta^2 + \Phi \right] - \frac{1}{g} \int d^3x \Phi \quad (3.33)$$

then clearly

$$S_2 = \mathcal{A}[V] - \mathcal{A}[0] \quad (3.34)$$

where V is determined by the constraint (1.10)

$$\left. \frac{\delta \mathcal{A}}{\delta \Phi} \right|_{\Phi=V} = 0 \quad (3.35)$$

We perform a functional expansion of \mathcal{A} in powers of Φ

$$\begin{aligned} \mathcal{A}[\Phi] &= \mathcal{A}[0] + \int d^3x \mathcal{A}_1(x) \Phi(x) + \frac{1}{2!} \int d^3x d^3y \mathcal{A}_2(x, y) \Phi(x) \Phi(y) \\ &\quad + \frac{1}{3!} \int d^3x d^3y d^3z \mathcal{A}_3(x, y, z) \Phi(x) \Phi(y) \Phi(z) + \dots \end{aligned} \quad (3.36)$$

where all functional derivatives are evaluated at $\Phi = 0$. We now use Eqn (3.35) to obtain

$$\mathcal{A}_1(x) = - \int d^3y \mathcal{A}_2(x, y) V(y) - \frac{1}{2!} \int d^3y d^3z \mathcal{A}_3(x, y, z) V(y) V(z) + \dots \quad (3.37)$$

Inserting this back into Eqn (3.36) for \mathcal{A} and using Eqn (3.34) we obtain a modified expression for S_2

$$S_2 = -\frac{1}{2!} \int d^3x d^3y \mathcal{A}_2(x, y) V(x) V(y) - \frac{2}{3!} \int d^3x d^3y d^3z \mathcal{A}_3(x, y, z) V(x) V(y) V(z) + \dots \quad (3.38)$$

Thus after expressing the phase shifts η_ℓ in successive powers of V

$$\eta_\ell = \eta_\ell^{(1)} + \eta_\ell^{(2)} + \eta_\ell^{(3)} + \eta_\ell^{(4)} + \dots \quad (3.39)$$

we obtain the quantity $\tilde{\eta}_\ell$ given by

$$\tilde{\eta}_\ell = -\eta_\ell^{(2)} - 2\eta_\ell^{(3)} - 3\eta_\ell^{(4)} + \dots \quad (3.40)$$

which appears in our final expression for S_2

$$S_2 = -\frac{2}{\pi} \sum_{\ell=q/2}^{\infty} (2\ell + 1) \int_0^{\infty} dk \tilde{\eta}_\ell(k) \left[\frac{k}{k^2 + \Delta^2} - \frac{k}{k^2 + \Lambda^2} \right] \quad (3.41)$$

III.C Calculation of Phase Shifts

We will begin by estimating the contribution of the large and small r regions of $V(r)$ for large ℓ and k/Δ . For large ℓ we may safely use the semiclassical expression for the phase-shift [26]

$$\eta_\ell(k) = \int_{r_0}^{\infty} dr \sqrt{k^2 - \frac{\kappa_\ell^2}{r^2} - V(r)} - \int_{\tilde{r}_0}^{\infty} dr \sqrt{k^2 - \frac{\kappa_\ell^2}{r^2}} \quad (3.42)$$

where \tilde{r}_0 and r_0 are the classical turning- points *i.e.* the points where the expressions under the radical vanish, and κ_ℓ ($\sim \ell$ for large ℓ) was defined in Eqn (3.6). From the results in Section III.A it is clear that in the entire region $r \gg 1/\Lambda$ $V(r)$ can be written in the

form $V(r) = \Delta^2 F(\Delta r)$ where the function $F(x)$ satisfies $F(x) \sim 1/x^2$ for small x , and $F(x) \sim 1/x^4$ for large x . Inserting this functional form into the expression above, and changing the variable of integration to $\xi = kr/\kappa_\ell$ we obtain

$$\eta_\ell(k) = \kappa_\ell \int_{\xi_0}^{\infty} d\xi \sqrt{1 - \frac{1}{\xi^2} - \frac{1}{(k/\Delta)^2} F\left(\frac{\kappa_\ell}{k/\Delta} \xi\right)} - \kappa_\ell \int_1^{\infty} dr \sqrt{1 - \frac{1}{\xi^2}} \quad (3.43)$$

The subsequent analysis is different in the two regimes $\kappa_\ell \sim \ell \geq k/\Delta$ and $\ell \leq k/\Delta$ which we consider separately

