

## Far From Equilibrium Dynamics of the Bose Gas

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### 1. Introduction

The recent successful observations of Bose condensation in neutral, trapped atomic gases [1] and excitons in  $Cu_2O$  [2] have taken the experiments into a heretofore inaccessible regime, as pointed out by several speakers at this conference. These also have led theorists to ask and study questions regarding the Bose gas which, without these recent success stories, would have been merely academic exercises. One such exciting field is to understand the time-dependent nonequilibrium phenomena in the Bose gas. In this paper, we study theoretically one particular non-equilibrium question which we hope would be possible to investigate experimentally in the near future. The question is the following: Following a rapid quench of the Bose gas from a high temperature disordered state to a low temperature ordered state, how does the condensate density grow from its initial value zero to its equilibrium value corresponding to the final temperature? A few recent papers [3,4] addressing this question have focused on the early-time (of the order of a few collision times) nonuniversal dynamics. However, as also noted in Ref. [5], the interesting experimental questions are instead associated with the late time dynamics that involves the coarsening of the Bose condensate order parameter field. Our analytical and numerical studies of this late time dynamics of the Bose gas shows, in addition to answering the question on the growth of condensate density, that this dynamics represents a new universality class of phase ordering kinetics.

Before elaborating on the nonequilibrium question, it would be useful to recapitulate briefly the equilibrium behaviour of the Bose gas. A dilute Bose gas (with repulsive interactions) at equilibrium undergoes a phase transitions from a high temperature “normal” state to a low temperature “superfluid” state at a nonzero critical temperature  $T_c$  in space dimensions  $d > 2$ . This low temperature superfluid state

has true long range order, i.e., the correlation function of the complex Bose order parameter field  $\psi(r)$  approaches a nonzero constant,  $\langle \psi^*(0)\psi(r) \rangle \rightarrow |\langle \psi \rangle|^2$  as  $r \rightarrow \infty$ . On the other hand, in  $d = 2$ , the Bose gas undergoes a Kosterlitz-Thouless transition at  $T = T_{KT}$ . For  $T < T_{KT}$ , the system has quasi long range order, i.e., the correlation function decays to zero as  $r \rightarrow \infty$  but in a slow power law fashion,  $\langle \psi^*(0)\psi(r) \rangle \sim r^{-\eta(T)}$  where the exponent  $\eta(T)$  depends continuously on temperature ranging from  $\eta(0) = 0$  to  $\eta(T_{KT}) = 1/4$ . Associated with this phase transition what also happens in the low temperature ordered phase is Bose condensation, namely, the average number of particles in the  $k = 0$  mode becomes macroscopic, i.e.,  $\langle n(k=0) \rangle = \langle |\psi(k=0)|^2 \rangle \sim L^d$  for  $d > 2$  and  $\sim L^{2-\eta(T)}$  for  $d = 2$ . It has been argued [6] that the statics of the Bose gas is in the same universality class as that of the classical  $XY$  model of ferromagnets.

We now come to the non-equilibrium question. Suppose that we prepare our Bose gas at equilibrium at a very high initial temperature  $T_i$  where the condensate density is zero and then rapidly quench the system to a temperature  $T_f$  below the transition temperature. We start the clock immediately after the quench and monitor the condensate density  $\rho_0(t) = \langle n(k=0, t) \rangle / L^{d'}$  (where  $d' = d$  for  $d > 2$  and  $d' = 2 - \eta(T)$  for  $d = 2$ ) as a function of time. As time progresses, this density is expected to rise from its initial value 0 and then eventually saturate to its equilibrium value (a nonzero number of  $\sim O(1)$ ). We would like to know the precise form of this temporal evolution. We show below that this time evolution of the condensate density can be described in a natural and precise language by using the phenomenology of phase ordering kinetics developed recently for dissipative classical spin systems, as reviewed by Bray [7]. As a byproduct of this study, we find that while the statics of the Bose gas and classical  $XY$  model are in the same universality class, the non-equilibrium dynamics of these two systems belong to different universality classes.

In Section-2, we briefly review the scaling hypotheses of the phase ordering theory and discuss the scaling predictions drawn from this general theory for the Bose gas. In Section-3, we introduce a solvable toy model of coarsening of Bose gas in  $d = 2$  and present its exact solutions to illustrate the importance of nondissipative Poisson bracket terms in the equations of motion. Section-4 contains a discussion on the Gross-Pitaevski (GP) equation that describes the evolution of the Bose gas in a ‘‘microcanonical’’ ensemble. Section-5 contains the numerical results on the GP equation in  $d = 2$  and 3 that verifies the scaling predictions of Section-2. Finally, we summarize and conclude in Section-6. A shorter version of this work has appeared elsewhere [8].