(i) $\ell \geq k/\Delta$

In this case the argument of F is always bigger than unity, and we may replace $F(x)$ by its asymptotic form $\sim 1/x^4$. We will analyze in general a functional form $F(x) \sim 1/x^p$ ($p \geq 4$) to estimate the contributions of the leading and subleading terms of $F(x)$. Integrals of the form appearing in Eqn (3.43) have been examined in general in Appendix B. For the case $F(x) = c/x^p$, we identify in Eqn (B.1) the functions $A(\xi) = 1 - 1/\xi^2$, $B(\xi) = c/\xi^p$, $X = \infty$ and

$$\epsilon = \epsilon_p = \left(\frac{k/\Delta}{\kappa_\ell}\right)^{p-2} \frac{1}{\kappa_\ell^2} \quad (3.44)$$

The parameter ϵ is always small for the parameter ranges under consideration and an expansion in ϵ can be safely performed. A typical term contributing to $\eta_\ell^{(2)}$ is estimated to be of order

$$\eta_\ell^{(2)}(k) \sim \kappa_\ell \epsilon_{p_1} \epsilon_{p_2} \sim \frac{(k/\Delta)^{(p_1+p_2-4)}}{\ell^{(p_1+p_2-1)}} \quad (3.45)$$

where $p_1, p_2 \geq 4$. The expression (3.41) involves the sum $\sum_\ell (2\ell + 1) \eta_\ell^{(2)}(k)$. We estimate the contribution of this term to the sum

$$\sum_{\ell \sim k/\Delta}^{\infty} \ell \eta_\ell^{(2)}(k) \sim \frac{\Delta}{k} \quad (3.46)$$

for all $p_1, p_2 \geq 4$. Inserting this dependence into the integral over k in the phase-shift expression for S_2 (Eqn (3.41)) we see that the final result is finite in the limit $\Delta/\Lambda \rightarrow 0$. Such contributions to S_2 are subdominant to S_1 and can be safely ignored. Thus there is *no contribution* to the leading asymptotic result for the core action from the range $\ell \geq k/\Delta$.

(ii) $\ell \leq k/\Delta$

Now the argument of F is small except for $\xi > (k/\Delta)/\kappa_\ell$. The contribution of the range $\xi > (k/\Delta)/\kappa_\ell$ can be shown, as above, to be negligible; we will therefore concentrate on the region $\xi < (k/\Delta)/\kappa_\ell$. In this case we know from Section III.A that $F(x) \sim -\alpha_q/x^2$. As in (i) we will consider the general functional form $F(x) = c'/x^s$ with $s \leq 2$ to estimate the contributions of leading and sub-leading terms in $F(x)$. Comparing the expression for the phase-shift (3.43) with the canonical form (B.1) in Appendix B we obtain $A(\xi) = 1 - 1/\xi^2$, $B(\xi) = c'/\xi^s$, $X \sim (k/\Delta)/\kappa_\ell$ and

$$\epsilon = \epsilon_s = \left(\frac{\kappa_\ell}{k/\Delta} \right)^{2-s} \frac{1}{\kappa_\ell^2} \quad (3.47)$$

The parameter ϵ is small, except for the physically interesting case of $s = 2$ and small ℓ : in this case the quasiclassical approximation breaks down, and other methods will have to be used. Continuing our analysis, but now with $s < 2$, we obtain from Eqn (B.7) the following estimate for a typical term in the second-order phase-shift

$$\eta_\ell^{(2)}(k) \sim \begin{cases} \kappa_\ell \epsilon_{s_1} \epsilon_{s_2} & \sim (k/\Delta)^{(s_1+s_2-4)} \ell^{(1-s_1-s_2)} & \text{if } 1 < s_1 + s_2 < 4 \\ \kappa_\ell \epsilon_{s_1} \epsilon_{s_2} X^{(1-s_1-s_2)} & \sim (k/\Delta)^{-3} & \text{if } s_1 + s_2 < 1 \end{cases} \quad (3.48)$$

As before, we will need the quantity $\sum_\ell (2\ell + 1) \eta_\ell^{(2)}(k)$ in the expression (3.41) for S_2 . We estimate the contribution of $\eta_\ell^{(2)}(k)$ to this sum

$$\sum_{\ell=q/2}^{k/\Delta} \ell \eta_\ell^{(2)}(k) \sim \begin{cases} (k/\Delta)^{(s_1+s_2-4)} & \text{if } 3 < s_1 + s_2 < 4 \\ (k/\Delta)^{-1} & \text{if } s_1 + s_2 < 3 \end{cases} \quad (3.49)$$

Inserting this dependence into the integral over k in the phase-shift expression for S_2 (Eqn (3.41)) we see that, for all $s_1 + s_2 < 4$ the final result is finite in the limit $\Delta/\Lambda \rightarrow 0$. Thus *all the sub-leading* terms in $V(r)$ at small r have *no contribution* to the leading asymptotic result for the core action from the range $\ell \leq k/\Delta$.

Thus the main conclusion of the rather involved calculations discussed so far in this section is rather simple and remarkable: the only part of $V(r)$ which can possibly contribute a $\ln(\Lambda/\Delta)$ (or more divergent) term to the core action, S_2 , is the $-\alpha_q/r^2$ term at small r . The quasi-classical method breaks down for such a potential and a different method is used below to obtain an *exact* expression for phase shift.

The strategy we shall follow is rather analogous to that used in Section III.A in the calculation of $V(r)$ and $\langle |z^\alpha|^2 \rangle$. We use the following form for $V(r)$

$$V(r) = \begin{cases} -\frac{\alpha_q}{r^2} & \text{for } r < \mu/\Delta \\ 0 & \text{for } r > \mu/\Delta \end{cases} \quad (3.50)$$

where μ is a constant of order 1: the final answer will turn out to be insensitive to the precise value of μ . In the presence of $V(r)$ the eigenfunctions of the operator \mathcal{M} (Eqn (3.1)), introduced in Section III.A, have the form of Eqn (3.25) where the radial function $f(r)$ is $f(r) = J_{\nu'_\ell}(kr)$ for $r < \mu/\Delta$ and $f(r) = J_{\nu_\ell}(kr) + cY_{\nu_\ell}(kr)$ for $r > \mu/\Delta$. Here the J_ν and Y_ν are Bessel functions of order ν , the constant ν'_ℓ was defined in Eqn (3.21), while ν_ℓ was defined in Eqn (3.8). In the absence of $V(r)$, the radial wavefunction $f_0(r)$ is $J_{\nu_\ell}(kr)$. Using the well known asymptotic form of the Bessel functions and definition of the phase-shift $\eta_\ell(k)$ in Eqns (3.27) and (3.28) we obtain the following exact expression for the phase-shift

$$\tan \eta_\ell(k) = \frac{J'_{\nu'_\ell}(\mu k/\Delta)J_{\nu'_\ell}(\mu k/\Delta) - J_{\nu_\ell}(\mu k/\Delta)J'_{\nu'_\ell}(\mu k/\Delta)}{Y'_{\nu_\ell}(\mu k/\Delta)J_{\nu'_\ell}(\mu k/\Delta) - Y_{\nu_\ell}(\mu k/\Delta)J'_{\nu'_\ell}(\mu k/\Delta)} \quad (3.51)$$

This expression can be expanded in powers of α_q and the entire series is equivalent, term by term, to a power series expansion for the phase-shift in terms of $V(r)$. This enables us

to identify $\eta_\ell^{(2)}$, $\eta_\ell^{(3)}$, $\eta_\ell^{(4)}$... as defined in Eqn (3.39) and obtain the following closed-form expression for the quantity $\tilde{\eta}_\ell$ defined in Eqn (3.40)

$$\tilde{\eta}_\ell(k) = \eta_\ell(k) - \alpha_q \frac{d\eta_\ell}{d\alpha_q}(k) \quad (3.52)$$

For the range of parameters $\ell \ll k/\Delta$, the result (3.51) simplifies to

$$\eta_\ell(k) = \frac{\pi}{2}(\nu_\ell - \nu'_\ell) \quad (3.53)$$

$$= \frac{\pi}{2} \left(\sqrt{\left(\ell + \frac{1}{2}\right)^2 - \frac{q^2}{4}} - \sqrt{\left(\ell + \frac{1}{2}\right)^2 - \frac{q^2}{4} - \alpha_q} \right) \quad (3.54)$$

and the quantity $\tilde{\eta}_\ell(k)$ is

$$\tilde{\eta}_\ell = \frac{\pi}{2} \left(\nu_\ell - \nu'_\ell - \frac{\alpha_q}{2\nu'_\ell} \right) \quad (3.55)$$

Note that $\tilde{\eta}_\ell(k)$ is independent of k and has the leading dependence $\sim 1/\ell^3$; this is consistent with quasi-classical in Eqn (3.48) with $s_1 + s_2 = 4$. In the opposite limit $\ell \gg k/\Delta$, it is easy to show from (3.51) that both η_ℓ and $\tilde{\eta}_\ell$ are *exponentially* small in ℓ . Before inserting $\tilde{\eta}_\ell$ into the expression (3.41) for S_2 we perform the sum over ℓ , and obtain for large k/Δ :

$$\begin{aligned} \frac{2}{\pi} \sum_{\ell=q/2}^{\infty} (2\ell + 1) \tilde{\eta}_\ell(k) &= \sum_{\ell=q/2}^{\sim k/\Delta} (2\ell + 1) \left(\nu_\ell - \nu'_\ell - \frac{\alpha_q}{2\nu'_\ell} \right) + \mathcal{O}\left(\frac{\Delta}{k}\right) \\ &= \sum_{\ell=q/2}^{\infty} (2\ell + 1) \left(\nu_\ell - \nu'_\ell - \frac{\alpha_q}{2\nu'_\ell} \right) + \mathcal{O}\left(\frac{\Delta}{k}\right) \\ &\equiv -\Xi_q + \mathcal{O}\left(\frac{\Delta}{k}\right) \end{aligned} \quad (3.56)$$

where the last equation defines the constants Ξ_q . We find $\Xi_1 = 0.027784776098205850\dots$, and $\Xi_2 = 0.084667877126678540\dots$. Inserting this into the expression (3.41) for S_2 we obtain our final result

$$S_2 = \Xi_q \ln\left(\frac{\Lambda}{\Delta}\right) + \mathcal{O}\left(\frac{\Delta}{\Lambda}\right) \quad (3.57)$$