## 2. Scaling Predictions From Phase Ordering Theory

In the theory of phase ordering kinetics [7], one considers the evolution of a classical spin system (such as Ising model) after a rapid quench from a high temperature disordered phase to a low temperature ordered phase. The dynamics is assumed to be overdamped and purely relaxational in which each spin simply moves along the steepest downhill direction in the instantaneous energy landscape. Locally ordered regions will appear immediately after the quench, but the orientation of the spins in each region will be different. The coarsening process is then one of alignment of neighbouring regions, usually controlled by the motion and annihilation of topo-

logical defects (domain walls for Ising spins, vortices for  $XY$  spins, etc.). As time progresses, these domains grow in size as the system tries to achieve local ordering on larger and larger scales. A key step in the theory is the introduction of a single length scale  $l(t)$ , a monotonically increasing function of the time  $t$ , which is about the size of a typical ordered domain at time  $t$ . At late times when  $l(t)$  is larger than the microscopic length scales such as the range of interactions or the lattice spacing, it is believed that the late stage morphology of the system is completely characterized by  $l(t)$ , and is independent of microscopic details or initial conditions (as long as the initial condition is short ranged), i.e., it is universal. At late times,  $l(t)$  typically grows like a power law,  $l(t) \sim t^{1/z}$  where the exponent  $z$  depends upon the various conservation laws satisfied by the dynamics. For nonconserved Ising or  $XY$  spin dynamics, it is well established that  $z = 2$  [7].

The morphology of growing domains is characterized by various time dependent correlation functions which exhibit universal scaling behaviour. For the purpose of comparison with the Bose gas, let us illustrate these scaling predictions for the 2 component classical  $XY$  model. Let  $\psi(r, t)$  denote the order parameter field for the  $XY$  model. Then in an infinite system, the scaling hypothesis of phase ordering kinetics predicts a scaling form for the equal time correlation function,  $G(r, t) = \langle \psi(0, t)\psi(r, t) \rangle \sim r^{-\eta}g[rt^{-1/z}]$  where  $\eta = 0$  for  $d > 2$  and in  $d = 2$ ,  $\eta = \eta(T)$ , the usual temperature dependent exponent associated with the final equilibrium state. This means that the structure factor, the Fourier transform of  $G(r, t)$ , will scale as  $S(k, t) \sim t^{(d-\eta)/z}\tilde{g}(kt^{1/z})$ . In particular, the  $k = 0$  mode will grow as  $S(0, t) \sim t^{(d-\eta)/z}$  in an infinite system.

However, in a finite system of linear size  $L$ , the system will stop coarsening and attain the equilibrium ordered state when  $l(t) \sim L$ . In that case, these scaling behaviour of the infinite system would be modified by finite size scaling (FSS). For example, it would then predict a FSS form for the equal time correlation function,  $G(r, t) \sim L^{-\eta}\Phi_G[r/L, t/L^z]$  and hence  $S(0, t) \sim L^{d-\eta}\Phi[t/L^z]$ . This last scaling function  $\Phi$  goes to a constant for  $t \gg L^z$  and the system attains equilibrium after  $t \sim L^z$ . The value of  $z$  is known to be 2 for the classical  $XY$  model [7] apart from the logarithmic corrections in  $d = 2$  [15,16].

Now consider the Bose gas. The order parameter in this case is the boson annihilatin field  $\psi(r, t)$  which is complex; the phase of the expectation value of  $\psi$  is aligned across the system in the equilibrium Bose-condensed state. A key point is that after relatively few atomic collisions, once the domain size  $l(t)$  is large enough (e.g., larger than the de Broglie wavelength), it is permissible [4] to treat  $\psi(r, t)$  as a *classical* field which obeys Hamilton-Jacobi equations of motion ( for a related discussion on the emergence of classical dynamics in the equilibrium properties of an antiferromagnet, see Ref. [9]). It must be kept in mind that it is only the equations of motion for the collective order parameter which are classical-the very existence of the complex order parameter is due entirely to quantum mechanics, and the fact that there is a wavefunction for the condensate..

If we believe that the late time ordering dynamics of the Bose field can be described by the scaling phenomenology of the phase ordering theory described above (even though the equation of motion for the Bose field is different from that of the classical spins), then one obtains the prediction that the  $k = 0$  mode of the structure function, which in the Bose gas case is just the number of particles in the  $k = 0$

mode, will scale as  $S(0, t) \sim L^{d-\eta}\Phi[t/L^z]$ . Then once we know the exponent  $z$  and the scaling function  $\Phi$ , it will give us the temporal evolution of the condensate density, the question we started out with.

Of course the value of the exponent  $z$  and the scaling function  $\Phi$  may be different from the classical  $XY$  case as they depend on the conservation laws satisfied by the equations of motion of the order parameter. Indeed one of the central results of this paper is to establish that while this scaling prediction holds for the Bose gas, the exponent  $z$  in the Bose gas (evolving via the GP equation as discussed later) is different from that of the  $XY$  model. This is due to the important property of the equations of motion for  $\psi$ , discussed in Section-4, that they are not simply relaxational. Instead, they contain nondissipative, kinematical “streaming” or “Poisson bracket” terms [10]. One such term is responsible for the Josephson precession of the phase of  $\psi$  at a rate determined by the local chemical potential. One of our main objectives is to understand the consequences of such terms on the phase ordering theories. We will argue that the Josephson term constitutes a relevant perturbation on the dynamics and that the resulting coarsening process belongs to a new universality class.

In Section-5, we give numerical evidence supporting the scaling of  $S(0, t)$  as predicted above and also determine the exponent  $z$  numerically. But before doing that, in the next Section, we consider a solvable toy model of coarsening which will prove that indeed the “streaming terms” in the equation of motion are relevant perturbations. This will prepare us to expect that the value of  $z$  in the Bose gas dynamics might be different from the classical  $z = 2$  of the  $XY$  model and also give us important physical insights as to why they are different.

### 3. A Solvable Toy Model of Coarsening

In this Section we consider a simple toy model of coarsening that illustrates the possible consequences of the Josephson term in a simple setting. Consider the Bose gas in  $d = 2$ . As mentioned earlier, for  $T < T_{KT}$ , the Bose gas is superfluid. Now consider the phase ordering process in which the Bose gas is rapidly moved at time  $t = 0$  from contact with a reservoir at an initial  $T = T_i$ , to a reservoir with a final  $T = T_f$ , such that  $T_f < T_i < T_{KT}$ . A similar quench was considered in Ref. [12] for the purely dissipative  $XY$  model. As time progresses, the system will approach the equilibrium configurations corresponding to  $T = T_f$  starting from the initial configurations corresponding to  $T_i$ ; this ordering will proceed simply via the spin wave dynamics. This is in contrast to the ordering via annihilation of vortex-antivortex pairs as in the case of quench from a temperature above  $T_{KT}$ . Since there are no vortices in the initial configurations, they won’t be generated because the system will only reduce its energy.

Indeed in the long time limit, all vortices and fluctuations in the amplitude of  $\psi$  can be neglected, and we may parametrize  $\psi = \exp(i\phi)$ . The free energy density in the purely dissipative  $XY$  model is now determined simply by the gradients of the phase  $\sim (\nabla\phi)^2$ . In the case of the Bose gas, it is also necessary to take the conserved number density into account. Let  $m$  be proportional to the deviation of the particle

density from its mean value; then the free energy density we shall work with is

$$\mathcal{F} = \frac{1}{2} \int d^2r [(\nabla\phi)^2 + m^2]. \quad (1)$$

We have rescaled spatial coordinates and  $m$  to obtain convenient coefficients in  $\mathcal{F}$ . Note that the fields  $m$  and  $\phi$  are not independent but are related via the Poisson bracket

$$\{m(r), \phi(r')\} = g_0 \delta(r - r'), \quad (2)$$

where  $g_0$  is a constant. The origin of the Josephson precession term, whose effects on dynamics we wish to study, lies in this Poisson bracket. The method reviewed in Ref. [10] now leads to the *linear* equations of motion [6]

$$\frac{\partial\phi}{\partial t} = \Gamma_0 \nabla^2 \phi + g_0 m + \theta, \quad \frac{\partial m}{\partial t} = \lambda_0 \nabla^2 m + g_0 \nabla^2 \phi + \zeta, \quad (3)$$

where the coefficients  $\Gamma_0, \lambda_0 > 0$  represent the dissipation arising from coupling of the system to the reservoir. The effects of the reservoir are also contained in the Gaussian thermal noise sources  $\theta$  and  $\zeta$  with zero mean and (for  $t > 0$ ) correlations appropriate to  $T = T_f$ :  $\langle \theta(r, t) \theta(r', t') \rangle = 2\Gamma_0 T_f \delta(r - r') \delta(t - t')$ ,  $\langle \zeta(r, t) \zeta(r', t') \rangle = -2\lambda_0 T_f \nabla^2 \delta(r - r') \delta(t - t')$ , and  $\langle \zeta(r, t) \theta(r', t') \rangle = 0$  ( $k_B = 1$ ). Equations (3) are linear, can be easily integrated and all correlations can be computed exactly.