IV. Conclusions

We recapitulate the main results of this paper. We examined instanton tunneling events in the 2+1 dimensional CP^{N-1} model in the large N limit. This amounts to looking for saddle points of the action

$$S_{eff} = \text{Tr} \ln [-(\partial_\mu - iA_\mu)^2 + i\lambda] - i\frac{1}{g} \int d^3x \lambda \quad (4.1)$$

which is a functional of the fields A_μ and λ . A charge q instanton at $x = 0$ has the gauge field specified by

$$\epsilon_{\mu\nu\lambda} \frac{\partial A_\nu^i}{\partial x_\lambda} = \frac{q x_\mu}{2 x^3} \quad (4.2)$$

which is the vector potential of a Dirac monopole of with total flux $2\pi q$. The saddle point value of λ is given by

$$i\lambda = \Delta^2 + V(r) \quad (4.3)$$

where $\xi = 1/\Delta$ is the spin correlation length and $V(r)$ is an attractive potential determined in this paper (Eqn (1.11)). The form of $V(r)$ maintains the constraint on the matter fields z^α of the CP^{N-1} model.

After determination of $V(r)$, the action $S_{eff} = S_{core}/N$ was evaluated. We found

$$\lim_{\xi/a \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{S_{core}}{N} = 2\varrho_q \ln \left(\frac{\xi}{a} \right) \quad (4.4)$$

where a is the lattice spacing, ξ is the spin correlation length, and the ϱ_q are a set of universal numbers. We find $\varrho_1 = 0.06229609020041076\dots$ and $\varrho_2 = 0.15554762255599117\dots$

The core-action implies that free instantons occur with a density $n_q a^3 \sim (\xi/a)^{-2N\varrho_q}$. Instanton effects are known to lead to exponentially decaying correlation functions for the gauge field A_μ [17] and a confining force between the z quanta. Our results imply that these

phenomena occur at a length-scale ξ_C given by

$$\frac{\xi_C}{a} = \left(\frac{\xi}{a}\right)^{Ne_1} \tag{4.5}$$

in the limit $N \rightarrow \infty$ with ξ large, but finite. Our results also have important consequences for the disordered phase of two-dimensional $SU(N)$ quantum antiferromagnets. In particular the instanton action is a crucial quantity determining the magnitude of the spin-Peierls ordering in this phase. These results are discussed in Refs [8, 9] and were briefly mentioned in Section I.A.

An important question left unanswered in this paper is that of the critical behavior of the transition between magnetically ordered and disordered phases of $SU(N)$ antiferromagnets and the CP^{N-1} model. This amounts to examining the limit $\xi \rightarrow \infty$ while keeping N fixed. This paper has taken the limits in the opposite order and it is not clear that the limits commute. An understanding of this issue requires examination of the $1/N$ corrections to the instanton core-action. An important question which this analysis must settle is whether terms higher order in $1/N$ are more singular in the limit $\xi \rightarrow \infty$. Existing $1/N$ [27] and $2 + \epsilon$ [16] expansion treatments of the transition in the CP^{N-1} model have simply ignored the instanton tunnelling events.

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Appendix

A. Green's functions in a monopole field

In this appendix we evaluate the Green's function G_0 , obtained in Eqn (3.9), in the presence of the monopole field A_μ^i but with a space-independent mass Δ^2 . We will concentrate on large r . We recall Eqn (3.9)

$$G_0(x, x) = \sum_{\ell=q/2}^{\infty} \frac{2\ell+1}{4\pi r} (I_{\nu_\ell}(\Delta r)K_{\nu_\ell}(\Delta r) - I_{\nu_\ell}(\Lambda r)K_{\nu_\ell}(\Lambda r)) \quad (\text{A.1})$$