Let us first recall the structure of the solutions expected from naive scaling [7] for  $d = 2$ . One expects a single length scale growing as  $l(t) \sim t^{1/z}$ . Also the morphology of the evolving patterns are characterized by two types of correlation functions: (i) The equal time correlator  $G(r, t) = \langle \psi^*(r, t) \psi(0, t) \rangle$  is expected to scale as  $G(r, t) \sim r^{-\eta_f} g(r/t^{1/z})$  where  $g$  is a universal scaling function and  $\eta_f$  is the equilibrium exponent of the quasi long range order at  $T = T_f$  as mentioned in Section-2. (ii) The unequal-time correlation function  $C(r, t) = \langle \psi^*(r, t) \psi(0, 0) \rangle$  for which we expect for large  $r$  and  $t$ ,  $C(r, t) \sim t^{-\lambda/z} f(r/t^{1/z})$  where  $f$  is a universal scaling function, and  $\lambda$  is the autocorrelation exponent.

It turns out that our model  $\mathcal{F}$  does not completely obey the simple scaling hypotheses as stated above. This becomes clear upon considering the two-time correlator  $C$  whose explicit exact solution turns out to depend upon two time-dependent length scales  $l_1(t) \sim (at)^{1/2}$  and  $l_2(t) \sim g_0 t$  (with  $a = (\Gamma_0 + \lambda_0)/2$ ). It actually obeys the scaling form  $C(r, t) \sim t^{-(3\eta_i + \eta_f)/4} \tilde{f}[r/(at)^{1/2}, r/(g_0 t)]$  (where  $\eta_i = T_i/2\pi$ ). The dependence of these scales on  $g_0$  suggests that  $g_0$  is a relevant perturbation with renormalization group eigenvalue 1, in the language of Ref. [7]. The scaling function  $\tilde{f}$  is found to be

$$\tilde{f}(x_1, x_2) = \exp\left[-\eta_i \int_0^\infty \frac{dy}{y} \{1 - J_0(y)\} \cos(y/x_2) e^{-y^2/x_1^2}\right]. \quad (4)$$

For  $r \sim l_1(t)$ , using  $\tilde{f}(x_1, x_2 \rightarrow 0) = 1$ , we find that the autocorrelation  $C(0, t) \sim t^{-(3\eta_i + \eta_f)/4}$  in contrast to the result in the model of Ref. [12]  $C(0, t) \sim t^{-(\eta_i + \eta_f)/4}$ . On the contrary, one could insist on a scaling picture using only the single larger length scale  $r \sim l_2(t)$ , and would then need  $\tilde{f}(x_1 \rightarrow \infty, x_2)$  which equals  $[1 + \sqrt{1 - x_2^2}]^{-\eta_i}$  for  $x_2 < 1$  and equals  $x_2^{-\eta_i}$  for  $x_2 \gg 1$ . It can also be checked

that one recovers the initial equal time equilibrium result for  $C(r, t)$  when  $r \rightarrow \infty$  with  $t$  large but fixed. We also note that the relevance of  $g_0$  is evident in the auto-correlations of  $m$ . We find  $\langle m(0, t)m(0, 0) \rangle \sim \frac{1}{t} f_1(g_0 \sqrt{t/a})$  where

$$f_1(\tau) = 4\pi^2 \eta_i \left[ 1 - \int_0^\infty \sin ye^{-y^2/2\tau^2} dy \right]; \quad (5)$$

clearly, for  $g_0 = 0$ , this autocorrelator decays as  $1/t$  for large  $t$ , while for nonzero  $g_0$  it decays faster as  $t^{-2}$ . Finally, results on the equal-time  $\psi$  correlator  $G$  are as follows. It has a crossover time  $t_1 \sim \tilde{a}/g_0^2$  with  $\tilde{a} = |\Gamma_0 - \lambda_0|$ ; this time is similar to the crossover time in  $\langle m(0, t)m(0, 0) \rangle$ , except that  $\tilde{a}$  has replaced  $a$ . Both for  $t \ll t_1$  and for  $t \gg t_1$ ,  $G$  obeys a scaling form similar to that obtained in Ref. [12] (which has  $g_0 = 0$ ):  $G(r, t) \sim r^{-\eta_f} g(r/(\gamma t)^{1/2})$  where  $g$  is the scaling function described in [12]; however, the rate  $\gamma = \Gamma_0$  for  $t \ll t_1$  and  $\gamma = a$  for  $t \gg t_1$ .