For large r , it will become clear from the following that the sum is dominated by large values of ℓ . This motivates use of the following large order/argument expansion of the Bessel functions [25]

$$I_\nu(z)K_\nu(z) \sim \frac{1}{2\sqrt{\nu^2+z^2}} \left(1 + \frac{v_1(t)}{z^2} + \frac{v_2(t)}{z^4} + \dots \right) \quad (\text{A.2})$$

with

$$\begin{aligned} v_1(t) &= \frac{1}{8(1+t^2)^3} (1-4t^2) \\ v_2(t) &= \frac{1}{128(1+t^2)^6} (27-472t^2+592t^4-64t^6) \end{aligned} \quad (\text{A.3})$$

and

$$t = \frac{\nu}{z} \quad (\text{A.4})$$

We begin by inserting the first term in the asymptotic expansion (A.2) into the expression for G_0 . The resulting sum, J_1 , is given by

$$J_1 = \frac{1}{4\pi r} \sum_{\ell=q/2}^{L_{max}} (2\ell+1) \left(\frac{1}{\sqrt{(2\ell+1)^2 - 4q^2 - 4\Delta^2 r^2}} - \frac{1}{\sqrt{(2\ell+1)^2 - 4q^2 - 4\Lambda^2 r^2}} \right) \quad (\text{A.5})$$

where we have introduced an upper limit, L_{max} , on the ℓ summation which will eventually be set to infinity. For large r , this sum can be evaluated in inverse powers of r by the Euler-Maclaurian expansion. We omit the tedious, but straightforward, details and state the final

result below. The term in the two curly brackets represent the contributions of the first and second terms in Eqn (A.5) respectively

$$J_1 = \left\{ \frac{L_{max} + 1 - \Delta r}{4\pi r} + \frac{1}{96\pi\Delta r^2} + \frac{1}{384\pi\Delta^3 r^4} \left(-q^2 + \frac{7}{20} \right) + \mathcal{O}\left(\frac{1}{\Delta^5 r^6}\right) \right\} - \left\{ \frac{L_{max} + 1 - \Lambda r}{4\pi r} + \mathcal{O}\left(\frac{1}{\Lambda r^2}\right) \right\} \quad (\text{A.6})$$

Similarly the second term from Eqn (A.2) yields the following contribution, J_2 , to G_0

$$J_2 = -\frac{1}{96\pi\Delta r^2} + \frac{1}{768\pi\Delta^3 r^4} + \mathcal{O}\left(\frac{1}{\Delta^5 r^6}\right) + \mathcal{O}\left(\frac{1}{\Lambda r^2}\right). \quad (\text{A.7})$$

Finally the third term from (A.2) contributes J_3

$$J_3 = -\frac{17}{7680\pi\Delta^3 r^4} + \mathcal{O}\left(\frac{1}{\Delta^5 r^6}\right) + \mathcal{O}\left(\frac{1}{\Lambda r^2}\right). \quad (\text{A.8})$$

Combining J_1 , J_2 , and J_3 , we obtain

$$G_0(x, x) = \frac{\Lambda - \Delta}{4\pi} - \frac{q^2}{384\pi\Delta^3 r^4} + \mathcal{O}\left(\frac{1}{\Delta^5 r^6}\right) + \mathcal{O}\left(\frac{1}{\Lambda^3 r^4}\right) \quad (\text{A.9})$$

which was the result quoted in Eqn (3.10). Note that the $1/r^2$ terms in J_1 and J_2 have canceled against each other. Moreover for $q = 0$ the expression for G_0 is independent of r : this is clearly a consequence of the translational invariance of the system in the absence of instantons. The fact that we have reproduced this trivial limit is a highly non-trivial check on the correctness of our manipulations.

B. Phase Shift Integrals

In this appendix we present the expansion of the semi-classical phase-shift in powers of the scattering potential. The integrals which occur in Section III.C can all be transformed into the following canonical form

$$I = \int_{\xi_0}^X d\xi \sqrt{A(\xi) + \epsilon B(\xi)} - \int_1^X d\xi \sqrt{A(\xi)} \quad (\text{B.1})$$

where the integrands vanish at the lower limit of integration: $A(1) = 0$ and $A(\xi_0) + \epsilon B(\xi_0) = 0$. We shall obtain an expansion of I in powers of ϵ

$$I = \epsilon I_1 + \epsilon^2 I_2 + \dots \quad (\text{B.2})$$

We begin by determining ξ_0 in powers of ϵ : a straightforward calculation yields

$$\xi_0 = 1 - \epsilon \frac{B(1)}{A'(1)} + \epsilon^2 \frac{B(1)}{A'^3(1)} \left(A'(1)B'(1) - \frac{1}{2}A''(1)B(1) \right) + \dots \quad (\text{B.3})$$

We now introduce the new dummy variable of integration $\tilde{\xi} = \xi - (\xi_0 - 1)$ in the first integral and obtain

$$I = \int_1^{X-\xi_0+1} d\tilde{\xi} \sqrt{A(\tilde{\xi} + \xi_0 - 1) + \epsilon B(\tilde{\xi} + \xi_0 - 1)} - \int_1^X d\xi \sqrt{A(\xi)} \quad (\text{B.4})$$