While this phase only model  $\mathcal{F}$  is not relevant for studying quenches from above the transition temperature (since it neglects the nonlinear terms and hence the vortices which are the elementary defects for the quench from high temperatures), the exact solution of this linear model is quite instructive. It clearly emphasizes the importance of the nondissipative Josephson coupling term. In fact as seen above, the presence of this term ( $g_0 \neq 0$ ) changes the universality class of the dynamics. Thus it is reasonable to expect that even for quenches from above the transition temperature, this term would play an important role. In fact this is what we demonstrate in the next section by studying the full nonlinear equation that describes the time evolution of the order parameter.

#### 4. Coarsening of Bose Gas: A Deterministic Microcanonical Approach

Consider the quench of the Bose gas from above the transition temperature. In this case, the initial configuration, being a typical high temperature configuration, contains several different types of topological defects (excitations). As the system dissipates energy with time, most of these defects will disappear after a short transient time leaving behind only the elementary excitations. In  $d = 2$ , these elementary excitations are point vortices and in  $d = 3$  they are vortex lines. As time progresses, these elementary defects move around and annihilate each other upon meeting (and thereby release energy) and the system becomes more and more ordered. To study this coarsening process that proceeds via the annealing of defects it is necessary to study the evolution of both the phase and amplitude of  $\psi$ .

This growth of long range order in the system can be studied in two ways. In one case one considers the deterministic time evolution of an *isolated* Bose gas, not in contact with a heat bath. What we find in our study is that *though the dynamics in this case is nondissipative, the system still exhibits an irreversible approach to the equilibrium*. In the other case, the Bose gas is in contact with a heat bath and its evolution equations are therefore necessarily stochastic. These are the analogues of *microcanonical* and *canonical* ensembles in equilibrium statistical mechanics. Most previous studies on coarsening have been done in the stochastic “canonical” ensemble. While it may be reasonable to expect that both descriptions may lead to same results for the universal scaling properties, a word of caution, however, is warranted since this

equivalence is well established only for equal time properties of equilibrium systems. Equilibrium and nonequilibrium dynamics may be more subtle; indeed, in a recent study [11] of unequal time dynamics of the quantum Ising chain in a transverse field it was found that the underlying deterministic quantum dynamics did not map onto any known classical stochastic model.

In this paper, we use only the deterministic “microcanonical” approach and do not make any statement about the “canonical” results. To the best of our knowledge, this deterministic “microcanonical” approach has never been used before to study coarsening in any system. The use of this approach is not just cosmetic, in fact it has some advantages over the “canonical” approach, atleast for the Bose gas. As we will see below, the dynamics in the “microcanonical” approach is completely specified by the Hamiltonian of the system with no additional phenomenological parameters. The “canonical” dynamics, on the other hand, needs several phenomenological constants as input parameters. Therefore, the “microcanonical” dynamics is much easier to implement numerically and one does not need to do a “time consuming” search in a rather big parameter space as in the “canonical” case.

For the isolated Bose gas (“microcanonical” ensemble), an excellent approximation for the total energy of an order parameter configuration  $\psi(r, t)$  is  $\mathcal{H} = \int d^d r [|\nabla\psi|^2 + \frac{u}{2}|\psi|^4]$ , where the length scales have been rescaled to make the coefficient of the gradient term unity, and  $u > 0$  is the two-particle  $T$  matrix at low momentum, representing the strength of the repulsive onsite interaction. The standard Hamilton-Jacobi equation of motion for  $\psi$  follows using the Poisson bracket  $\{\psi, \psi^*\} = i$

$$i\frac{\partial\psi}{\partial t} = [-\nabla^2 + u|\psi|^2]\psi, \quad (6)$$

and is well known [13] Gross-Pitaevski (GP) or nonlinear Schrodinger equation. We can also add a quadratic  $|\psi|^2$  term to  $\mathcal{H}$ , and it leads to a term linear in  $\psi$  in the GP equation; however this linear term can be eliminated by an innocuous global phase change in  $\psi$ . The GP equation conserves the total number of particles  $\int d^d r |\psi|^2$ , the total momentum, and  $\mathcal{H}$ , and hence there is no global dissipation of energy. Nevertheless, in the thermodynamic limit, the GP equation does display irreversible coarsening, as will be abundantly clear from our numerical results to be described in the next Section. A random initial state with a negligible number of particles in the zero momentum ( $k$ ) state (i.e., short range initial correlations), evolves eventually to a state with a condensate fraction equal to that expected at equilibrium in the microcanonical ensemble at the total energy of the initial state. Basically while the total energy of the system is conserved, there is nevertheless an energy flow from the low momentum states to high momentum states thus effectively making the system more and more ordered as time progresses.