We now expand the integrand in powers of ϵ . The crucial advantage of this choice of variables is that the coefficient of every power of ϵ is guaranteed to vanish at $\tilde{\xi} = 1$; expansion of the radical in powers of ϵ then necessarily yields integrals which are convergent at $\tilde{\xi} = 1$. To first order in ϵ we obtain

$$I_1 = \int_1^X d\tilde{\xi} \left(\frac{B(\tilde{\xi})}{2\sqrt{A(\tilde{\xi})}} - \frac{B(1)A'(\tilde{\xi})}{2A'(1)\sqrt{A(\tilde{\xi})}} \right) + \frac{B(1)\sqrt{A(X)}}{A'(1)} \quad (\text{B.5})$$

In this, and the subsequent equations we have assumed that $A'(X)$ and $B(X)$ can be safely neglected. The second term in the integrand above is an exact derivative and we obtain the final expected result [26]

$$I_1 = \int_1^X d\tilde{\xi} \frac{B(\tilde{\xi})}{2\sqrt{A(\tilde{\xi})}} \quad (\text{B.6})$$

A similar analysis can be performed at the next order in ϵ , yielding

$$I_2 = \int_1^X d\tilde{\xi} \left[-\frac{1}{8(A(\tilde{\xi}))^{3/2}} \left(B(\tilde{\xi}) - \frac{B(1)A'(\tilde{\xi})}{A'(1)} \right)^2 + \frac{1}{2\sqrt{A(\tilde{\xi})}} \left(\frac{A''(\tilde{\xi})B^2(1)}{2A'^2(1)} - \frac{B'(\tilde{\xi})B(1)}{A'(1)} \right) \right] \quad (\text{B.7})$$

All the integrals are convergent at $\tilde{\xi} = 1$. Note also that every term is second order in B . It is clear that this procedure can be iterated to all orders to yield a well defined power series in ϵ .

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Tables

| | | Δa | | | | | |
|-----|----|------------|-------|-------|--------|--------|--------|
| | | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 |
| L | 6 | | | | 0.2112 | 0.1712 | 0.1425 |
| | 7 | | 0.329 | 0.247 | 0.1966 | 0.1623 | 0.1368 |
| | 8 | 0.430 | 0.301 | 0.232 | 0.1883 | 0.1573 | 0.1336 |
| | 9 | 0.393 | 0.283 | 0.223 | 0.1834 | 0.1543 | 0.1317 |
| | 10 | 0.367 | 0.272 | 0.217 | 0.1802 | 0.1525 | 0.1304 |
| | 12 | 0.337 | 0.259 | 0.211 | 0.1767 | 0.1503 | 0.1291 |
| | 15 | 0.316 | 0.251 | 0.207 | 0.1746 | 0.1491 | 0.1286 |

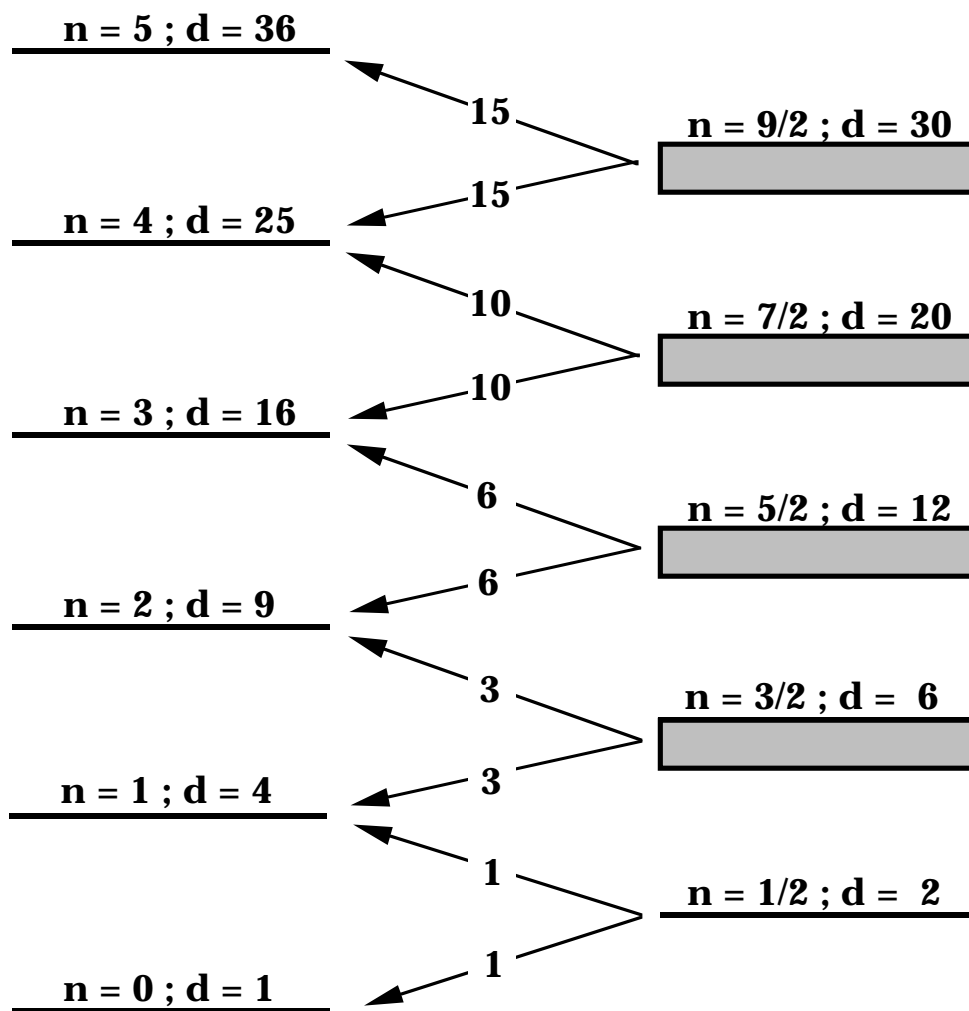
Table 1: Numerical results for the action, $S(L)$, of a instanton anti-instanton pair separated by a distance $2La$ in a box of $2L \times 2L \times 4L$ lattice points as a function of the inverse correlation length Δ (a is the lattice spacing). We reiterate that the mass Δ^2 was *space-independent* in the numerical calculation. The action was obtained by numerical evaluation of the determinant on a lattice with lattice-spacing 1. The normal derivative of the field on the surface of the box was made to vanish by placing image charges outside the box (Fig. 2).

| Δa | $2S_1$ |
|------------|-------------------|
| 0.15 | 0.325 ± 0.023 |
| 0.20 | 0.269 ± 0.010 |
| 0.25 | 0.221 ± 0.009 |
| 0.30 | 0.185 ± 0.009 |
| 0.35 | 0.156 ± 0.008 |
| 0.40 | 0.133 ± 0.005 |

Table 2: Infinite L limit of the action of the instanton anti- instanton pair obtained by the extrapolation procedure discussed in the Section II.D. Here S_1 is the core-action of a single instanton with a *space-independent* mass Δ^2 .

Figure Captions

1. Schematic of the eigenvalues of the operator $(\vec{\nabla} - i\vec{A}^i)^2$ on S^3 in the cases $\vec{A}^i = 0$ and \vec{A}^i due to a instanton anti-instanton pair. The eigenvalues are labeled by a principle quantum number n . In the absence of instantons, all states with the same value of n are degenerate; the number of such states is denoted by d . In the presence of instantons, states with the same value of n form clusters shown by the shaded boxes; now d denotes the total number of states in the cluster. Note that the spacing of the levels is purely schematic and is not scaled to the eigenvalues.
2. Two-dimensional section of the configuration of instanton charges. A instanton anti-instanton pair is placed inside the box B . An infinite set of image charges is placed outside the box to ensure that the *normal* component of the field on the surfaces of B is zero; the charge positions can be obtained by repeated translations of the fragment shown above. All field lines emerging from the positive charge go to the negative charge without leaving B .
3. The filled circles represent the action, $2S_1$, for an instanton anti-instanton pair obtained by numerical evaluation of the determinant for the case of a *space-independent* mass Δ^2 . The full line is a fit of the function $\alpha \ln(\beta/\Delta a)$ to the points with the four largest values of Δa ; we find $\alpha = 0.094$. The dashed line is a similar fit to the function $\alpha' + \beta'/\Delta$.