In the “canonical” approach on the other hand, it is permissible to add dissipative terms to the equation of motion of  $\psi$ . A simple additional damping term to the GP equation leads to a model expected to be in the same universality class of the so-called Model-A [10,7]; this model is, however, not acceptable: it violates the local conservation of the particle density, and, as discussed before Eq. (3), it is necessary [10,14] to introduce the density fluctuation field,  $m(r, t)$ ; the value of  $|\psi(r, t)|^2$  is then the contribution to the particle density from the low momentum states, while  $m(r, t)$  represents the density fluctuation from all states; the Poisson bracket in this case is

$\{m(r), \psi(r')\} = ig_0\psi(r)\delta(r - r')$ . This is model F in the language of Ref. [10]. (It is probably also necessary to introduce additional fields to account for other conserved quantities: a momentum density as in Model H or an energy density as in Model C of Ref. [10].) Note that the strength of the crucial precession term in the dynamics is controlled by  $g_0$  which is an adjustable phenomenological parameter (however, in the Hamiltonian dynamics of the microcanonical approach, there is no such freedom). Numerical study of coarsening using model  $F$  could thus be complicated by crossover effects associated with the adjustable value of  $g_0$  ( $g_0 = 0$  corresponds to the purely dissipative model-A dynamics, which is clearly in a different universality class).

We therefore restrict our numerical study here to the “microcanonical” approach to coarsening using the GP equation. These results are the subject of the following Section.

## 5. Numerical Results

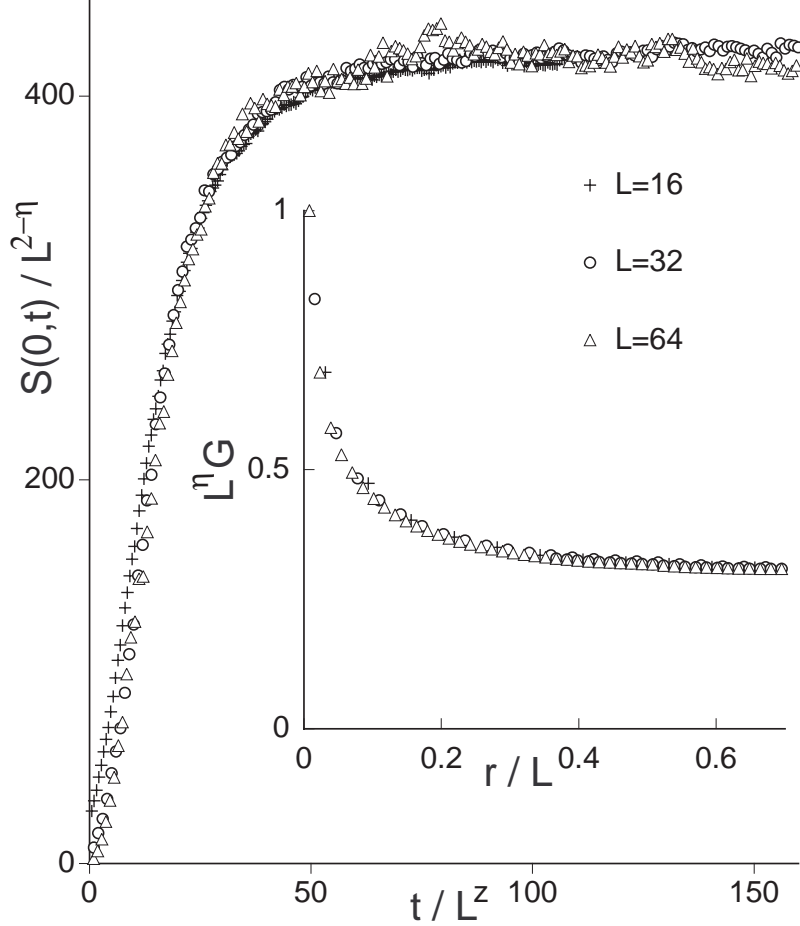
All of the numerical results obtained so far are consistent with the simplest naive scaling hypotheses described earlier, and do not require the introduction of two length scales, as was necessary in the linear model of Section-3 (though we have not yet obtained numerical results on unequal-time correlations, for which the linear model  $\mathcal{F}$  clearly displayed two length scales). We will present results both in  $d = 2$  and  $d = 3$ . The  $d = 2$  system allowed us to study larger sizes with better finite-size scaling properties.