no instantons

instanton
anti-instanton
pair

Figure 1

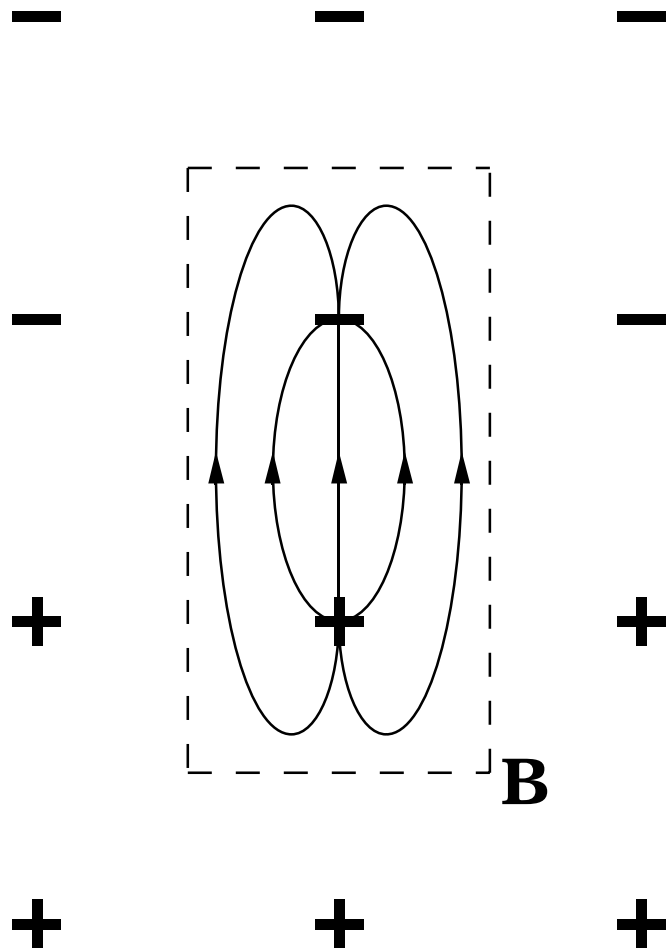


Figure 2

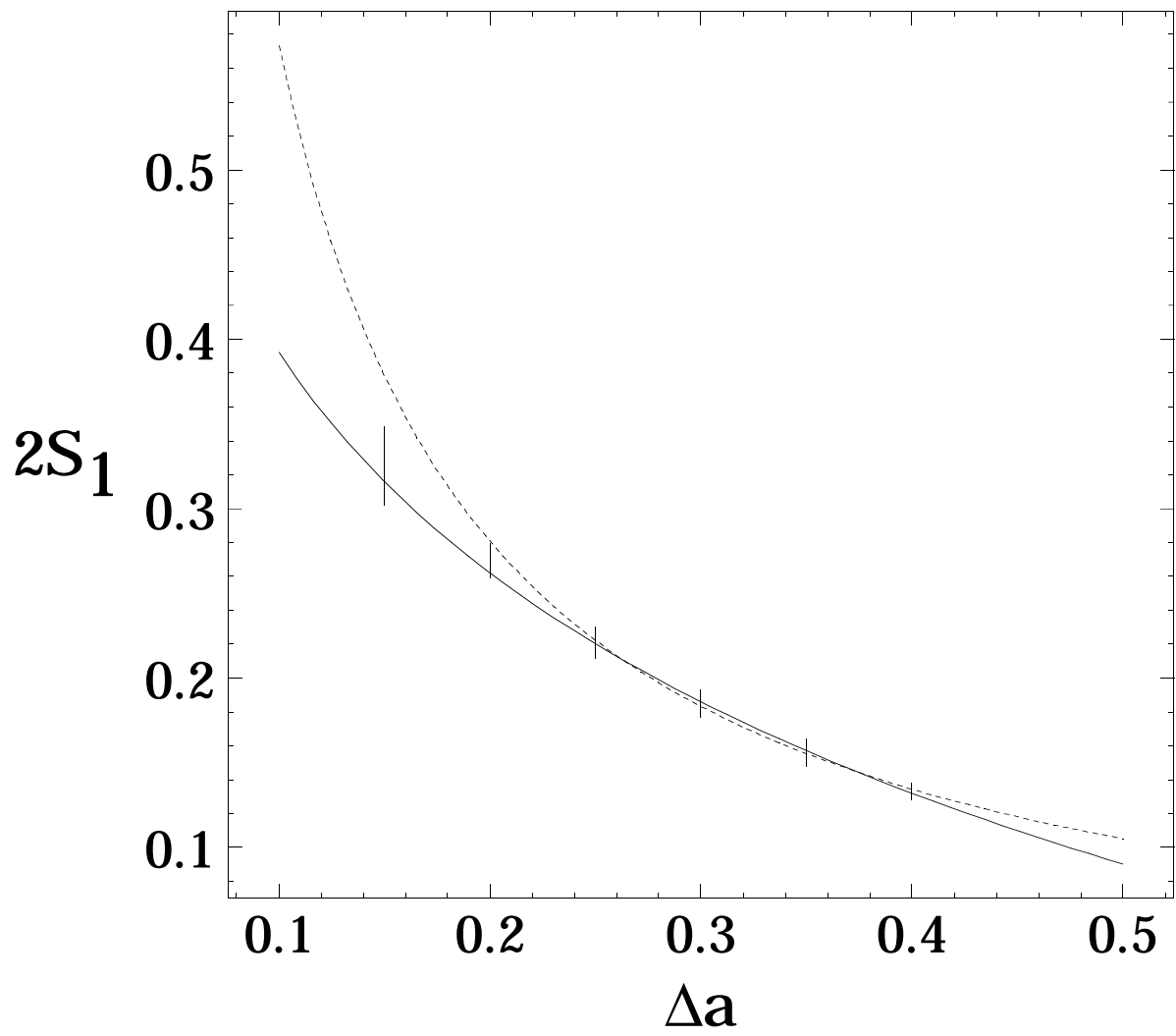


Figure 3