We discretized Eq. (6) on a lattice, and integrated in time using a fast Fourier transform based algorithm which conserved energy and particle number to a high accuracy. We work in units where the lattice spacing is unity and choose the scale of the lattice field to make the number density unity also. We set  $u$  to be approximately 0.25 so that we are considering a dilute gas. We choose an ensemble of initial conditions with a narrow distribution of energy, whose width goes to zero in the thermodynamic limit. We assign initial values to the Fourier components  $\psi(k, 0)$  as follows:  $\psi(k, 0) = \sqrt{n_0(k)} \exp[i\phi(k)]$  where the  $\phi(k)$ 's are independent random variables chosen from a uniform distribution with range  $[0, 2\pi]$  and the function  $n_0(k)$  is chosen to ensure that initial real-space correlations are short ranged (corresponding to a “high-temperature” configuration) while still having low enough energy so that the equilibrium state corresponding to this energy is superfluid. Though the ensemble of initial conditions is not strictly the Gibbs distribution at any temperature, it is however expected that the precise details of the initial conditions do not matter for the late time universal properties as long as the initial correlations are short ranged. More specifically we chose

$$n_0(k) = \frac{c}{[\epsilon(k) + \mu_1]} \frac{1}{[1 + \exp\{(\epsilon(k) - \mu_2)/T\}]}, \quad (7)$$

where  $\epsilon(k)$  is the Fourier representation of the lattice version of the Laplacian and  $c$  sets the overall scale of  $n_0(k)$ . Here one chooses the parameters  $\mu_1$ ,  $\mu_2$ , and  $T$  to achieve the appropriate trade-off between energy and correlation length. Note that this careful choice of initial conditions is needed as the GP equation does not have any explicit dissipation and the system evolves in the phase space on a constant



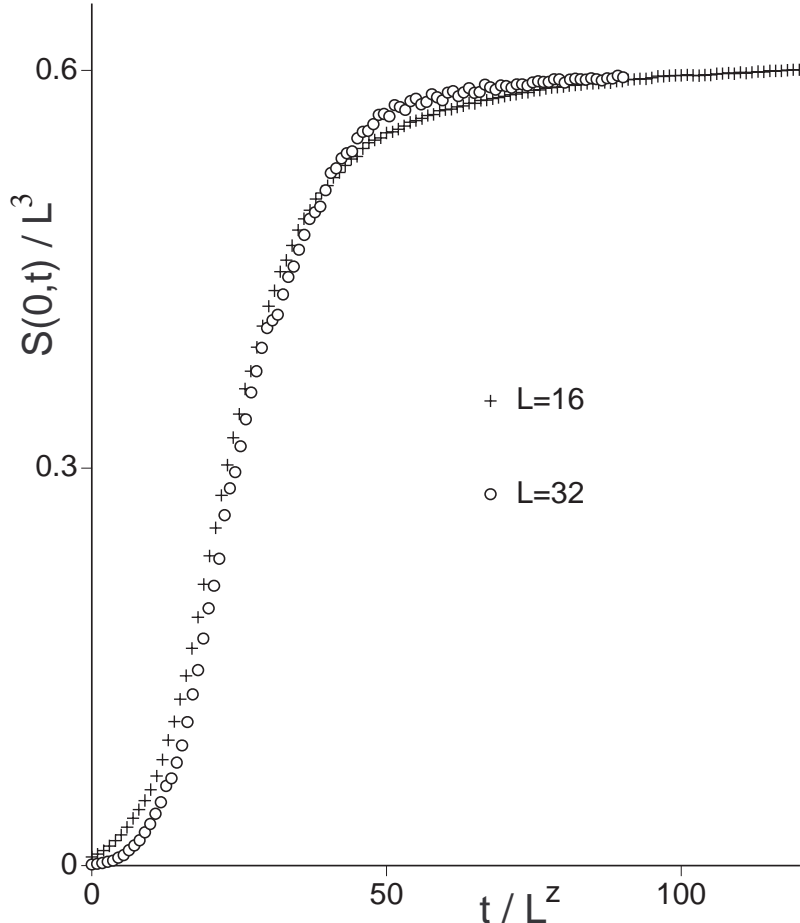


**Figure 1.** Numerical results from the simulation of GP equation in  $d = 2$ . The number of particles in the zero momentum state is  $S(0, t)$  and the figure shows its scaling properties as a function of the system size  $L$  and time  $t$ . The inset shows the scaling of the equilibrium equal-time correlation function  $G(r, t \rightarrow \infty)$ . The best scaling collapse was obtained in both plots for  $\eta \approx 0.27$  and  $z \approx 1.1$ . The scale of all axes (except the values of  $r/L$ ) are arbitrary.

energy surface. So, one has to choose this constant energy surface via tuning the initial conditions in such a way that ensures that there are indeed some long-ranged configurations on this surface where the system can finally go to. The point is that if there are such long-ranged configurations on the constant energy manifold then the dynamics of the system evolving via the GP equation takes the system automatically and irreversibly (in the thermodynamic limit) to those long-ranged configurations. So the choice of this complicated initial condition just ensures that there are indeed such long-ranged configurations on the constant energy surface.

We tested the finite size scaling form mentioned in Section-2 for the equal time correlator:  $G(r, t) = L^{-\eta} \Phi_G[r/L, t/L^z]$  where  $\eta = 0$  in  $d = 3$  and in  $d = 2$ ,  $\eta$  is the exponent associated with the final equilibrium state. We also computed the number of particles in the  $k = 0$  mode  $S(0, t)$  expected to scale as  $S(0, t) \sim L^{2-\eta} \Phi[t/L^z]$  as mentioned also in Section-2.

Results for  $d = 2$  are shown in Fig. (1). We performed finite-size scaling



**Figure 2.** Numerical results for the GP equation in  $d = 3$ . The notation is as in Fig. 1, with the exponent  $z \approx 1.15$ .

analysis for  $L = 16, 32$ , and  $64$  and found reasonable data collapse with  $\eta \approx 0.27$  and  $z \approx 1.1$ . The value of  $\eta$  indicates that we are at a nonzero temperature close to  $T_{KT}$ ; strictly speaking we must have  $\eta \leq 1/4$ , but the value of  $\eta$  is relatively  $T$  independent near  $T_{KT}$ , and the discrepancy is within our numerical errors. The value of  $z$  is in sharp contrast to the  $z = 2$  (with logarithmic corrections) result obtained by various groups [15,16] for the purely dissipative Model-A dynamics [10] (obtained from Model F by setting  $g_0 = 0$  and ignoring the  $m$  field) of classical  $XY$  spins. Although we have determined the value of  $z$  for a quench to a particular temperature  $T_f = 2\pi\eta \approx 1.695$  (in units of  $k_B = 1$ ), we expect that  $z$  is same for all  $0 < T_f < T_{KT}$ . Results for  $d = 3$  are shown in Fig. (2) for linear sizes  $L = 16$  and  $32$ . The data collapse is not as good as in  $d = 2$ , but again we obtained a  $z \approx 1.1$ . Thus our numerical results, both in  $d = 2$  and  $3$ , are consistent with a value of  $z = 1$ , which is also the result suggested by the exact calculation in the phase only model  $\mathcal{F}$  of Section-3.

## 6. Conclusion

We close with some physical discussion on reasons for the difference between the deterministic GP model, and quenches in the stochastic and purely dissipative Model A [15,16]. The dynamics in the GP model proceeds via the annihilation

of nearby vortex-antivortex pairs (in  $d = 2$ ) as in Model A. However there is an important difference between the two in details of the vortex motion. In Model A, oppositely charged vortices attract each other with a force that falls off as the inverse of their separation (apart from logarithmic corrections). The overdamped dynamics causes the vortices to then move towards each other with a velocity proportional to attractive force, and this implies  $l(t) \sim t^{1/2}$ . In the GP model, on the other hand, the situation is much more complex. In addition to vortices, the system also has a propagating “spin-wave” mode arising from the “streaming” terms in the equation of motion. The finite velocity of this propagating mode gives rise to the linear length scale  $l(t) \sim t$ . A pair of oppositely charged vortices, apart from interacting with the spin-wave background, also has an attractive force between them. However, now the underlying dynamics causes the pair to move with uniform velocity in a direction perpendicular to the line joining them (the force leading to this motion is often called the Magnus force). These qualitative differences in the nature of the defect dynamics change the universality class of the coarsening process of the Bose evolving via the GP equation.

In summary, we have presented evidence, both analytical and numerical, that the phase-ordering dynamics of an isolated Bose gas belongs to a new universality class. A particular conclusion of this work is that the condensate density of the Bose gas, following a sudden quench from the normal to the superfluid phase in dimensions  $d \geq 2$ , will grow at late times in a power law fashion as  $t^{d/z}$  before saturating to its final equilibrium value. Our work, both analytical and numerical, provide evidence that  $z = 1$  for the Bose gas evolving via the GP equation. Whether this value of  $z$  is same as that of the “canonical” Model-F of Ref. [10] remains an open question. In fact, recent numerical results on “canonical”, stochastic Model-F do indicate that the value of  $z$  might be 2 for that case [17]. (We speculate this may be because while Model-F has accounted for the conserved number density, it has not accounted for the conserved energy and momentum densities of the GP dynamics.) While the value of  $z$  may be different for different dynamics, the scaling prediction of the power law growth of the condensate density  $\sim t^{d/z}$  at late times remains valid and needs to be tested experimentally. How this simple scaling gets modified in presence of harmonic traps has been discussed in Ref. [18].

Given the latest advances in the experiments on Bose systems as we heard from various exciting talks in this conference, we may conclude with the hope that experimental verifications of our theoretical predictions summarized in the preceding paragraph may not be far off.

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